To Extract the Cube Root see preface page 15

A

TREATISE

ON

MENSURATION,

BOTH IN

THEORY AND PRACTICE.

THE SECOND EDITION, WITH MANY ADDITIONS.

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LONDON:

PRINTED FOR G. G. J. AND J. ROBINSON, AND R. BALDWIN, PATERNOSTER-ROW; AND G. AND T. WILKIE, ST. PAUL'S CHURCH-YARD.

MDCCLXXXVIII.

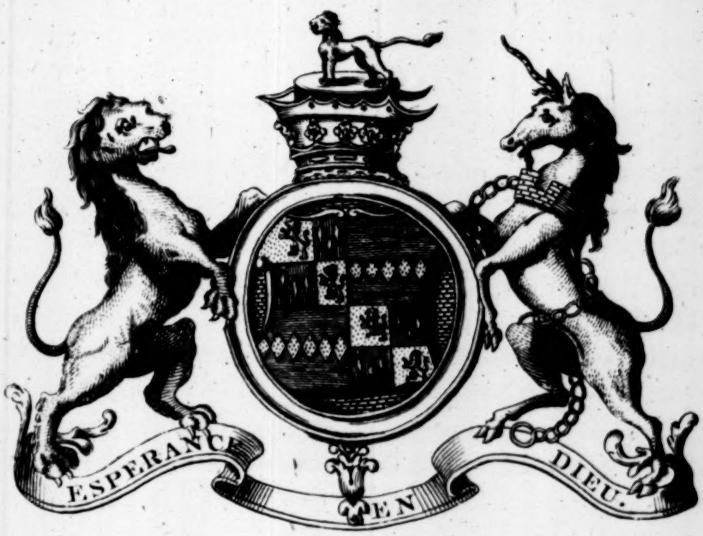
Measures of Surface

The imperial square yard contains 9 imperial square feet and the imperial square foot 144 impl. square inches. The circular foot (that is, a circle whose diameter is one foot) cont-ains 183.346 circular inches (that is circles w[hose] diameters are each 1 inch. The French squ[are] foot contains 163.563 impl. square inches & the square decimetre 15.506 impl. square in[ches]

Measures of Volume

The imperial cubic (or solid yard) contains 2[7] imperial cubic feet, and the imperial cubic foot contains 1728 imperial cubic inches. The cylindrical foot ~~contains~~ (that is - a cylinder 1 foot long & 1 foot in diameter) contain[s] 1357.17 cubic inches. The spherical foot (th[at] is a sphere 1 foot in diameter) contains 904.78 cubic inches; and a conical foot (that is a cone 1 foot in height and 1 foot in diamete[r] at the base) contains 452.39 cubic inches. The cubic foot contains very nearly 22[00] cylindrical inches (that is cylind~~ers~~ - 1 inch long and 1 inch in diameter) it con-ains very nearly 3300 spherical inche[s] that is, spheres 1 inch in diameter; & it con-ins very nearly 6600 conical inches, th[at] is cones 1 inch in height & 1 inch in diameter at the base

277.274 or 277¼ cubic inch 1 imperial ga[llon]
2218.192 or 2218⅕ do 1 do Bush[el]
A cubic foot contains 6.232 Gallons
An imperial gallon of Distilled Wate[r] weighs 10 pounds avoirdupoise.



TO

THE MOST NOBLE AND PUISSANT PRINCE,

H U G H,

DUKE AND EARL OF NORTHUMBERLAND,

EARL AND BARON PERCY, LUCY, POYNINGS, FITZPAYNE, BRYAN, LATIMER, AND WARKWORTH, AND BARONET;

LORD LIEUTENANT AND CUSTOS ROTULORUM OF THE COUNTY OF NORTHUMBERLAND, AND OF THE TOWN AND COUNTY OF THE TOWN OF NEWCASTLE UPON TYNE; VICE ADMIRAL OF THE COUNTY OF NORTHUMBERLAND; AND LIEUTENANT GENERAL OF HIS MAJESTY'S FORCES, COLONEL OF THE SECOND TROOP OF HORSE GRENADIER GUARDS,

&c. &c. &c.

MY LORD,

IF the high honour this Work formerly received in the countenance of Your Grace's illustrious Father, had not particularly encouraged the Author's presumption, in thus seeking the protection of his noble Representative, he is convinced that he could not easily have found among the great and elevated, a character under whose

auſpices he ſhould more earneſtly have wiſhed to preſent it to the world in this its improved ſtate.

The great progreſs of all the uſeful arts and ſciences in this country, ſince the happy æra of the acceſſion of our auguſt Sovereign, muſt be aſcribed, next to his benign and munificent influence, to that eminent countenance which ſome of the greateſt characters have ſhewn, by their perſonal example in the cultivation of them, not leſs than by their patronage and protection.

To the hereditary fame of Your Moſt Noble Progenitors, Your Grace's perſonal bravery and military talents have added great and diſtinguiſhed luſtre. The public confidence and eſteem which neceſſarily follow ſuch accompliſhments, ſtamp a value on every thing, however inconſiderable, which is honoured with Your Grace's patronage: And the accurate judgment which Your Grace is known to poſſeſs on all ſubjects of extenſive practical utility, makes it the wiſh of ſuch as cultivate them, ardently to ſeek Your Grace's protection.

With theſe impreſſions, I preſume to lay the following performance at Your Grace's feet; and am, with the profoundeſt reſpect,

My Lord,

Your Grace's

moſt obedient, and

moſt devoted humble ſervant,

CHARLES HUTTON.

PREFACE.

BY Menfuration I underftand the art and fcience which is concerned about the meafure of extenfion, or the magnitude of figures; and it is, next to arithmetic, a fubject of the greateft ufe and importance, both in affairs that are abfolutely neceffary in human life, and in every branch of the mathematics: a fubject by which fciences are eftablifhed, and commerce is conducted; by whofe aid we manage our bufinefs, and inform ourfelves of the wonderful operations of nature; by which we meafure the heavens and the earth, eftimate the capacities of all veffels, and bulks of all bodies: gauge our liquors, build edifices, meafure our lands and the works of artificers, buy and fell an infinite variety of things neceffary in life, and are fupplied with the means of making the calculations which are neceffary for the conftruction of almoft all machines.

It is evident that the clofe connection of this fubject with the ordinary affairs of life, would very early evince its importance to mankind; and accordingly we find, that the moft celebrated philofophers have paid the greateft attention to it; and the chief and moft effential difcoveries in geometry in all ages, have been made in confequence of their attempts to improve this fcience. Socrates thought that the prime ufe of geometry was to meafure the ground, and indeed this bufinefs gave name to the fubject; and moft of the ancients feem to have had no other end befides menfuration in view, in all their laboured geometrical difquifitions. Euclid's Elements are almoft entirely devoted to it; and although there be contained in them many properties of geometrical figures which may be applied to other purpofes, and indeed of which the moderns have made the greateft ufe in moft other fciences; yet Euclid himfelf feems to have adapted them entirely to this purpofe. For, if it be confidered that his Elements contain a continued chain of reafoning, and of truths, of which the former are fucceffively applied to the difcovery of the latter, one propofition depending on another, and the fucceeding propofitions ftill approximating towards fome particular object near the end of each book; and when at the laft we find that object to be the equality, proportion, or relation between the magnitudes of figures, both plane and folid; it is fcarcely

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poſſible to avoid allowing this to have been Euclid's grand object. And accordingly he determined the chief properties in the menſuration of rectilineal plane and ſolid figures; and ſquared all ſuch planes, and cubed all ſuch ſolids. The only curve figures which he attempted, are the circle and ſphere; and when he could not accurately determine their meaſures, he gave an excellent method of approximating to them, by ſhewing how in a circle to inſcribe a regular polygon, which ſhould not touch another circle, concentric with the former, although their circumferences ſhould be ever ſo near together; and, in like manner, between any two concentric ſpheres to deſcribe a polyhedron, which ſhould not any-where touch the inner one: and approximations to their meaſures are all that have hitherto been given. But although he could not ſquare the circle, nor cube the ſphere, he determined the proportion of one circle to another, and of one ſphere to another, as well as the proportions of all rectilineal ſimilar figures to one another.

Archimedes took up menſuration where Euclid left it, and carried it a great length. He was the firſt who ſquared a curvilineal ſpace; unleſs Hypocrates muſt be excepted on account of his lunes. In his time the conic ſections were admitted into geometry, and he applied himſelf cloſely to the meaſuring of them, as well as other figures. Accordingly he determined the relations of ſpheres, ſpheroids, and conoids, to cylinders and cones; and the relations of parabolas to rectilineal planes whoſe quadratures had long before been determined by Euclid. He hath left us alſo his attempts upon the circle: he proved that a circle is equal to a right-angled triangle, whoſe baſe is equal to the circumference, and its altitude equal to the radius; and conſequently, that its area is equal to the rectangle of the radius, and half the circumference; and ſo reduced the quadrature of the circle to the determination of the ratio of the diameter to the circumference; but which, however, hath not yet been done. Being diſappointed of the *exact* quadrature of the circle, for want of the rectification of its circumference, which all his methods would not effect, he proceeded to aſſign an uſeful approximation to it; this he effected by the numeral calculation of the perimeters of the inſcribed and circumſcribed polygons; from which calculation it appears, that the perimeter of the circumſcribed regular polygon of 192 ſides, is to the diameter, in a leſs ratio than that of $3\frac{1}{7}$ ($3\frac{10}{70}$) to 1, and that the inſcribed polygon of 96 ſides, is to the diameter, in a greater ratio than that of $3\frac{10}{71}$ to 1; and conſequently much more that the circumference of the circle is to the diameter, in a leſs ratio than that of $3\frac{1}{7}$ to 1, but greater than that of $3\frac{10}{71}$ to 1: The firſt ratio of $3\frac{1}{7}$ to 1, reduced to whole numbers, gives that of 22 to 7, for $3\frac{1}{7} : 1 :: 22 : 7$, which therefore is nearly the ratio of the circumference to the diameter. From this ratio of the

the circumference to the diameter, he computed the approximate *area* of the circle, and found that it is to the ſquare of the diameter, as 11 is to 14.—He likewiſe determined the relation between the circle and ellipſe, with that of their ſimilar parts. It is highly probable that he likewiſe attempted the hyperbola; but it is not to be imagined that he met with any ſucceſs, ſince approximations to its area are all that can be given by the various methods that have ſince been invented.

Beſides theſe figures, he hath left us a treatiſe on the ſpiral, deſcribed by a point moving uniformly along a right line, which at the ſame time moves with an uniform angular motion; and he determined the proportion of its area to that of the circumſcribed circle; as alſo the proportion of their ſectors.

Throughout the whole works of this great man, which are chiefly on menſuration, he every where diſcovers the deepeſt deſign, and the fineſt invention; and ſeems to have been, with Euclid, exceedingly careful of admitting into his demonſtrations nothing but principles perfectly geometrical and unexceptionable: and although his moſt general method of demonſtrating the relations of curved figures to ſtraight ones, be by inſcribing polygons in them; yet to determine thoſe relations, he does not increaſe the number, and diminiſh the magnitude, of the ſides of the polygon *in infinitum*; but from this plain fundamental principle, allowed in Euclid's Elements, viz. that any quantity may be ſo often multiplied, or added to itſelf, as that the reſult ſhould exceed any propoſed finite quantity of the ſame kind, he proves that to deny his figures to have the propoſed relations, would involve an abſurdity.

He demonſtrated alſo many properties, particularly in the parabola, by means of certain numeral progreſſions, whoſe terms are ſimilar to the inſcribed figures; but ſtill without conſidering ſuch ſeries as continued *in infinitum*, and then ſumming up the terms of ſuch infinite ſeries.

He had another very curious and ſingular contrivance for determining the meaſures of figures, in which he proceeds as it were mechanically, by weighing them, or from the properties of the center of gravity.

Several other eminent men among the ancients wrote upon this ſubject, both before and after Euclid and Archimedes; but their attempts were uſually confined to particular parts of it, and made according to methods not eſſentially different from theirs. Among theſe are to be reckoned Thales, Anaxagoras, Pythagoras, Bryſon, Antiphon, Hypocrates of Chios, Plato, Apollonius, Philo, and Ptolemy; moſt of whom wrote of the quadrature of the circle; and thoſe after Archimedes, by his method, uſually extended the approximation to a greater degree of accuracy.

Many of the moderns alſo have proſecuted the ſame problem of the quadrature of the circle, after the ſame methods, to greater lengths; ſuch are Vieta, and Metius, whoſe proportion between the diameter and circumference, is that of 113 to 355, which is within about $\frac{3}{100000000}$ of the true ratio; but above all, Ludolph van Collen, or a Ceulen, who, with an amazing degree of induſtry and patience, by the ſame methods, extended the ratio to 36 places of figures, making the ratio to be that of 1 to 3·14159265358979323846264338327950288 + or 9 −. And the ſame was repeated and confirmed by his editor Snellius.

The firſt material deviation from the principles uſed by the ancients, in geometrical demonſtrations, was made by Cavalerius: the ſides of their inſcribed and circumſcribed figures, they always ſuppoſed to be of a finite and aſſignable number and length; he introduced the doctrine of indiviſibles, a method which was very general and extenſive, and which, with great eaſe and expedition, ſerved to meaſure and compare geometrical figures. Very little new matter, however, was added to geometry by this method, its facility being its chief advantage. But there was great danger in uſing it, and it ſoon led the way to infinitely ſmall elements, and infiniteſimals of endleſs orders; methods which were very uſeful in reſolving difficult problems, and in inveſtigating or demonſtrating theories that are general and extenſive; but ſometimes led their incautious followers into errors and miſtakes, which occaſioned diſputes and animoſities amongſt them. There were now, however, many excellent things performed in this ſcience; not only many new properties were diſcovered concerning the old figures, but new curves were meaſured; and although ſeveral of them could not be exactly ſquared or cubed, yet general and infinite approximating ſeries were aſſigned, of which the laws of their continuation were manifeſt, and in ſome of which the terms were independent of each other. Dr. Wallis, Mr. Huygens, and Mr. James Gregory, performed wonders: Huygens in particular muſt always be admired for his ſolid, accurate, and very maſterly works.

During the preceding ſtate of things, ſeveral men, whoſe vanity ſeemed to have overcome their regard for truth, aſſerted, that they had diſcovered the quadrature of the circle, and publiſhed their attempts in the form of ſtrict geometrical demonſtrations, with ſuch aſſurance and ambiguity, as ſtaggered and miſled many who could not ſo well judge for themſelves, and perceive the fallacy of their principles and arguments. Among thoſe were Longomontanus, and our countryman Hobbes, who obſtinately refuſed all conviction of his errors.

The uſe of infinites was, however, diſliked by ſeveral people, and particularly by Sir Iſaac Newton, who, among his nume-

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rous and great diſcoveries, hath given us that of the method of fluxions; a diſcovery of the greateſt importance, both in philoſophy and mathematics; being a method ſo general and extenſive, as to include all inveſtigations concerning magnitude, diſtance, motion, velocity, time, &c. with wonderful eaſe and brevity; a method eſtabliſhed by its great author upon true and inconteſtable principles; principles perfectly conſiſtent with thoſe of the ancients, and which were free from the imperfections and abſurdities attending ſome that had lately been introduced by the moderns: he rejected no quantities as infinitely ſmall, nor ſuppoſed any parts of curves to coincide with right lines; but propoſed it in ſuch a form as admits of a ſtrict geometrical demonſtration. Upon the introduction of this method, moſt ſciences aſſumed a different appearance, and the moſt abſtruſe problems became eaſy and familiar to every one; things which before ſeemed to be inſuperable, became eaſy examples, or particular caſes, of theories ſtill more general and extenſive; rectifications, quadratures, cubatures, tangencies, caſes *de maximis & minimis*, and many other ſubjects, became general problems, and were delivered in the form of general theories, which included all particular caſes: thus, in quadratures, a formula was aſſigned, which would expreſs the areas of all poſſible curves whatever, both known and unknown, and which, by proper ſubſtitutions, gave the area for any particular caſe, either in finite terms, or in infinite ſeries, of which any term, or any number of terms, could be eaſily aſſigned; and the like in other things. And although no curve, whoſe quadrature was unſucceſsfully attempted by the ancients, became by this method perfectly quadrable, yet many general methods were diſcovered for approximating to their areas, of which in all probability the ancients had not the leaſt idea or hope; and innumerable curves were ſquared which were utterly unknown to them.

The excellency of this method revived ſome hopes of ſquaring the circle; and its quadrature was attempted with eagerneſs. The quadrature of a ſpace was now reduced to the finding of the fluent of a given fluxion; but this problem, however, was found to be incapable of a general ſolution in finite terms: the fluxion of every fluent was found to be always aſſignable, but the reverſe of this problem could be effected only in particular caſes: among the exceptions, to the great mortification of geometricians, was included the caſe of the circle, with regard to all the forms of fluxions attending it.

Another method of obtaining the area was tried: of the quantity expreſſing the fluxion of any area, in general, the fluent could always be aſſigned in the form of an infinite ſeries; which ſeries, therefore, defined all areas in general, and which, on ſubſtituting for particular caſes, was often found to break off and terminate, and to afford an area in finite terms: but here again

again the caſe of the circle failed, its area being ſtill an infinite ſeries.

All hopes of the quadrature of the circle being now at an end, the geometricians employed themſelves in diſcovering and ſelecting the beſt forms of infinite ſeries for determining its area; among which it is evident, that thoſe were to be preferred which were ſimple, and would converge quickly; but it commonly happened that theſe two properties were divided, the ſame ſeries very rarely including them both. The mathematicians in moſt parts of Europe now applied themſelves diligently to theſe new diſcoveries, and many ſeries were aſſigned on all hands, ſome admired for their ſimplicity, and others for their rate of convergency; thoſe which converged the quickeſt, and were at the ſame time ſimpleſt, which therefore were moſt uſeful in computing the area of the circle in numbers, were thoſe in which, beſides the radius, the tangent of ſome certain arc of the circle, was the quantity by whoſe powers the ſeries converged; and from ſome of theſe ſeries, the area hath been computed to a great extent of figures. Dr. Edmund Halley gave a remarkable one, from the tangent of 30 degrees, by means of which the very induſtrious Mr. Abraham Sharp computed the area of the circle to 72 places of figures; but even this was afterwards far exceeded by Mr. John Machin, who, by means hereafter deſcribed in this book, compoſed a ſeries ſo ſimple, and which converged ſo quickly, that by it, in a very little time, he extended the quadrature of the circle to 100 places of figures; from which it appears, that if the diameter be 1, the circumference will be

3·1415926535,8979323846,2643383279,5028841971,6939937510,
5820974944,5923078164,0628620899,8628034825,3421170679,

and conſequently the area will be

·7853981633,9744830961,5660845819,8757210492,9234984377,
6455243736,1480769541,0157155224,9657008706,3355292669.

And I have lately given, in the Philoſophical Tranſactions, various other ſeries for the ſame purpoſe, which are ſtill ſimpler in their form, and converge more readily than thoſe above mentioned.

Whilſt I have been giving the preceding account of the progreſs of this ſubject, I have at the ſame time unawares been writing its panegyric; for, from hence it appears, that moſt of the material improvements or inventions in the ſcience of geometry, have been principally made for the improvement of menſuration; which ſufficiently ſhews the dignity of this ſubject, a ſubject which, as Dr. Barrow ſays, "deſerves to be more curiouſly weighed, becauſe from hence a name is impoſed upon that mother and miſtreſs of the reſt of the mathematical ſciences, which is employed about magnitudes, and which is

wont to be called geometry (a word taken from ancient uſe, becauſe it was firſt applied only to meaſuring the earth, and fixing the limits of poſſeſſions), though the name ſeemed very ridiculous to Plato, who ſubſtitutes in its place that more extenſive name of Metrics or Menſuration; and others after him gave it the title of Pantometry, becauſe it teaches the method of meaſuring all kinds of magnitudes."

But notwithſtanding the dignity and importance of this ſubject, the books which have lately been offered to the public under the title of menſuration, have treated it in a very unworthy manner, and have brought it into much contempt. It is intended in this work, therefore, to place the ſubject in a more favourable light, and in ſome meaſure to endeavour to retrieve its reputation.

The work conſiſts of five principal parts, each part being divided into ſeveral ſections, every ſection containing ſeveral problems or propoſitions, which are all demonſtrated after the ſimpleſt and ſhorteſt methods that could be deviſed; the proceſs being ſometimes by pure geometry, and ſometimes by algebra, the method of fluxions, the method of increments, &c, according as the one or the other appeared beſt ſuited to the purpoſe. And thus, by uſing various modes of demonſtration, I was enabled to make this work much more complete than it could otherwiſe have been; not only by employing that particular method which would perform the buſineſs in the ſhorteſt or cleareſt manner, but which alſo would render the ſubject ſtill more extenſive. However, I had not *always* my choice of the method of demonſtration, the ſubject ſometimes requiring one mode, and ſometimes another; for although the method of fluxions be generally the moſt conciſe, and, in other reſpects, the moſt proper for inveſtigating the meaſure of extenſions, yet there are ſome things in this buſineſs which are too ſimple for it, as well as ſome others which riſe above its reach: thus, the method of fluxions cannot determine the meaſure of a rectangle, that determination being below its root, being prior to the fluxionary method of determining areas; for in this mode, that determination is ſuppoſed, and its reſult aſſumed in the very notation; for the rectangle which is conſidered as the fluxion of a ſurface, is denoted by its length drawn into its breadth: and on this account thoſe fluxioniſts, who, in their chapter of quadratures, attempt to determine the meaſure of a rectangle, argue in a circle, or deduce as a concluſion what was neceſſarily ſuppoſed in the premiſes: the ſame may be ſaid with regard to priſms. And, on the other hand, that method alone would not extend to the determination of the equality of a triangle and hyperbola of the ſame baſe, and of an infinite length, it being there neceſſary to call in the aſſiſtance of the method of increments.

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As extenſions are of three kinds, longitudinal, ſuperficial, and ſolid, ſo in this work the ſcience is treated of as diſtinguiſhed by nature into theſe three principal parts; that is, ſo far as the nature of geometrical figures would conveniently admit. With regard to right lines, plane ſurfaces, and ſolids, the diſtinction is general; but it does not obtain with regard to curved lines and ſurfaces. For, by preſerving this diſtinction ſo entire, as to determine the meaſure of all kinds of lines in the firſt part, and of all kinds of ſurfaces in the ſecond, I immediately perceived that the book could not be ſo conveniently adapted to the proper ſtudy of the generality of readers: on which account, in the firſt part, I have treated only of the meaſure of right lines; and in the ſecond part, only of the meaſure of planes which are bounded by right and circular lines, without any curve ſurfaces, excepting that of the ſphere. In the third part, which treats of ſolids, I have generally placed the problems which relate to the meaſures of the lines, ſurfaces, and ſolidities of each particular figure, immediately after each other, becauſe the knowledge of the one commonly led to *that* of the others. The latter parts of the book are employed chiefly about the applications of the general problems to ſeveral intereſting practical ſubjects in life.

So much for the diſtribution and order of the parts in general. I ſhall now proceed to deſcribe the contents of the parts themſelves more particularly.

The whole work conſiſts of five parts.

PART I. Contains the menſuration of right lines and right-angled angles, and is divided into three ſections.

Sect. 1. Contains ſeveral geometrical definitions and problems; ſome of which are new, and, it is preſumed, they are all more complete, and leſs exceptionable, than thoſe inſtead of which they have been ſubſtituted. The problems will prepare the learner for making the ſeveral figures which are afterwards treated of.

Sect. 2. Contains plane Trigonometry, or the meaſuring of lines and angles. All the caſes of trigonometry, both right and oblique angled, are here reduced to three only; by which means they are eaſier to be remembered, and more clearly underſtood. Beſides theſe caſes, which perform the buſineſs in the common way, by means of the ſines, tangents, and ſecants of angles, I have given a new and extenſive method, by which all the caſes of trigonometry are performed independent of ſines, tangents, and ſecants, and without any kind of tables.

Sect. 3. Contains the application of trigonometry to the determination of heights and diſtances; in which a great variety of caſes and methods concerning this curious ſubject are explained.

PART II. Treats of ſuperficial menſuration, or the menſuration of plane figures, and is divided into two ſections.

Sect.

Sect. 1. Treats of the areas, &c. of right-lined and circular figures; in which, besides many things that are new and curious, are given an explanation of Professor Machin's celebrated quadrature of the circle, and the demonstrations of some useful approximations to the measures of circular arcs and areas, which had been given by Mr. Huygens and Sir Isaac Newton, without demonstrations.

Sect. 2. Contains a curious and useful collection of questions concerning areas, promiscuously placed, and resolved by the rules in the former sections.

PART III. Contains the measuring of solids, and is divided into 8 sections.

Sect. 1. Treats of bodies that are bounded by right or circular lines, viz. prisms, pyramids, the sphere, and the circular spindle.

Sect. 2. Treats of the five regular solids or bodies.

Sect. 3. Treats of solid rings.

Sect. 4. Treats of the conic sections in general; and though it be short, it contains several things that are new and of great importance.

Sect. 5. Treats of the ellipse and the figures generated by it, viz. spheroids and elliptic spindles.

Sect. 6. In like manner treats of parabolic lines, areas, surfaces, and solidities. And

Sect. 7. Of hyperbolic lines, areas, surfaces, and solidities.

In these sections the several figures and bodies are very extensively and particularly handled, many of the rules, &c. both here and throughout the whole book, being new and interesting; and I have given throughout many neat approximations to the values of several things which cannot be truly expressed otherwise than by an infinite series; which approximations are mostly new, excepting two or three that were given by Sir I. Newton, and which I have demonstrated here for the first time.

Sect. 8. Or the last of this part, contains a promiscuous collection of questions concerning solids, to exercise the learner in the foregoing rules.

PART IV. Contains, in 3 sections, several subjects relating to mensuration in general.

Sect. 1. Contains a treatise on the true quadrature and cubature of curves in general. In which are contained some of the most universal and important propositions that can be made in the subject.

Sect. 2. Contains the equidistant-ordinate method; or, the approximate quadrature and cubature of curves in general, by means of equidistant ordinates or sections. A subject by which general and finite rules are discovered for all figures; for some of which they are accurately true, and for the others they are very near approximations; which are often the most useful rules that can be applied to many things in real practice.

Sect.

Sect. 3. Contains, in a very concise but copious treatise, the relations between the areas and solidities of figures, and the centers of gravity of their generating lines and planes.

Then the

Fifth and last PART, in four sections, contains the application of the general rules to the most useful subjects of measuring that happen in ordinary life. In these subjects very material improvements are almost every where made, both with respect to the matters and the disposition of them.

Sect. 1. Contains a very simple treatise of land surveying; explaining the use of the instruments, the methods of surveying, of planning, of computing the contents, of reducing plans, and of dividing the ground.

Sect. 2. Contains a very curious and complete treatise on gauging. As in like manner doth

Sect. 3. On the measuring of artificers works; viz, Bricklayers, Masons, Carpenters and Joiners, Slaters, and Tilers, Plasterers, Painters, Glaziers, Pavers, and Plumbers. Containing the description of the carpenter's rule, the several measures used by each, with the methods of taking the dimensions, and of squaring and summing them up. The whole illustrated by a real case of a building, in which are shewn the methods of entering the dimensions and contents in the pocket book, of drawing out the abstracts, and from them drawing out the forms of the bills.

Sect. 4. Contains a curious treatise on timber measuring; in which, among several other things, is given a new rule for measuring round timber, which not only gives the content very exact, but it is at the same time as easy in the operation as the common false one, either by the pen or the sliding rule. It contains also some curious rules for cutting timber to the most advantage.

The book then concludes with a large table, of the areas of circular segments, extended to ten times the usual length.

It may be necessary to remark that, in this book, where a curve or a space is said to be non-quadrable, or it is said that the value of a thing cannot be expressed except by an infinite series, or any such-like expression is used; the meaning is, that it is not geometrically quadrable, or that its area or value cannot be expressed in a finite number of terms, by any method yet known, or by the method there used; but not that it is a thing naturally impossible in itself. For although a space be not quadrable, by the methods yet known, it does not therefore follow that its quadrature is an impossible thing, or that some method may not hereafter be discovered by which it may be squared. All the methods used by the geometricians before Archimedes, were insufficient for the quadrature of any curve space whatever; but were they therefore to infer that no curve could by any means be squared? Archimedes discovered a method by which he squared the parabola; and by the lately-discovered method of fluxions, we can find

find as many quadrable curves as we please. It is true we have not yet found the area of the circle, and several other figures, in finite terms; yet for each of these we can assign infinite series whose laws of progression are visible; which is more than the ancients could do, or perhaps ever expected could be done, if they even at all thought of such things. And, perhaps, hereafter a method may be discovered of squaring any figure whatever. Which is the chief problem in geometry.

In this edition have been made many large and useful additions, in almost every section of the work; and it is presumed that the whole is arranged in a more regular and perfect order than before.

Royal Military Academy,
Jan. 24, 1788.

An easy Rule to find the Cube Root with an example say what is the cube root of 12812904

Divide the given number into periods of 3 figures, begining at the place of units, place the cube root of the 1st period 2, in the quotient & subtract its cube 8 from the first period & bring down the next period for a dividend which is 4812 to find a divisor, multiply the square of the figure placed in the quotient by 300 = 1200 find how oftens this is contained in the dividend viz. 3 times, place the 3 in the quotient for the 2nd figure of the root, Multiply the part of the Root formerly found viz. 2 by the last figure placed in the root, viz 3, & the product by 30 = 180 add this & the square of the last figure placed in the root to the divisor viz. 1200 Multiply the sum of these 1389 by the last figure placed in the root, 3, and subtract the product 4167 from the dividend 4812, bring down another period for a new dividend & proceed in the same manner

12,812,904 (234 Ansr.
8
4812

2² = 4 × 300 = 1200
3 = 6 × 30 = 180
3 = —— = 9
1389 × 3 = 4167
645904

² × 300 = 158700
× 4 × 30 = 2760
4² = 16
161476 × 4 = 645904

CONTENTS.

A TREATISE ON MENSURATION.

PART I.

OF LINEAL AND ANGULAR MENSURATION, OR THE MENSURATION OF LINES AND ANGLES.

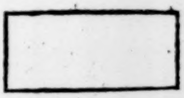
SECTION I.

GEOMETRICAL DEFINITIONS AND PROBLEMS.

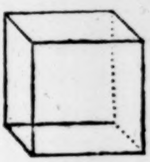
1. A POINT has no parts nor dimenſions, neither length, breadth, nor thickneſs.

2. A line is length, without breadth or thickneſs.

3. A ſurface, or ſuperficies, is an extenſion, or a figure, of two dimenſions, length and breadth; but without thickneſs.



4. A body or ſolid, is a figure of three dimenſions, namely, length, breadth, and thickneſs.

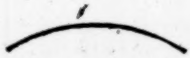


Hence surfaces are the extremities of solids; lines the extremities of surfaces; and points the extremities of lines.

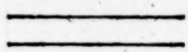
5. Lines are either right, or curved, or mixed of these two.

6. A right line, or straight line, lies all in the same direction, between its extremities; and is the shortest distance between two points.

7. A curve continually changes its direction, between its extreme points.



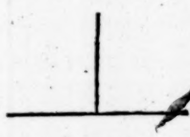
8. Lines are either parallel, oblique, perpendicular, or tangential.



9. Parallel lines are always at the same distance; and never meet, though ever so far produced.

10. Oblique right lines change their distance, and would meet, if produced, on the side of the least distance.

11. One line is perpendicular to another, when it inclines not more on the one side than on the other.



12. One line is tangential, or a tangent to another, when it touches it, without cutting, when both are produced.

13. An angle is the inclination, or opening, of two lines, having different directions, and meeting in a point.

14. Angles are right or oblique, acute or obtuse.

15. A right angle, is that which is made by one line perpendicular to another. Or when the angles on each side are equal to one another, they are right angles.

16 An oblique angle is that which is made by two oblique lines; and is either less or greater than a right angle.

17. An acute angle is less than a right angle.

18 An obtuse angle is greater than a right angle.

19. Superficies are either plane or curved.

20. A plane, or plane superficies, is that with which a right line may, every way, coincide.—But if not, it is curved.

21. Plane figures are bounded either by right lines or curves.

22. Plane figures bounded by right lines, have names according to the number of their sides, or of their angles; for they have as many sides as angles; the least number being three.

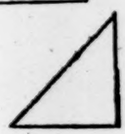
23. A figure of three sides and angles, is called a triangle. And it receives particular denominations from the relations of its sides and angles.

24. An equilateral triangle, is that whose three sides are all equal.

25. An isosceles triangle, is that which has two sides equal.

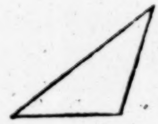
26. A scalene triangle, is that whose three sides are all unequal.

27. A right-angled triangle, is that which has one right angle.



28. Other triangles are oblique-angled, and are either obtuse or acute.

29. An obtuse-angled triangle has one obtuse angle.

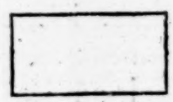


30. An acute-angled triangle has all its three angles acute.

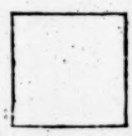
31. A figure of four sides and angles, is called a quadrangle, or a quadrilateral.

32. A parallelogram is a quadrilateral which has both its pairs of opposite sides parallel. And it takes the following particular names.

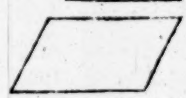
33. A rectangle is a parallelogram, having all its angles right ones.



34. A square is an equilateral rectangle; having all its sides equal, and all its angles right ones.

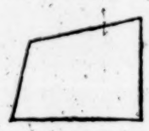


35. A rhomboid is an oblique-angled parallelogram.



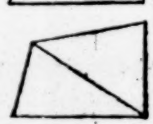
36. A rhombus is an equilateral rhomboid; having all its sides equal, but its angles oblique.

37. A trapezium is a quadrilateral which has not both its pairs of opposite sides parallel.



38. A trapezoid has only one pair of opposite sides parallel.

39. A diagonal is a right line joining any two opposite angles of a quadrilateral.



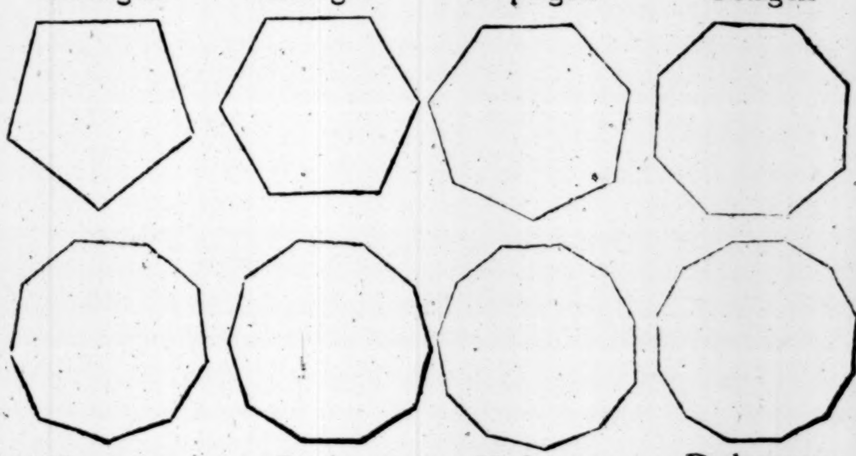
40. Plane figures having more than four ſides are, in general, called polygons: and they receive other particular names according to the number of their ſides or angles.

41. A pentagon is a polygon of five ſides; a hexagon has ſix ſides; a heptagon, ſeven; an octagon, eight; a nonagon, nine; a decagon, ten; an undecagon, eleven; and a dodecagon has twelve ſides.

42. A regular polygon has all its ſides and all its angles equal.—If they are not both equal, the polygon is irregular.

43. An equilateral triangle is alſo a regular figure of three ſides, and the ſquare is one of four: the former being alſo called a trigon, and the latter a tetragon.

Pentagon Hexagon Heptagon Octagon



Nonagon Decagon Undecagon Dodecagon

44. A circle is a plane figure bounded by a curve line, called the circumference, which is every where equi-diſtant from a certain point within, called its center.

N. B. The circumference itſelf is often called a circle.

45. The radius of a circle, is a right line drawn from the center to the circumference.

46. The diameter of a circle, is a right line drawn through the center, and terminating in the circumference on both ſides.

47. An arc of a circle, is any part of the circumference.

48. A chord is a right line joining the extremities of an arc.

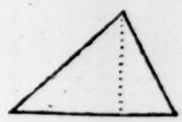
49. A ſegment is any part of a circle bounded by an arc and its chord.

50. A ſemicircle is half the circle, or a ſegment cut off by a diameter.

51. A ſector is any part of a circle, bounded by an arc, and two radii, drawn to its extremities.

52. A quadrant, or quarter of a circle, is a ſector having a quarter of the circumference for its arc, and its two radii are perpendicular to each other.

53. The height or altitude of a figure, is a perpendicular let fall from an angle, or its vertex, to the opposite side, called the base.



54. In a right-angled triangle, the side opposite the right angle, is called the hypotenuse; and the other two sides the legs, or sometimes the base and perpendicular.

55. When an angle is denoted by three letters, of which one stands at the angular point, and the other two on the two sides, that which stands at the angular point is read in the middle.

56. The circumference of every circle is supposed to be divided into 360 equal parts, called degrees; and each degree into 60 minutes, each minute into 60 seconds, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

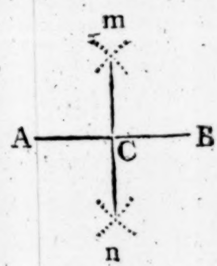
57. The measure of a right-lined angle, is an arc of any circle contained between the two lines which form that angle, the angular point being the center; and it is estimated by the number of degrees contained in that arc. Hence a right-angle is an angle of 90 degrees.

The definition of solids, or bodies, will be given afterwards, when we come to treat of the mensuration of solids.

PROBLEMS.

PROBLEM I.

To divide a given Line A B *into two Equal Parts.*



From the centers A and B, with any radius greater than half AB, describe arcs cutting each other in m and n. Draw the line m c n, and it will cut the given line into two equal parts in the middle point c.

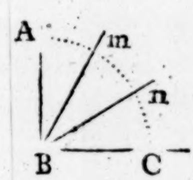
PROBLEM II.

To divide a given Angle A B C *into two Equal Parts.*

From the center B, with any radius, describe the arc AC. From A and C, with one and the same radius, describe arcs intersecting in m. Draw the line B m, and it will bisect the angle as required.

PROBLEM III.

To divide a Right Angle A B C *into three Equal Parts.*



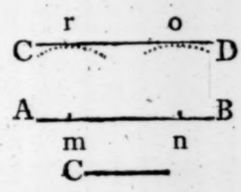
From the center B, with any radius, describe the arc AC. From the center A, with the same radius, cross the arc AC in n. And with the center C, and the same radius, cut the arc AC in m. Then through the points m and n draw B m and B n, and they will trisect the angle as required.

PROBLEM IV.

To draw a Line Parallel to a given Line AB.

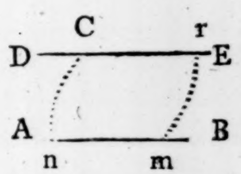
CASE 1. *When the Parallel Line is to be at a given Distance* C.

From any two points m and n, in the line AB, with a radius equal to C, describe the arcs r and o:—Draw CD to touch these arcs, without cutting them, and it will be the parallel required.



CASE 2. *When the Parallel Line is to pass through a given Point* C.

From any point m, in the line AB, with the radius mC, describe the arc Cn.—From the center C with the same radius, describe the arc mr.—Take the arc Cn in the compasses, and apply it from m to r.—Through C and r draw DE, the parallel required.



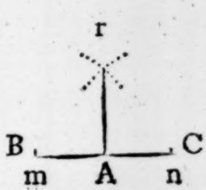
N. B. This problem is more easily effected with a parallel ruler.

PROBLEM V.

To erect a Perpendicular from a given Point A *in a given Line* BC.

CASE 1. *When the Point is near the Middle of the Line.*

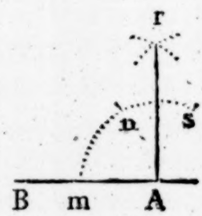
On each side of the point A take any two equal distances Am, An. From the centers m, n, with any radius greater than Am or An, describe two arcs intersecting in r.—Through A and r draw the line Ar, and it will be the perpendicular as required,



CASE

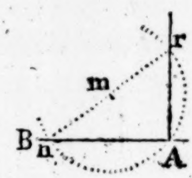
CASE 2. *When the Point is too near the End of the Line.*

With the center A, and any radius, describe the arc mns.—From the point m, with the same radius, turn the compasses twice over on the arc at n and s.—Again, with the centers n and s, describe arcs intersecting in r. Then draw A r, and it will be perpendicular as required.



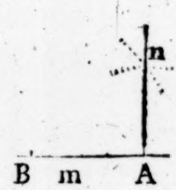
Another Method.

From any point m as a center, with the radius or distance m A, describe an arc cutting the given line in n and A.—Through n and m draw a right line cutting the arc in r.—Lastly, draw A r, and it will be the perpendicular as required.



Another Method.

From any plane scale of equal parts set off A m equal to 4 parts.—With center A, and radius of 3 parts, describe an arc.—And with center m, and radius of 5 parts, cross it at n.—Draw A n for the perpendicular required.



Or any other numbers in the same proportion as 3, 4, 5, will do the same.

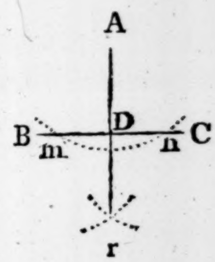
PRO-

PROBLEM VI.

From a given Point A, *out of a given Line* B C, *to let fall a Perpendicular.*

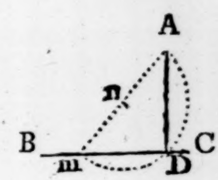
CASE 1. *When the Point is nearly opposite the Middle of the Line.*

With the center A, and any radius, describe an arc cutting B C in m and n.—With the centers m and n, and the same, or any other radius, describe arcs intersecting in r.—Draw A D r, for the perpendicular required.



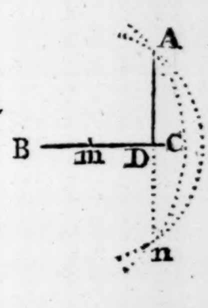
CASE 2. *When the Point is nearly opposite the End of the Line.*

From A draw any line A m to meet B C, in any point m.—Bisect A m at n, and with the center n, and radius A n or m n, describe an arc, cutting B C in D.—Draw A D the perpendicular as required.



Another Method.

From B or any point in B C, as a center, describe an arc through the point A.—From any other center m in B C, describe another arc through A, and cutting the former arc again in n.—Through A and n draw the line A D n; and A D will be the perpendicular as required.

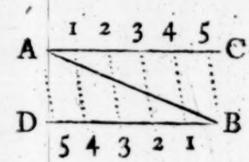


N. B.

N. B. Perpendiculars may be more readily raised and let fall, in practice, by means of a square, or other fit instrument.

PROBLEM VII.

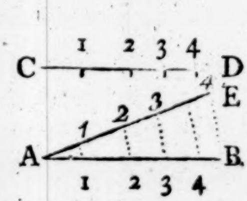
To divide a given Line AB *into any proposed Number of Equal Parts.*



From A draw any line AC at random, and from B draw BD parallel to it.—On each of these lines, beginning at A and B, set off as many equal parts, of any length, as AB is to be divided into.—Join the opposite points of division by the lines A5, 14, 23, &c; and they will divide AB as required.

PROBLEM VIII.

To divide a given Line AB *in the same Proportion as another Line* CD *is divided.*



From A draw any line AE equal to CD, and upon it transfer the divisions of the line CD.—Join BE, and parallel to it draw the lines 11, 22, 33, &c; and they will divide AB as required.

PRO-

PROBLEM IX.

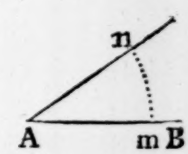
At a given Point A, *in a given Line* AB, *to make an Angle equal to a given Angle* C.

With the center C, and any radius, describe an arc mn.—With the center A, and the same radius, describe the arc rs.—Take the distance mn between the compasses, and apply it from r to s.—Then a line drawn through A and s, will make the angle A equal to the angle C as required.

PROBLEM X.

At a given Point A, *in a given Line* AB, *to make an Angle of any proposed Number of Degrees.*

With the center A, and radius equal to 60 degrees taken from a scale of chords, describe an arc, cutting AB in m.—Then take between the compasses, the proposed number of degrees from the same scale of chords, and apply them from m to n.—Through the point n draw An, and it will make the angle A of the number of degrees proposed.



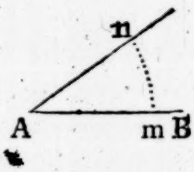
Note. Angles of more than 90 degrees are usually taken off at twice.

Or the angle may be made with any divided arch or instrument, by laying the center to the point A, and its radius along AB; then make a mark n at the proposed number of degrees, through which draw the line An as before.

PROBLEM XI.

To measure a given Angle A.

Describe the arc mn with the chord of 60 degrees, as in the last Problem. Take the arc mn between the compasses, and that extent, applied to the chords, will shew the degrees in the given angle.



Note. When the distance mn exceeds 90 degrees, it must be taken off at twice, as before.

Or the angle may be measured by applying the radius of a graduated arc, of any instrument, to AB, as in the last problem; and then noting the degrees cut off by the other leg An of the angle.

PROBLEM XII.

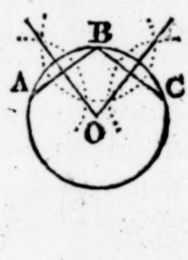
To find the Center of a Circle.

Draw any chord AB; and bisect it perpendicularly with CD, which will be a diameter.—Bisect CD in the point O, which will be the center.

PROBLEM XIII.

To describe the Circumference of a Circle, through three given Points, A, B, C.

From the middle point B draw chords to the other two points.—Bisect these chords perpendicularly by lines meeting in O, which will be the center.—Then from the center O, at the distance OA, or OB, or OC, describe the circle.



Note.

Note. In the same manner may the center of an arc of a circle be found.

PROBLEM XIV.

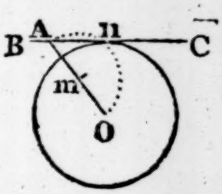
Through a given Point A, *to draw a Tangent to a given Circle.*

CASE 1. *When* A *is in the Circumference of the Circle.*

From the given point A, draw AO to the center of the circle.—Then through A draw BC perpendicular to AO, and it will be the tangent as required.

CASE 2. *When* A *is out of the Circumference.*

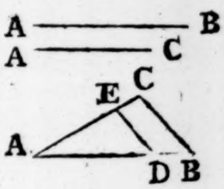
From the given point A, draw AO to the center, which bisect in the point m.—With the center m, and radius mA or mo, describe an arc cutting the given circle in n.—Through the points A and n, draw the tangent BC.



PROBLEM XV.

To find a Third Proportional to two given Lines AB, AC.

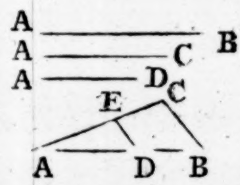
Place the two given lines, AB, AC, making any angle at A, and join BC.—In AB take AD equal to AC, and draw DE parallel to BC. So shall AE be the third proportional to AB and AC.



That is, AB : AC :: AC : AE.

PROBLEM XVI.

To find a Fourth Proportional to three given Lines, AB, AC, AD.

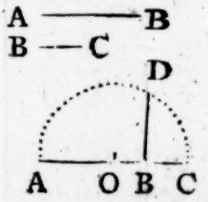


Place two of them, AB, AC, making any angle at A, and join BC. Place AD on AB, and draw DE parallel to BC. So ſhall AE be the fourth proportional required.

That is AB : AC : : AD : AE.

PROBLEM XVII.

To find a Mean Proportional between two given Lines, AB, BC.



Join AB and BC in one ſtraight line AC, and biſect it in the point O.—With the center O, and radius OA or OC, deſcribe a ſemicircle.—Erect the perpendicular BD, and it will be the mean proportional required.

That is AB : BD : : BD : BC.

PROBLEM XVIII.

To divide a Line AB *in Extreme-and-Mean Ratio.*

Raiſe BC perpendicular to AB, and equal to half AB. Join AC. With center C, and radius CB, croſs AC in D. Laſtly, with center A, and radius AD, croſs AB in E, which will divide the line AB in extreme and mean ratio, namely, ſo that the whole line is to the greater part, as the greater part is to the leſs part. That is, AB : AE : : AE : EB.

PRO-

PROBLEM XIX.

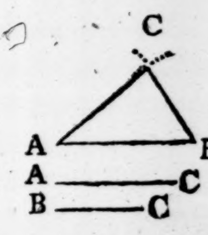
To make an Equilateral Triangle on a given Line AB.

From the centers A and B, with the radius AB, describe arcs, intersecting in C.—Draw AC and BC, and it is done.

Note. An isosceles triangle may be made in the same manner, taking for the radius the given length of one of the equal sides.

PROBLEM XX.

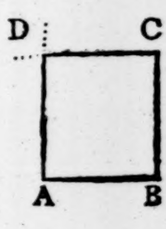
To make a Triangle with three given Lines AB, AC, BC.



With the center A and radius AC, describe an arc.—With the center B, and radius BC, describe another arc, cutting the former in C.—Draw AC and BC, and ABC is the triangle required.

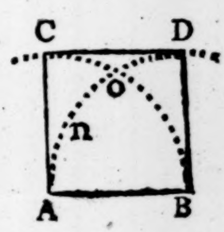
PROBLEM XXI.

To make a Square upon a given Line AB.



Draw BC perpendicular and equal to AB.—From A and C with the radius AB, describe arcs intersecting in D.—Draw AD and CD, and it is done.

Another Way.

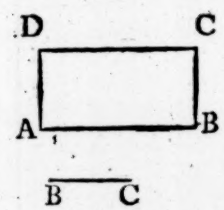


On the centers A and B, with the radius AB, describe arcs crossing at o.—Bisect AO in n.—With center o, and radius on, cross the two arcs in C and D.—Then draw AC, BD, CD.

PROBLEM XXII.

To describe a Rectangle, or a Parallelogram, of a given Length and Breadth.

Place BC perpendicular to AB.—With center A, and radius AC, describe an arc.—With center C, and radius AB, describe another arc, cutting the former in D.—Draw AD and CD, and it is done.

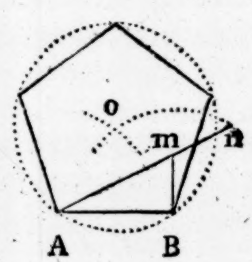


Note. In the same manner is described any oblique parallelogram, only drawing BC, making the given oblique angle with AB, instead of perpendicular to it.

PROBLEM XXIII.

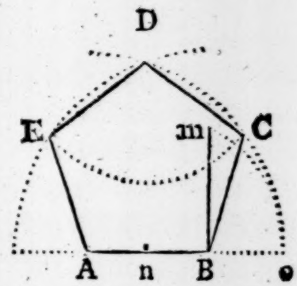
To make a regular Pentagon on a given Line AB.

Make B m perpendicular and equal to half AB.—Draw A m, and produce it till m n be equal to B m.—With centers A and B, and radius B n, describe arcs intersecting in o, which will be the center of the circumscribing circle.—Then with the center o, and the same radius, describe the circle; and about the circumference of it apply AB the proper number of times.



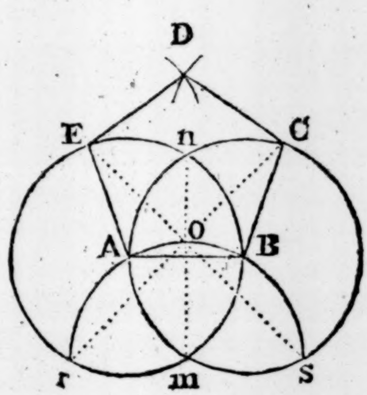
Another

Another Method.



Make BM perpendicular and equal to AB.—Bisect AB in n; then with the center n, and radius nm, cross AB produced in o.—With the centers A and B, and radius AO, describe arcs intersecting in D, the opposite angle of the pentagon. Lastly, with center D, and radius AB, cross those arcs again in C and E, the other two angles of the figure. Then draw the lines from angle to angle, to complete the figure.

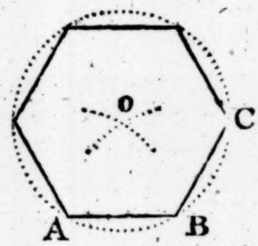
A third Method, nearly true.



On the centers A and B, with the radius AB, describe two circles, intersecting in m and n.—With the same radius, and the center m, describe rAOBs, and draw mn cutting it in o.—Draw roc and sob, which will give two angles of the pentagon.—Lastly, with radius AB, and centers C and E, describe arcs intersecting in D, the other angle of the pentagon nearly.

PROBLEM XXIV.

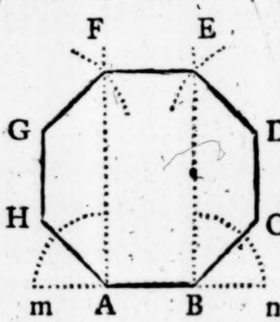
To make a Hexagon on a given Line AB.



With the radius AB, and the centers A and B, deſcribe arcs interſecting in o.—With the ſame radius, and center o, deſcribe a circle, which will circumſcribe the hexagon.—Then apply the line AB ſix times round the circumference, marking out the angular points, which connect with right lines.

PROBLEM XXV.

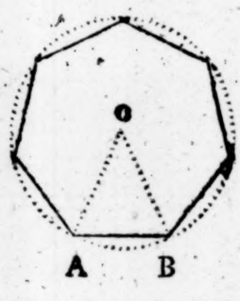
To make an Octagon on a given Line AB.



Erect AF and BE perpendicular to AB.—Produce AB both ways, and biſect the angles mAF and nBE with the lines AH and BC, each equal to AB.—Draw CD and HG parallel to AF or BE, and each equal to AB.—With radius AB, and centers G and D, croſs AF and BE in F and E.—Then join GF, FE, ED, and it is done.

PROBLEM XXVI.

To make any regular Polygon on a given Line AB.



Draw AO and BO making the angles A and B each equal to half the angle of the polygon.—With the center O and radius OA, deſcribe a circle.—Then apply the line AB continually round the circumference the proper number of times, and it is done.

Note. The angle of any polygon, of which the angles OAB and OBA are each one half, is found thus: divide the whole 360 degrees by the number of ſides, and the quotient will be the angle at the center O; then ſubtract that from 180 degrees, and the remainder will be the angle of the polygon, and is double of OAB or of OBA. And thus you will find the following table, containing the degrees in the angle O at the center, and the angle of the polygon, for all the regular figures from 3 to 12 ſides.

No. of ſides.	Name of the polygon.	Angle O at the center	Angle of the polyg	Angle OAB or OBA
3	Trigon	120°	60°	30°
4	Tetragon	90	90	45
5	Pentagon	72	108	54
6	Hexagon	60	120	60
7	Heptagon	$51\frac{3}{7}$	$128\frac{4}{7}$	$64\frac{2}{7}$
8	Octagon	45	135	$67\frac{1}{2}$
9	Nonagon	40	140	70
10	Decagon	36	144	72
11	Undecagon	$32\frac{8}{11}$	$147\frac{3}{11}$	$73\frac{7}{11}$
12	Dodecagon	30	150	75

PROBLEM XXVII.

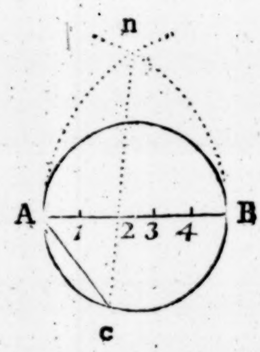
In a given Circle to inſcribe any regular Polygon; or, to divide the Circumference into any number of equal Parts.

(See the laſt figure.)

At the center O make an angle equal to the angle at the center of the polygon, as contained in the third column of the above table of polygons.—Then the diſtance AB will be one ſide of the polygon; which being carried round the circumference the proper number of times, will complete the figure.—Or, the arc AB will be one of the equal parts of the circumference.

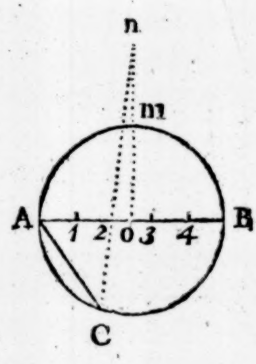
Another Method, nearly true.

Draw the diameter AB, which divide into as many equal parts as the figure has sides.—With the radius AB, and centers A and B, describe arcs crossing at n; from whence draw n c through the second division on the diameter; so shall AC be a side of the polygon nearly.



Another Method, still nearer.

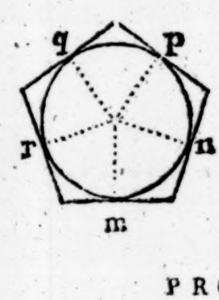
Divide the diameter AB into as many equal parts as the figure has sides, as before.—From the center o raise the perpendicular om, which produce till m n be three-fourths of the radius om.—From n draw n c through the second division of the diameter, and the line AC will be the side of the polygon still nearer than before; or the arc AC one of the equal parts into which the circumference is to be divided.



PROBLEM XXVIII.

About a given Circle to circumscribe any Polygon.

Find the points m, n, p, &c, as in the last problem, to which draw radii mo, no, &c, to the center of the circle.—Then through these points, m, n, &c, and perpendicular to these radii, draw the sides of the polygon.

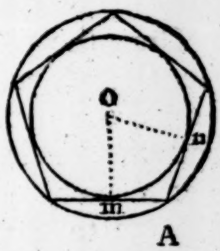


PRO-

PROBLEM XXIX.

To find the Center of a given Polygon, or the Center of its inscribed or circumscribed Circle.

Bisect any two sides with the perpendiculars mo, no; and their intersection will be the center.—Then with the center o, and the distance om, describe the inscribed circle; or with the distance to one of the angles, as A, describe the circumscribing circle.

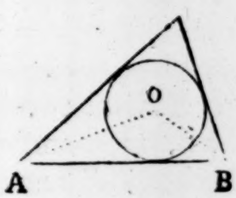


Note. This method will also circumscribe a circle about any given oblique triangle.

PROBLEM XXX.

In any given Triangle to inscribe a Circle.

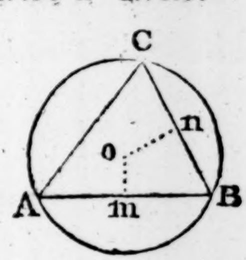
Bisect any two of the angles with the lines AO, BO, and O will be the center of the circle. —Then with the center o, and radius the nearest distance to any one of the sides, describe the circle.



PROBLEM XXXI.

About any given Triangle to circumscribe a Circle.

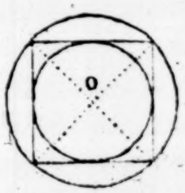
Bisect any two of the sides AB, BC, with the perpendiculars mo, no.—With the center c, and distance to any one of the angles, describe the circle.



PROBLEM XXXII.

In, or about, a given Square, to deſcribe a Circle.

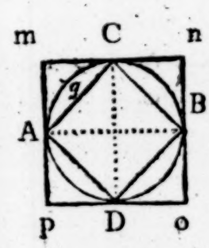
Draw the two diagonals of the ſquare, and their interſection o will be the center of both the circles.—Then with that center, and the neareſt diſtance to one ſide, deſcribe the inner circle; and with the diſtance to one angle, deſcribe the outer circle.



PROBLEM XXXIII.

In, or about, a given Circle, to deſcribe a Square, or an Octagon.

Draw two diameters AB, CD, perpendicular to each other.—Then connect their extremities, and they will give the inſcribed ſquare ACBD.—Alſo through their extremities draw tangents parallel to them, and they will form the outer ſquare mnop.

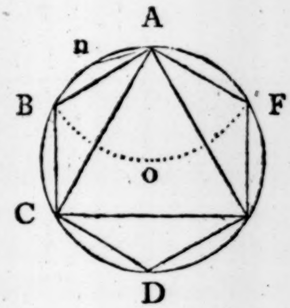


Note. If any quadrant, as AC, be biſected in q, it will give one-eighth of the circumference, or the ſide of the octagon.

PRO-

PROBLEM XXXIV.

In a given Circle, to inscribe a Trigon, a Hexagon, or a Dodecagon.

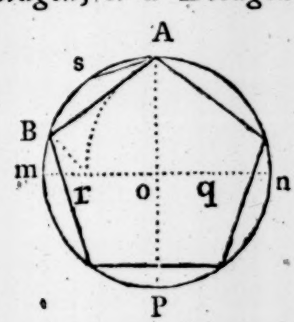


The radius is the side of the hexagon. Therefore from any point A in the circumference, with the distance of the radius, describe the arc BOF. Then is AB the side of the hexagon; and therefore carrying it six times round will form the hexagon, or divide the circumference into six equal parts, each containing 60 degrees.—The second of these C, will give AC the side of the trigon or equilateral triangle; and the arc AC one-third of the circumference, or 120 degrees.—Also the half of AB, or AN is one-12th of the circumference, or 30 degrees, and gives the side of the dodecagon.

Note. If tangents to the circle be drawn through all the angular points of any inscribed figure, they will form the sides of a like circumscribing figure.

PROBLEM XXXV.

In a given Circle to inscribe a Pentagon, or a Decagon.



Draw the two diameters AP, mn perpendicular to each other, and bisect the radius on at q.—With the center q and distance qA, describe the arc Ar; and with the center A, and radius Ar, describe the arc rB. Then is AB one-fifth of the circumference; and AB carried five times over will form the pentagon.—Also the arc AB bisected in s, will give AS the tenth part of the circumference, or the side of the decagon.

Note.

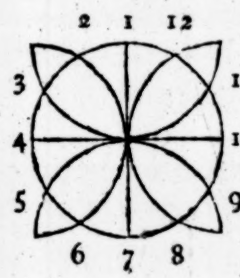
Note. Tangents being drawn through the angular points, will form the circumſcribing pentagon or decagon.

PROBLEM XXXVI.

To divide the Circumference of a given Circle into 12 *equal Parts, each of* 30 *Degrees.*

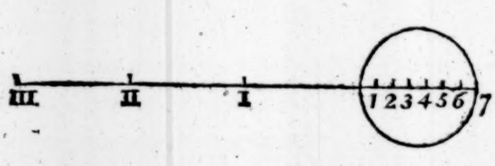
Or to inſcribe a Dodecagon by another Method.

Draw two diameters 1 7 and 4 10 perpendicular to each other.—Then with the radius of the circle, and the four extremities 1, 4, 7, 10, as centers, deſcribe arcs through the center of the circle; and they will cut the circumference in the points required, dividing it into 12 equal parts, at the points marked with the numbers.



PROBLEM XXXVII.

To draw a right Line equal to the Circumference of a given Circle.

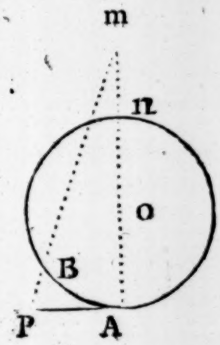


Take III 1 equal to 3 times the diameter and $\frac{1}{7}$ part more; and it will be equal to the circumference, very nearly.

PRO-

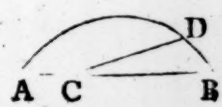
PROBLEM XXXVIII.

To find a Right Line equal to any given Arc AB *of a Circle.*



Through the point A and the center draw A m, making m n equal to $\frac{3}{4}$ of the radius n o.—Alſo draw the indefinite tangent AP perpendicular to it.—Then through m and B, draw m P; ſo ſhall AP be equal to the arc AB very nearly.

Otherwiſe.



Divide the chord AB into 4 equal parts.—Set one part AC on the arc from B to D.—Draw CD, and the double of it will be nearly equal to the arc ADB.

PROBLEM XXXIX.

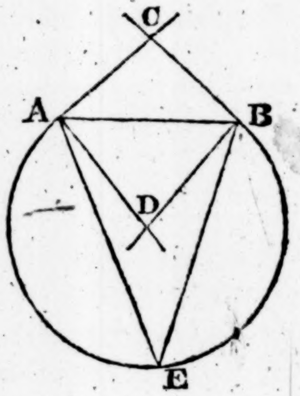
To divide a given Circle into any propoſed Number of Parts by equal Lines, ſo that thoſe Parts ſhall be mutually equal, both in Area and Perimeter.

Divide the diameter AB into the propoſed number of equal parts at the points a, b, c, &c,—Then on A a, A b, A c, &c, as diameters, deſcribe ſemicircles on one ſide of the diameter AB; and on B d, B c, B b, &c, deſcribe ſemicircles on the other ſide of the diameter. So ſhall the correſponding joining ſemicircles divide the given circle in

in the manner propoſed. And in like manner we may proceed when the ſpaces are to be in any given proportion.—As to the perimeters, they are always equal, whatever the proportion of the ſpaces is.

PROBLEM XL.

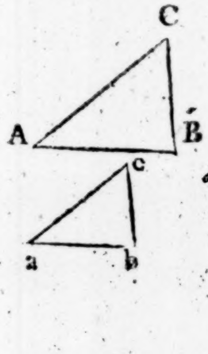
On a given Line AB *to deſcribe the Segment of a Circle capable of containing a given Angle.*



Draw AC and BC making the angles BAC and ABC each equal the given angle.—Draw AD perpendicular to AC, and BD perpendicular to BC. With center D, and radius DA or DB, deſcribe the ſegment AEB. Then any angle, as E, made in that ſegment, will be equal to the given angle.

PROBLEM XLI.

To make a Triangle ſimilar to a given Triangle ABC.



Let AB be one ſide of the required triangle. Make the angle a equal to the angle A, and the angle b equal to the angle B; then the triangle abc will be ſimilar to ABC as propoſed.

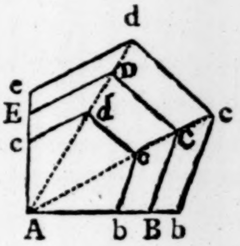
Note. If ab be equal to AB, the triangles will alſo be equal, as well as ſimilar.

PRO-

PROBLEM XLII.

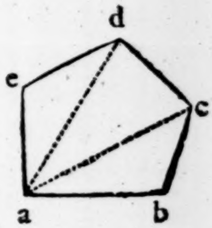
To make a Figure similar to any other given Figure ABCDE.

From any angle A draw diagonals to the other angles. —Take ab a side of the figure required. Then draw bc parallel to BC, and cd to CD, and de to DE, &c.



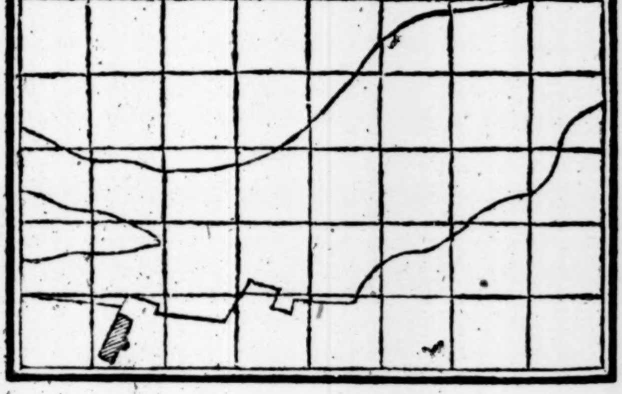
Otherwise.

Make the angles at a, b, e, respectively equal to the angles at A, B, E, and the lines will intersect in the corners of the figure required.

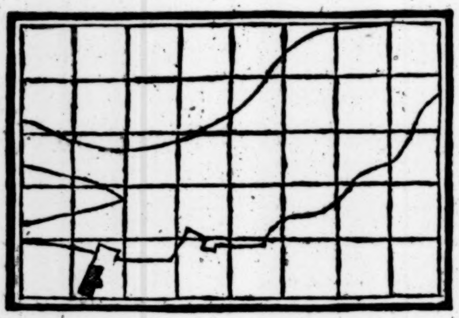


PROBLEM XLIII.

To reduce a Complex Figure from one Scale to another, mechanically by means of Squares.



Divide

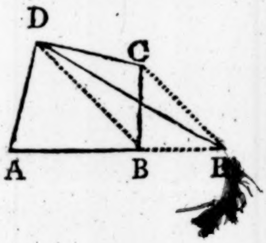


Divide the given figure, by croſs lines, into ſquares, as ſmall as may be thought neceſſary.—Then divide another paper into the ſame number of ſquares, and either greater, equal, or leſs, in the given proportion. —This done, obſerve what ſquares the ſeveral parts of the given figure are in, and draw, with a pencil, ſimilar parts in the correſponding ſquares of the new figure. And ſo proceed till the whole is copied.

PROBLEM XLIV.

To make a Triangle equal to a given Trapezium ABCD.

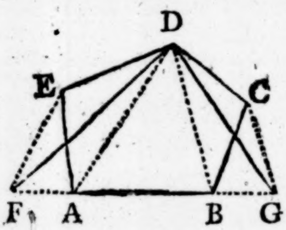
Draw the diagonal DB, and CE, parallel to it, meeting AB produced in E.—Join DE; ſo ſhall the triangle ADE be equal to the trapezium ABCD.



PROBLEM XLV.

To make a Triangle equal to the figure ABCDEA.

Draw the diagonals DA, DB, and the lines EF, CG, parallel to them, meeting the baſe AB, both ways produced, in F and G.—Join DF, DG; and DFG will be the triangle required equal to the given figure ABCDE.



Note. Nearly in the same manner may a triangle be made equal to any right-lined figure whatever.

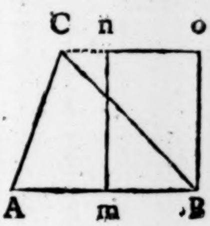
PROBLEM XLVI.

To make a Triangle equal to a given Circle.

Draw any radius AO, and the tangent AB perpendicular to it.—On which take AB equal to the circumference of the circle by problem XXXVII.—Join BO, so shall ABO be the triangle required, equal to the given circle.

PROBLEM XLVII.

To make a Rectangle, or a Parallelogram, equal to a given Triangle ABC.

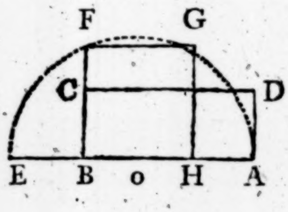


Bisect the base AB in m.—Through C draw cno parallel to AB.—Through m and B draw mn and BO parallel to each other, and either perpendicular to AB, or making any angle with it. And the rectangle or parallelogram mnoB will be equal to the triangle, as required.

PRO-

PROBLEM XLVIII.

To make a Square equal to a given Rectangle ABCD.

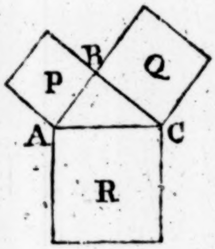


Produce one ſide, AB, till BE be equal to the other ſide BC.—Biſect AE in O; on which as a center, with radius AO deſcribe a ſemicircle, and produce BC to meet it at F.—On BF make the ſquare BFGH, and it will be equal to the rectangle ABCD, as required.

*** Thus the circle and all right-lined figures, have been reduced to equivalent ſquares.

PROBLEM XLIX.

To make a Square equal to two given Squares P *and* Q.



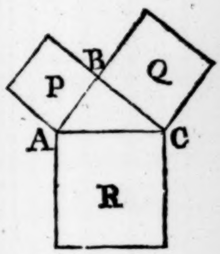
Set two ſides AB, BC, of the given ſquares, perpendicular to each other.—Join their extremities AC; ſo ſhall the ſquare Q, conſtructed on AC, be equal to the two P and Q taken together.

Note. Circles, or any other ſimilar figures, are added in the ſame manner. For, if AB and BC be the diameters of two circles, AC will be the diameter of a circle equal to both the other two. And if AB and BC be the like ſides of any two ſimilar figures, then AC will be the like ſide of another ſimilar figure equal to both the two former, and upon which the third figure may be conſtructed by problem XLII.

PRO-

PROBLEM L.

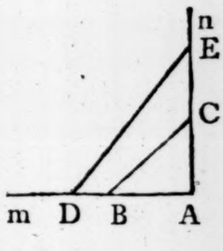
To make a Square equal to the Difference between two given Squares P R.



On the side AC of the greater square, as a diameter, describe a semicircle; in which apply AB the side of the less square.—Join BC, and it will be the side of a square equal to the difference between the two P and R, as required.

PROBLEM LI.

To make a Square equal to the Sum of any Number of Squares taken together.



Draw two indefinite lines Am, An, perpendicular to each other at the point A. On the one of these set off AB the side of one of the given squares, and on the other AC the side of another of them. Join BC, and it will be the side of a square equal to the two together.—Then take AD equal to BC, and AE equal to the side of the third given square. So shall DE be the side of a square equal to the sum of the three given squares.—And so on continually, always setting more sides of the given squares on the line An, and the sides of the successive sums on the other line Am.

Note. And thus any number of any sort of figures may be added together.

PROBLEM LII.

To make plain Diagonal Scales,

Draw any line as AB of any convenient length. Divide it into 11 equal parts*. Complete these into rectangles of a convenient height, by drawing parallel and perpendicular lines. Divide the altitude into 10 equal parts, if it be for a decimal scale for common numbers, or into 12 equal parts, if it be for feet and inches; and through these points of division draw as many parallel lines, the whole length of the scale.—Then divide the length of the first division AC into 10 equal parts, both above and below; and connect these points of division by diagonal lines, and the scale is finished, after being numbered as you please.

Note. These diagonal scales serve to take off dimensions or numbers of three figures. If the first large divisions be units; the second set of divisions along AC, will be 10th parts; and the divisions in the altitude, along AD, will be 100th parts. If CB be tens, AC will be units, and AD will be 10th parts. If CB be hundreds, AC will be tens, and AD units. If CB be thousands, AC will be hundreds, and AD will be tens. And so on, each set of divisions being tenth parts of the former ones.

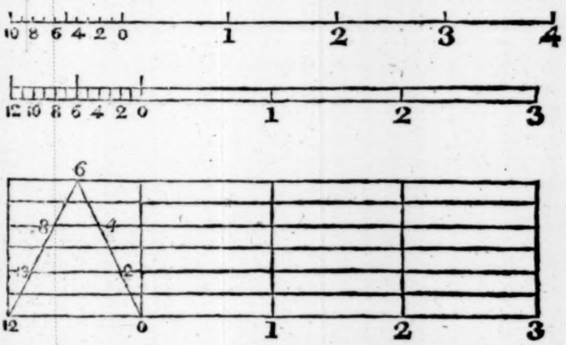
For example, suppose it were required to take off 243 from the scale. Fix one foot of the compasses at 2 of the greatest divisions, at the bottom of the scale, and extend the other to 4 of the second divisions, along

* Only 4 parts are here drawn, for want of room.

along the bottom; then, for the 3, slide up both points of the compasses by a parallel motion, till they fall upon the third longitudinal line; and in that position extend the second point of the compasses to the fourth diagonal line, and you have the extent of three figures as required.

Or if you have any line to measure the length of.—Take it between the compasses, and applying it to the scale, suppose it fall between 3 and 4 of the large divisions; or, more nearly, that it is 3 of the large divisions, or 3 hundreds, and between 5 and 6 of the second divisions, or 5 tens or 50, and a little more. Slide up the points of the compasses by a parallel motion, keeping one foot always on the vertical division of 3 hundred, till the other point fall exactly on one of the diagonal lines, which suppose to be 8, which is 8 units. Which shews that the length of the line, proposed to be measured, is 358.

PLANE SCALES FOR TWO FIGURES.



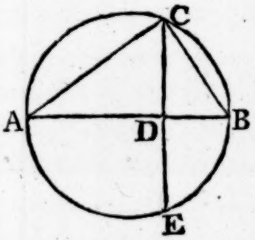
The above are three other forms of scales, the first of which is a decimal scale, for taking off common numbers consisting of two figures. The other two are duodecimal scales, and serve for feet and inches, &c.

These, and other scales, engraven on ivory, are fittest for practical use. And the most convenient

 form

form of a plane ſcale of equal diviſions, is on the very edge of the ivory, made thin at the edge, for laying along any line, and then marking on the paper oppoſite any diviſion required: which is better than taking lengths off a ſcale with compaſſes.

REMARKS.



Note 1. That in a circle, the half chord DC, is a mean proportional between the ſegments AD, DB of the diameter AB perpendicular to it. That is AD : DC : : DC : DB.

2. The chord AC is a mean proportional between AD and the diameter AB. And the chord BC a mean proportional between DB and AB.

That is, AD : AC : : AC : AB,

and BD : BC : : BC : AB.

3. The angle ACB, in a ſemicircle, is always a right angle.

4. The ſquare of the hypotenuſe of a right-angled triangle, is equal to the ſquares of both the ſides.

That is, $AC^2 = AD^2 + DC^2$,

and $BC^2 = BD^2 + DC^2$,

and $AB^2 = AC^2 + BC^2$.

5. Triangles that have all the three angles of the one, reſpectively equal to all the three of the other, are called equiangular triangles, or ſimilar triangles.

6. In ſimilar triangles, the like ſides, or ſides oppoſite the equal angles, are proportional.

7. The areas, or ſpaces, of ſimilar triangles, are to each other, as the ſquares of their like ſides.

SECT.

SECT. II.

PLANE TRIGONOMETRY.

PLANE Trigonometry teaches the relations and calculations of the ſides and angles of plane triangles.

The angles of triangles are meaſured by the number of degrees contained in the arc cut off by the legs of the angle, and whoſe center is the angular point. A right angle is therefore an angle of 90 degrees; and the ſum of the three angles of every triangle, or two right angles, is equal to 180°. Wherefore, in a right-angled triangle, the one acute angle being ſubtracted from 90°, the remainder will be the other; and the ſum of any two angles of a triangle being taken from 180°, will leave the third angle.

Degrees are marked at the top of the figure with a ſmall °, minutes with ′, ſeconds with ″, and ſo on. Thus 57° 30′ 12″, that is, 57 degrees 30 minutes and 12 ſeconds.

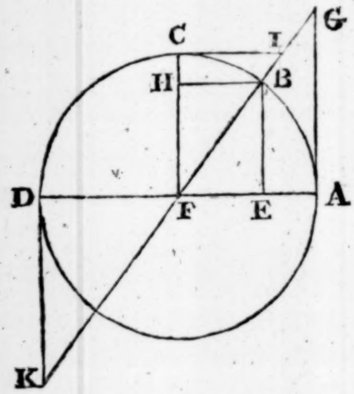
In a right-angled triangle, the ſide oppoſite the right angle, is called the hypotenuſe; and the other, the legs, or ſides, or ſometimes the baſe and perpendicular.

The complement of an arc is what it wants of a quadrant. So BC = 40° is the complement of AB = 50°.

The ſupplement of an arc is what it wants of a ſemicircle. So BCD = 130° is the complement of AB = 50°.

The ſine of an arc is the line drawn from one end of the arc perpendicularly upon the diameter drawn through the other end of the arc. So BE is the ſine of AB or of BCD.

 The



The versed sine of an arc is the part of the diameter between the sine and the beginning of the arc. So AE is the versed sine of AB, and DE the versed sine of BCD.

The tangent of an arc is the line drawn perpendicularly from one end of the diameter passing through one end of the arc, and terminated by the line drawn from the center through the other end of the arc. So AG or DK is the tangent of AB, or of BCD.

The secant of an arc is the line drawn from the center through the end of the arc, and terminated by the tangent. So FG or FK is the secant of AB, or of BCD.

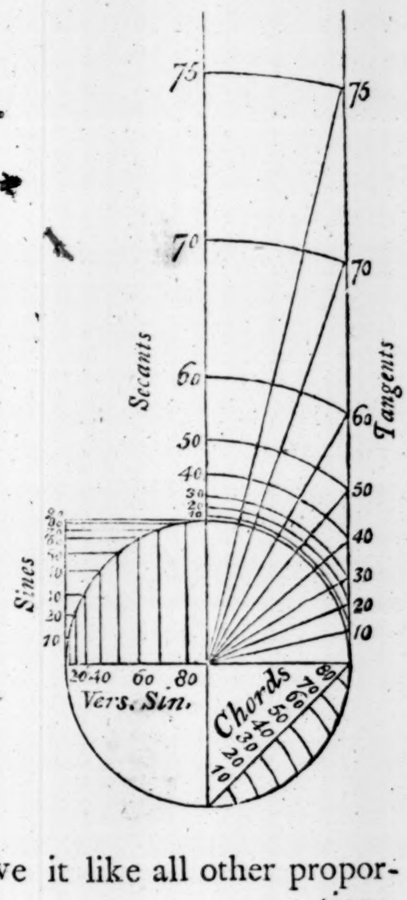
The cosine, cotangent, or cosecant of an arc, is the sine, tangent, or secant of the complement of that arc. So BH is the cosine, CI the cotangent, and FI the cosecant of AB.

From the definitions it is evident that the sine, tangent, and secant, are common to two arcs, which are the supplements of each other: so the sine, tangent, or secant of 50°, is also the sine, tangent, or secant of 130°.

The sine, tangent, or secant of an angle, is the sine, tangent, or secant of the arc, or the degrees, by which the angle is measured.

The ſine, tangent, and ſecant of every degree and minute in a quadrant, are calculated to the radius 1, and ranged in tables for uſe. But becauſe trigonometrical operations with theſe natural ſines, tangents, and ſecants, require tedious multiplications and diviſions, the logarithms of them are taken, and ranged in tables alſo; and the logarithmic ſines, tangents, and ſecants are commonly uſed, as they require only additions and ſubtractions, inſtead of the multiplications and diviſions.

The method of conſtructing the ſcales of chords, ſines, tangents, and ſecants, uſually placed on inſtruments, is exhibited in the following figure.



There are uſually three methods of reſolving triangles, or the caſes of trigonometry; namely, Geometrical conſtruction, Arithmetical computation, and Inſtrumental operation.

In the firſt method, let the triangle be conſtructed, by making the parts of the given magnitudes, ~~namely~~ the ſides from a ſcale of equal parts, and the angles from a ſcale of chords, or other inſtrument. Then meaſure the required parts by the ſame ſcale.

In the ſecond method, having ſtated the terms of the proportion according to the rule, reſolve it like all other propor-

 tions,

tions, in which a fourth term is to be found from three given terms, by multiplying the second and third together, and dividing the product by the first, in working with the natural numbers, whether they be sides, or sines, tangents, or secants of angles. Or, in working with the logarithms, add the log. of the 2d and 3d terms together, and from the sum subtract the log. of the 1st term; then the number answering to the remainder will be the 4th term required.

To work a stating instrumentally, as suppose by the log. lines on one side of the two-foot scales.—Extend the compasses from the first term to the second, or third, which happens to be of the same kind with it; then that extent will reach from the other term to the fourth, taking both extents towards the same side.

Note. For the sides of triangles you use the line of numbers (marked Num.) and for the angles, the lines of sines or tangents (marked Sin. and Tan.) according as the proportion respects sines or tangents.—If the extent upon the tangents reach beyond the line, set it so far back as it reaches over.

In a triangle there must be given three parts, one of which, at least, must be a side; because the same angles are common to an infinite number of triangles.

In plane trigonometry there are three cases or varieties only, *viz.*

1. When two of the three parts given, are a side and its opposite angle.

2. When there are given two sides and their included angle.

3. When the three sides are given.

A Table

A Table of Lineal Meaſures.

Inches	Feet	Yards	Poles	Chains	Furlongs	Mile
12	1					
36	3	1				
198	$16\frac{1}{2}$	$5\frac{1}{2}$	1			
792	66	22	4	1		
7920	660	220	40	10	1	
63360	5280	1760	320	80	8	1

Note alſo, An inch is ſuppoſed equal to 3 barley-corns in length,
4 inches—a hand,
6 feet, or 2 yards—a fathom,
3 miles—a league,
60 nautical or geographical miles—a degree, or $69\frac{1}{2}$ ſtatute miles nearly,
360 degrees, or 25000 miles nearly—is the circumference of the earth.

PROBLEM I.

Given Three Parts, ſuch, that an Angle and its Oppoſite Side are Two of them, to find the reſt.

In any plane triangle, the ſides are proportional to the ſines of their oppoſite angles*. That is,

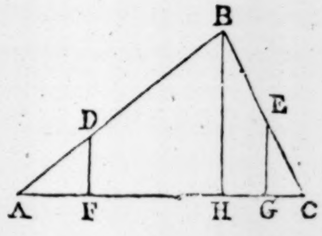
As one ſide :
Is to another ſide : :
So is ſin. angle opp. the former :
To ſin. angle opp. the latter.

Note

---

* DEMONSTRATION.

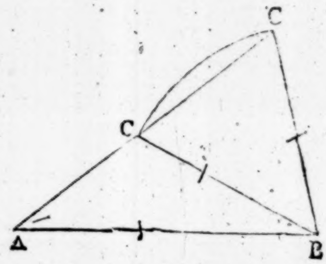
Let ABC be any triangle: in AB aſſume any point D, take CE = AD, and upon AC demit the perpendiculars DF, EG, BH; then will DF and EG be the ſines of the angles A, C, to the general radius AD or CE. Now, from ſimilar triangles, we ſhall have $\left\{ \begin{array}{l} AB : BH :: AD : DF \\ CB : BH :: AD\ (CE) : EG \end{array} \right\}$ and hence, of equality, AB : BC :: EG : DF.



Note 1. To find an angle, begin the proportion with a ſide oppoſite a given angle; and to find a ſide, begin with an angle oppoſite a given ſide.

2. An angle found by this rule is always ambiguous, except it be a right angle, or except that the magnitude of the given angle prevent the ambiguity; becauſe the ſine anſwers to two angles which are the ſupplements of each other: and accordingly the conſtruction gives two triangles with the ſame given parts; and when there is no reſtriction or limitation included in the propoſition, either of them may be taken. The degrees, in the table, anſwering to the ſine, is the acute angle; and if the angle be obtuſe, take thoſe degrees from 180°, and the remainder will be the obtuſe angle.—When the given angle is obtuſe, or right, there can be no ambiguity; for then neither of the other angles can be obtuſe, and the conſtruction will produce but one triangle.

EXAMPLE I.



In the plane triangle ABC,

Given $\left\{\begin{array}{l} \left.\begin{array}{l}\text{AB } 345\\ \text{BC } 232\end{array}\right\}\text{ yards} \\ \angle\text{A } 37^\circ\ 20' \end{array}\right.$

Required the other parts.

Geometrically.

1. Draw the line AB = 345 from ſome convenient ſcale of equal parts.

2. Make the angle A = 37° 20′.

3. With the center B and radius 232, taken from the ſame ſcale of equal parts, croſs AC in C.

4. Draw BC, and the triangle is conſtructed.

Then

Then the angles B and C, measured by the scale of chords, and the side AC, measured by the scale of equal parts, will be found to be as follows: viz.

∠ B 27°	∠ C $115\frac{1}{2}$	AC 174
or $78\frac{1}{4}$	or $64\frac{1}{2}$	or $374\frac{1}{2}$

Arithmetically.

As side BC	—	232	—	— 2·3654880
To side BA	—	345	—	— 2·5378191
So sine ∠ A	37° 20′		—	— 9·7827958
To sine ∠ C	115° 36′	or 64° 24		— 9·9551269
∠ A	37 20	37 20		
subtr.	152 56	or 101 44 sub.		
from	180 00	180 00		
leaves	27 04	or 78 16 the ∠ B.		

Then

As sine ∠ A	—	37° 20′	—	9·7827958
To sine ∠ B	—	27° 04′	—	9·6580371
		78 16	—	9·9908291
So side BC	—	232	—	2·3654880
To side AC	—	174·07	—	2·2407293
		374·56	—	2·5735213

Instrumentally.

In the first proportion, Extend from 232 to 345 upon the line of numbers; that extent will reach, upon the sines, from $37°\frac{1}{3}$ to $64°\frac{1}{2}$ the angle C.

In the second proportion, Extend from $37°\frac{1}{3}$ to 27° or $78°\frac{1}{4}$ upon the sines; that extent will reach, upon the numbers, from 232 to 174 or $374\frac{1}{2}$, for the side AC.

EX-

EXAMPLE II.

In the plane triangle ABC

$$\text{Given}\left\{\begin{array}{l}\left.\begin{array}{l}AB\ 162\\ AC\ 270\end{array}\right\}\text{chains}\\ \angle B\ 90^\circ\end{array}\right.\ \text{Required the other parts.}\Big\}\ \text{Anſ.}\left\{\begin{array}{l}BC\ 216\ \text{chains}\\ \angle C\ 36^\circ\ 52'\ 12''\\ \angle A\ 53\ \ 07\ \ 48\end{array}\right.$$

EXAMPLE III.

In the plane triangle ABC

$$\text{Given}\left\{\begin{array}{l}AB\ 365\ \text{poles}\\ \angle B\ 24^\circ\ 45'\\ \angle A\ 57\ \ 12\end{array}\right.\ \text{Required the other parts.}\Big\}\ \text{Anſ.}\left\{\begin{array}{l}\angle C\ 98^\circ\ 03'\\ \left.\begin{array}{l}AC\ 154{\cdot}33\\ BC\ 309{\cdot}86\end{array}\right\}\text{pol.}\end{array}\right.$$

EXAMPLE IV.

In the plane triangle ABC

$$\text{Given}\left\{\begin{array}{l}AB\ 53\ \text{miles}\\ \angle A\ 121^\circ\ 14'\\ \angle C\ 29^\circ\ 23'\end{array}\right.\ \text{Required the other parts.}\Big\}\ \text{Anſ.}\left\{\begin{array}{l}\angle B\ 29^\circ\ 23'\\ AC\ 53\ \text{miles}\\ BC\ 92{\cdot}36\ \text{miles}\end{array}\right.$$

EXAMPLE V.

In the plane triangle ABC

$$\text{Given}\left\{\begin{array}{l}AB\ 162\\ \angle C\ 36^\circ\ 52'\ 12''\\ \angle A\ 53\ \ 07\ \ 48\end{array}\right.\ \text{Required the other parts.}\Big\}\ \text{Anſ.}\left\{\begin{array}{l}\angle B\ 90^\circ\\ AC\ 270\ \text{chains}\\ BC\ 216\ \text{chains}\end{array}\right.$$

EXAMPLE VI.

In the plane triangle ABC

$$\text{Given}\left\{\begin{array}{l}AB\ 365\ \text{poles}\\ AC\ 154{\cdot}33\ \text{poles}\\ \angle C\ 98^\circ\ 03'\end{array}\right.\ \text{Required the other parts.}\Big\}\ \text{Anſ.}\left\{\begin{array}{l}\angle B\ 24^\circ\ 45'\\ \angle A\ 57\ \ 12\\ BC\ 309{\cdot}86\ \text{pol.}\end{array}\right.$$

EXAMPLE VII.

In the plane triangle ABC

Given	AC 120 feet BC 112 feet $\angle$A 57° 27′	Anf.	$\angle$B	64° 34′ 21″ 115 25 39	
Required the other par.			$\angle$C	57 58 39 7 07 21	
			AB	112·65 16·47	feet

PROBLEM II.

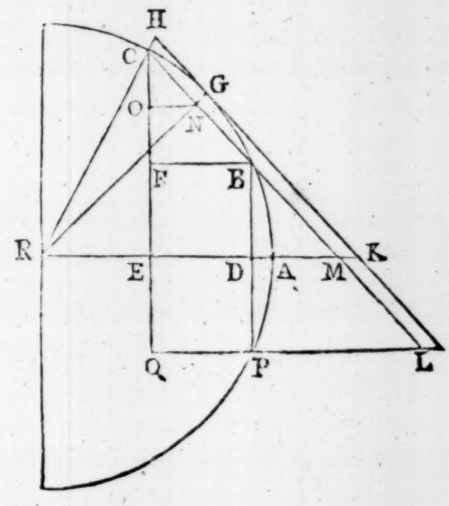
Given Two Sides and the Angle Included by them, to find the rest.

In a plane triangle, As the sum of any two sides is to their difference, so is the tangent of half the sum of their opposite angles, to the tangent of half their difference. Then the half difference added to the half sum of the angles, gives the greater; and subtracted, leaves the less angle*.

Then,

* DEMONSTRATION.

By the first problem, the sides are as the sines of their opposite angles, and consequently the sum of the sides will be to the difference of the sides, as the sum of the sines to the difference of the sines of the said opposite angles.



Wherefore we have only to prove that the sum of the sines is to the difference of the sines of two arcs, as the tangent of half the sum of those arcs is to the tangent of half their difference: In order to which, let BD, CE, be the sines of the arcs AB, AC; produce BD to the circumference at P, and produce CE till EQ be = DP; to the middle point G of the arc BC draw the tangent HGK, and draw CNBML parallel to it; join RH, RG, and draw ON, FB, and QL parallel to RAMK.

Now

Then, all the angles being known, find the unknown ſide by the firſt problem.

Note. When, in this caſe, the triangle is right-angled, the longeſt ſide will be found by extracting the ſquare root of the ſum of the ſquares of the other two ſides; and then the angles will be found by the firſt problem.

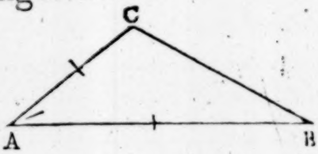
Note alſo, That inſtead of the tangent of the half ſum, we may uſe the cotangent of half the given angle, which is the ſame thing.

EXAMPLE I.

In the plane triangle ABC

Given { AB 345 yards; AC 174·07 yards; ∠A 37° 20′ }

Required the other parts.



Geometrically.

1. Draw AB equal to 345, from a ſcale of equal parts.

2. Make

Now it is evident that CQ is the ſum, and CF the difference of the ſines; and that GK is the tangent of half the ſum AG, and GH the tangent of half the difference CG, of the two arcs AB, AC; alſo NM is = ½CL, for BN = NC, and BM = ML: then, in the ſimilar triangles CQL, RNM, RGK, it will be as

CQ : CF :: (½ CQ or) OE : (½ CF or) OC,
and NM : NC :: OE : OC,
and NM : NC :: GK : GH,
theref. CQ : CF :: GK : GH.

Which was to be demonſtrated.

And that the half ſum increaſed and diminiſhed by the half difference, gives the greater and leſs angle reſpectively, is evident from the figure.—And that two quantities of any kind may be found, by the ſame rule, from their ſum and difference, may be proved thus: Let CN repreſent the leſs, and NL the greater, of any two quantities; and let B be the middle of the right line CL. Then it is evident that BL = BC is the half ſum, and BN the half difference, as alſo that LB + BN = NL the greater quantity, and CB − BN = NC the leſs.

2. Make the angle A equal to 37° 20′.

3. Make AC equal to 174·07, by the scale of equal parts.

4. Join B, C, and it is done.

Then, the parts being measured, we have the ∠C = 115°½, the ∠B = 27°, and BC = 232 yards.

Arithmetically.

As sum of sides AB + AC	519·07	—	2·7152259
To dif. of sides AB − AC	170·93	—	2·2328183
So tang. $\frac{\angle C + \angle B}{2}$ —	71° 20′	—	10·4712979
To tang. $\frac{\angle C - \angle B}{2}$ —	44 16	—	9·9888903
Their sum	115 36 ∠C		
Their dif.	27 04 ∠B		

Then

As sine ∠C 115° 36′ or 64° 24′	—	9·9551259
To sine ∠A — — 37° 20′	—	9·7827958
So side AB 345 — —	—	2·5378191
To side BC 232 — —	—	2·3654890

Instrumentally.

In the first proportion, Extend from 519 to 171 on the line of numbers; that extent will reach, upon the tangents, from 71°⅓ (the contrary way, because the tangents are set back again from 45°) a little beyond 45, which being set so far back from 45, falls upon 44°¼ the fourth term.

In the second proportion, Extend from 64°⅓ to 37°⅓ on the sines, that extent will reach, on the numbers, from 345 to 232 the fourth term required.

EXAMPLE II.

In the plane triangle ABC

$$\left.\begin{array}{l}\text{Given}\left\{\begin{array}{l}\left.\begin{array}{l}\text{AB } 162\\ \text{BC } 216\end{array}\right\}\text{chains}\\ \angle\text{B } 90^\circ\end{array}\right.\\ \text{Required the other parts.}\end{array}\right\}\text{Ans.}\left\{\begin{array}{l}\angle\text{A } 53^\circ\ 07'\ 48''\\ \angle\text{C } 36\ \ 52\ \ 12\\ \text{AC } 270 \text{ chains}\end{array}\right.$$

EXAMPLE III.

In the plane triangle ABC

$$\left.\begin{array}{l}\text{Given}\left\{\begin{array}{l}\left.\begin{array}{l}\text{AB } 53\\ \text{BC } 92\cdot36\end{array}\right\}\text{miles}\\ \angle\text{B } 29^\circ\ 23'\end{array}\right.\\ \text{Required the other parts.}\end{array}\right\}\text{Ans.}\left\{\begin{array}{l}\angle\text{A } 121^\circ\ 14'\\ \angle\text{C } \ \ 29\ \ 23\\ \text{AC } \ \ 53 \text{ miles.}\end{array}\right.$$

EXAMPLE IV.

In the plane triangle ABC

$$\left.\begin{array}{l}\text{Given}\left\{\begin{array}{l}\left.\begin{array}{l}\text{AC } 120\\ \text{BC } 112\end{array}\right\}\text{poles}\\ \angle\text{C } 57^\circ\ 58'\ 39''\end{array}\right.\\ \text{Required the other parts.}\end{array}\right\}\text{Ans.}\left\{\begin{array}{l}\angle\text{A } 57^\circ\ 27'\ 00''\\ \angle\text{B } 64\ \ 34\ \ 21\\ \text{AB } 112\cdot65 \text{ pol.}\end{array}\right.$$

PROBLEM III.

Given the Three Sides of a Triangle, to find the Angles.

* In any plane triangle, having let fall a perpendicular from the greatest angle upon the opposite side or

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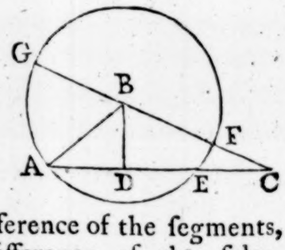
* DEMONSTRATION.



From one end B of the least side AB, of the triangle ABC, as a center, and radius AB, describe a circle cutting the other two sides in E and F; produce CB to the circle at G, and let fall the perpendicular BD.—Then is GB = BF = AB, and (by 3. III. Eucl.) AD = DE, and consequently EC = CD − DA the difference of the segments, FC = GB − BA the difference of the sides, and

or base, dividing it into two segments, and the whole triangle into two right-angled triangles; it will be

As the base, or sum of the segments :
Is to the sum of the other two sides : :
So is the difference of those sides :
To the difference of the segments of the base.

Then half the difference being added to, and subtracted from, half their sum, will give the greater and less segment.

Hence, in each of the right-angled triangles, will be known two sides, and the angle opposite to one of them; and consequently the other angles will be found by the first problem.

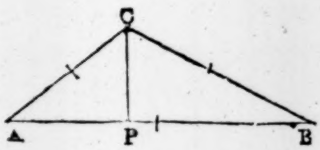
Note. In the above proportions, if half the difference of the sides be taken for the third term, then the fourth term will be half the difference of the segments. Which will commonly be more convenient to use than the whole differences.

EXAMPLE I.

In the plane triangle ABC

Given $\left\{\begin{array}{ll}\text{AB} & 345 \\ \text{AC} & 174{\cdot}07 \\ \text{BC} & 232\end{array}\right\}$ yards.

Required the angles.



E *Geometrically*.

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and GC = CB + BA the sum of the sides. But (by Cor. to 36. III. Eucl.) the rectangle CA × CE = CG × CF, or CA : CG : : CF : CE, that is AC : CB + BA : : CB − BA : CD − DA. *Q. E. D.*

And that the half sum of two quantities increased and diminished by their half difference, gives the greater and less quantities respectively, was proved in the last problem.

Geometrically.

1. Draw AB equal to 345 by a ſcale of equal parts.

2. With the centers A and B, and radii 174·07 and 232, taken from the ſame ſcale, deſcribe arcs interſecting in C.

3. Draw AC and BC, and it is done.

Then by meaſuring the angles, they appear to be nearly of the following dimenſions, *viz.* $\angle A = 37^\circ\frac{1}{3}$, $\angle B = 27^\circ$, and $\angle C = 115^\circ\frac{1}{2}$.

Arithmetically.

Having let fall the perpendicular CP, it will be As $AB = 345 : BC + CA = 406{\cdot}07 :: BC - CA = 57{\cdot}93 : \frac{406{\cdot}07 \times 57{\cdot}93}{345} = 68{\cdot}18 = BP - PA$.

$$\text{Hence, } \frac{345 + 68{\cdot}18}{2} = 206{\cdot}59 = BP,$$

$$\text{and } \frac{345 - 68{\cdot}18}{2} = 138{\cdot}41 = AP.$$

Then in the triangle APC, right-angled at P,

As AC	—	174·07	—	2·2407239
To AP	—	138·41	—	2·1411675
So s. ∠P	—	90° 00′	—	10·0000000
To s. ∠ACP	—	52 40	—	9·9004436
which being taken from		90 00		
leaves		37 20 ∠A		

And in the triangle BPC,

As BC	—	232	—	2·3654880
To BP	—	206·59	—	2·3151093
So s. ∠P	—	90° 00′	—	10·0000000
To s. ∠PCB	—	62 56	—	9·9496213
taken from		90 00		
leaves		27 04 ∠B		

Alſo

$$\begin{array}{rll} \text{Alſo} & 52^\circ\ 40' & \angle \text{ACP} \\ \text{added to} & 62\ \ \ 56 & \angle \text{BCP} \\ \hline \text{makes} & 115\ \ \ 36 & \angle \text{ACB} \\ \hline \end{array}$$

Whence the $\angle A = 37^\circ\ 20'$, the $\angle B = 27^\circ\ 04'$, and the $\angle C = 115^\circ\ 36'$.

Inſtrumentally.

In the firſt proportion.—Extend from 345 to 406, on the line of numbers; that extent will reach, upon the ſame line, from 58 to 68·2, the difference of the ſegments of the baſe.

In the ſecond proportion.—Extend from 174 to $138\frac{1}{2}$ on the numbers; that will reach, on the ſines, from 90° to $52^\circ\frac{2}{3}$.

In the third proportion.—Extend from 232 to $206\frac{1}{2}$, and that extent will reach from 90 to 63°.

EXAMPLE II.

In the plane triangle ABC

$$\text{Given}\left\{\begin{array}{ll} \text{AB} & 162 \\ \text{AC} & 270 \\ \text{BC} & 216 \end{array}\right. \quad \text{Required the angles.}\quad \text{Anſ.}\left\{\begin{array}{llll} \angle A & 53^\circ & 07' & 48'' \\ \angle B & 90 & 00 & 00 \\ \angle C & 36 & 52 & 12 \end{array}\right.$$

EXAMPLE III.

In the plane triangle ABC

$$\text{Given}\left\{\begin{array}{ll} \text{AB} & 112{\cdot}65 \\ \text{AC} & 120 \\ \text{BC} & 112 \end{array}\right. \quad \text{Required the angles.}\quad \text{Anſ.}\left\{\begin{array}{llll} \angle A & 57^\circ & 27' & 00'' \\ \angle B & 64 & 34 & 21 \\ \angle C & 57 & 58 & 39 \end{array}\right.$$

EXAMPLE IV.

In the plane triangle ABC

$$\text{Given}\left\{\begin{array}{ll}\text{AB} & 53\\ \text{AC} & 53\\ \text{BC} & 92\cdot36\end{array}\right.\quad\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}\ \text{Ans.}\left\{\begin{array}{lrr}\angle\text{A} & 121^\circ & 14'\\ \angle\text{B} & 29 & 23\\ \angle\text{C} & 29 & 23\end{array}\right.$$

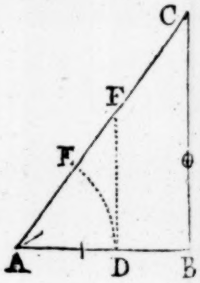
Required the angles.

These three problems include all the cases of plane triangles, as well right-angled as oblique; besides which there are some other theorems, suited to some particular forms of triangles, which are often more expeditious in use than the general ones; one of which, as the case for which it serves so often occurs, take as follows.

PROBLEM IV.

Given the Angles and a Leg, of a Right-angled Triangle; to find the Other Leg and the Hypotenuse.

* As the radius
Is to the given leg AB : :
So is tang. of angle A :
To the opposite leg BC : :
And so is secant of the same ∠A :
To the hypotenuse AC.



EX-

* DEMONSTRATION.

With the center A and any radius AD, describe an arc DE, and erect the perpendicular DF; which, it is evident, will be the tangent, and AF the secant of the arc DE, or angle A, to the radius AD.—And in the similar triangles ADF, ABC, it will be AD : AB :: DF : BC :: AF : AC. *Q. E. D.*

GENERAL SCHOLIUM.

Besides these rules, I shall here set down some new theorems concerning the relations of the sides and angles of triangles, independent of any tables.

PRO-

EXAMPLE.

In the plane triangle ABC, right-angled at B,

Given $\left\{\begin{array}{l} \text{AB } 162 \\ \angle\text{A } 53^\circ\ 07'\ 48'' \end{array}\right\}$ Required AC and BC.

PROPOSITION.

If $2a$ denote a side of any triangle, A the number of degrees contained in its opposite angle, and r the radius of the circle circumscribing the triangle: Then I say that A is equal to

$$57{\cdot}2957795 \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9} \ \&c.$$

For since $2a$ is the chord of the arc upon which the angle, whose measure is A, insists; a will be the sine of half that arc, or the sine of the angle to the radius r, since an angle in the circumference of a circle is measured by half the arc upon which it stands: now it appears, from rule 3. prob. 6. sect. 1. part 2. of this work, that the said half arc z is equal to

$a + \frac{a^3}{2.3r^2} + \frac{3a^5}{2.4.5r^4} + \frac{3.5a^7}{2.4.6.7r^6}$ &c; and, $3{\cdot}14159r$ denoting half the circumference of the same circle, or the arc of 180 degrees, we shall have $3{\cdot}14159r : 180^\circ :: z : \frac{180z}{3{\cdot}14159r} =$

$$\frac{57{\cdot}2957795z}{r} = 57{\cdot}2957795 \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} \&c$$

= the degrees in the angle or half arc.

Corollary 1. By reverting the above series we obtain

$$\frac{a}{r} = \frac{\text{A}}{n} - \frac{\text{A}^3}{2.3n^3} + \frac{\text{A}^5}{2.3.4.5n^5} - \frac{\text{A}^7}{2.3.4.5.6.7n^7} \ \&c,$$

$$\text{putting } n = 57{\cdot}2957795 = \frac{180}{3{\cdot}14159 \ \&c},$$

Corollary 2. If $2a$ be the hypotenuse of a right-angled triangle, a will be $= r$, and then the general series will become $n \times : a + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7}$ &c $= 90$, or $\frac{90}{n} = \frac{90 \times 3{\cdot}14159 \ \&c}{180} =$

$$\frac{3{\cdot}14159 \ \&c}{2} = 1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7} + \frac{3.5.7}{2.4.6.8.9} \ \&c.$$

Geometrically.

Make AB = 162, and the angle A = 53° 07′ 48″; then raise the perpendicular BC meeting AC in C. So shall AC measure 270, and BC 216.

Calcu-

Corollary 3. Since the chord of 60 degrees is = the radius, or the sine of 30 degrees = half the radius, putting a for $\frac{1}{2}r$ in the general series, will give $n \times : \frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \frac{3.5}{2.4.6.7.2^7}$ &c = 30; and hence the sum of the infinite series

$$\frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \frac{3.5}{2.4.6.7.2^7} \text{ \&c}$$

$$\text{is} = \frac{30}{n} = \frac{30 \times 3{\cdot}14159 \text{ \&c}}{180} = \frac{3{\cdot}14159 \text{ \&c}}{6} =$$

one-sixth of the circumference of the circle whose diameter is 1.

Corollary 4. It might easily be shewn, from the principles of common geometry, that the sine of 60 degrees is to the radius, as $\frac{1}{2}\sqrt{3}$ is to 1; substituting, then, $\frac{1}{2}r\sqrt{3}$ for a in the general series, we shall have $n\sqrt{3} \times : \frac{1}{2} + \frac{3}{2.3.2^3} + \frac{3.3^2}{2.4.5.2^5} + \frac{3.5.3^3}{2.4.6.7.2^7}$ &c = 60; and hence the sum of the infinite series

$$\frac{1}{2} + \frac{3}{2.3.2^3} + \frac{3.3^2}{2.4.5.2^5} + \frac{3.5.3^3}{2.4.6.7.2^7} \text{ \&c will be}$$

$= \frac{60}{n\sqrt{3}} = \frac{60 \times 3{\cdot}14159 \text{ \&c}}{180\sqrt{3}} = \frac{3.14159 \text{ \&c}}{3\sqrt{3}}$, and is, therefore, to the infinite series in the third corollary, as 2 is to $\sqrt{3}$.

Corollary 5. If b, c be the halves of the other two sides of the triangle, and B, C the degrees contained in their opposite angles; since $B = n \times : \frac{b}{r} + \frac{b^3}{2.3r^3} + \frac{3b^5}{2.4.5r^5}$ &c,

and $C = n \times : \frac{c}{r} + \frac{c^3}{2.3r^3}$ &c, and the 3 angles of any triangle are equal to 180 degrees; we shall have $180 = A + B + C = n \times : \frac{a+b+c}{r} + \frac{a^3+b^3+c^3}{2.3r^3}$ &c, or the sum of the infinite series

$$\frac{a+b+c}{r} + \frac{1}{2.3}.\frac{a^3+b^3+c^3}{r^3} + \frac{3}{2.4.5}.\frac{a^5+b^5+c^5}{r^5} + \frac{3.5}{2.4.6.7}.\frac{a^7+b^7+c^7}{r^7}$$

&c

Calculation.

As radius 1 : AB = 162 :: 1·3333284 = nat. tangent of $53^\circ\ 07'\ 48''$: 215·9992 = BC.

And, as 1 : 162 :: 1·6666628 = nat. ſecant of the ſame ∠ A : 269·99937 = AC.

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&c will be $= \frac{180}{n} = \frac{180 \times 3{\cdot}14159\ \&c}{180} = 3{\cdot}14159$ &c = the circumference of a circle whoſe diameter is 1; a, b, c being the halves of the three ſides of any triangle, and r the radius of its circumſcribing circle.

Corollary 6. Since, by prob. 3, $b : a + c :: a - c : \frac{aa - cc}{b} =$ half the difference of the ſegments of the baſe (b) made by a perpendicular demitted from its oppoſite angle, and

$b + \frac{aa-cc}{b} + \frac{aa+bb-cc}{b}$ = the ſegment adjoining to the ſide $2a$,

we ſhall have $\sqrt{4a^2 - \frac{(aa+bb-cc)^2}{bb}} = \frac{\sqrt{4a^2b^2 - (aa+bb-cc)^2}}{b}$

for the value of the ſaid perpendicular to the baſe, and hence

$\frac{\sqrt{4a^2b^2-(aa+bb-cc)^2}}{b} : 2a :: c : \frac{2abc}{\sqrt{4a^2b^2-(aa+bb-cc)^2}} = r$ the radius of the circumſcribing circle.

Having now found the value of r, we can calculate all the caſes of trigonometry without any tables, and without reducing oblique triangles to right-angled ones; for having any three parts (except the three angles) given, we can find the reſt from theſe five equations:

1. $r = \frac{2abc}{\sqrt{4a^2b^2-(aa+bb-cc)^2}}$.

2. $A = n \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9}$ &c.

3. $B = n \times : \frac{b}{r} + \frac{b^3}{2.3r^3} + \frac{3b^5}{2.4.5r^5} + \frac{3.5b^7}{2.4.6.7r^7} + \frac{3.5.7b^9}{2.4.6.8.9r^9}$ &c.

4. $C = n \times : \frac{c}{r} + \frac{c^3}{2.3r^3} + \frac{3c^5}{2.4.5r^5} + \frac{3.5c^7}{2.4.6.7r^7} + \frac{3.5.7c^9}{2.4.6.8.9r^9}$ &c.

5. $A + B + C = 180$.

And

Instrumentally.

The extent from 45° to 53° 08′, upon the tangents, will reach from 162 to 216 upon the numbers.

Note.

And for the more convenience we may add the three following, which are derived from the 2d, 3d, and 4th, by reversion of series.

$$6.\ a = r \times : \frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7} \ \&c.$$

$$7.\ b = r \times : \frac{B}{n} - \frac{B^3}{2.3n^3} + \frac{B^5}{2.3.4.5n^5} - \frac{B^7}{2.3.4.5.6.7n^7} \ \&c.$$

$$8.\ c = r \times : \frac{C}{n} - \frac{C^3}{2.3n^3} + \frac{C^5}{2.3.4.5n^5} - \frac{C^7}{2.3.4.5.6.7n^7} \ \&c.$$

Where $n = 57 \cdot 2957795$ &c.

EXAMPLE.

Suppose we take here the first example in prob. 1, in which are given two sides $2b = 345$, $2c = 232$, and the angle opposite to $2c = 37° \ 20' = 37\frac{1}{3}$ degrees $= C$.

Then, $\frac{C}{n}$ being $= \frac{37\frac{1}{3} \times 3 \cdot 14159 \ \&c}{180} = \cdot 651589587$, we have $c = \frac{232}{2} = 116 = r \times : \cdot 651589587 - \cdot 04610744 + \cdot 00097879 - \cdot 000009894 + \cdot 000000058 \ \&c = r \times (\cdot 652568435 - \cdot 046117334) = \cdot 6064511\,r$. Hence $r = \frac{116}{\cdot 6064511} = 191 \cdot 27677$.

And $\frac{b}{r} = \frac{345 \times \cdot 6064511}{2 \times 116} = \cdot 9018346$.

Again $B = 57 \cdot 2957795 \times 1 \cdot 12402$ (the sum of the series in the third equation) $= 64 \cdot 4016$ degrees $= 64° \ 24'$.

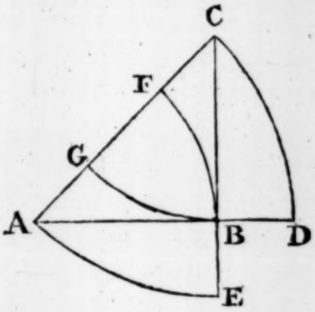
And $A = 180 - 37\frac{1}{3} - 64 \cdot 4016 = 180 - 101 \cdot 735 = 78 \cdot 265°$ $= 78° \ 16'$ nearly.

Lastly, $\frac{A}{n}$ being $= \frac{78 \cdot 265}{57 \cdot 2957795} = 1 \cdot 365982$, and $r = 191 \cdot 27677$, from the fifth equation we have $a = 191 \cdot 27677 \times : 1 \cdot 365982 - \cdot 4247992 + \cdot 0396379 - \cdot 0017607 + \cdot 0000288 - \cdot 0000005 = 191 \cdot 27677 \times \cdot 9790883 = 187 \cdot 27684$.

And hence $2a = 374 \cdot 55368 =$ the third side of the triangle.

Corollary

Note. It is common to add another method for right-angled triangles, which is this. ABC being the triangle, make a leg AB radius, that is, with center A and radius AB, describe an arc BF: then it is evident that the other leg BC represents the tangent, and the hypotenuse AC the secant of the angle A or arc BF.



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Corollary 7. As the series by which an angle is found, often converges very slowly, I have inserted the following approximation of it, viz.

$A = n \times (\frac{4}{3}\sqrt{2 - 2\sqrt{1 - \frac{aa}{rr}}} - \frac{a}{3r})$ nearly; where the letters denote the same quantities as in the above series.

For since $P = \sqrt{2 - 2\sqrt{1 - \frac{aa}{rr}}}$ is $= \frac{a}{r} + \frac{a^3}{2.4r^3} + \frac{7a^5}{2.4.16r^5}$ &c,

and $\frac{A}{n}$ is $= \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5}$ &c;

we shall have, by taking the former of these from the latter, $\frac{A}{n} - P = \frac{a^3}{24r^3} + \frac{13a^5}{640r^5}$ &c. But, from the first series, $\frac{1}{3}P - \frac{a}{3r} = \frac{a^3}{24r^3} + \frac{7a^5}{384r^5}$ &c; hence, by subtracting the latter from the former,

$\frac{A}{n} - P - \frac{1}{3}P + \frac{a}{3r} = \frac{A}{n} - \frac{4}{3}P + \frac{a}{3r} = \frac{a^5}{480r^5}$ &c; and

$A = n \times (\frac{4}{3}P - \frac{a}{3r}) = n \times (\frac{4}{3}\sqrt{2 - 2\sqrt{1 - \frac{aa}{rr}}} - \frac{a}{3r})$ nearly.

Corollary 8. And again, since $\frac{4}{105} \times (P - q - \frac{1}{8}q^3) = \frac{1}{480}q^5$ &c; (where q is $= \frac{a}{r}$), by subtracting this from $\frac{A}{n} - \frac{4P - q}{3} = \frac{1}{480}q^5$ &c, and reducing, there will be obtained $A = \frac{n}{105} \times$

(144

In like manner, if the leg BC be made radius; then the leg AB will repreſent the tangent, and AC the ſecant of the arc BG, or of the angle C.

But

$$(144p - 39q - \tfrac{1}{2}q^3) = \frac{n}{105} \times (144\sqrt{2 - 2\sqrt{1 - q^2}} - 39q - \tfrac{1}{2}q^3$$

which will commonly give the angle true to within a minute of the truth. Where note that the conſtant quantity $\frac{n}{105} =$ ·54567409. And from the whole may be drawn the following problem.

PROBLEM.

To perform all the Caſes of Trigonometry without any Tables.

HAVING any three parts of a triangle given, except the three angles, the other three parts may be found by ſome of the following ſix general theorems.

1. $A = \frac{1}{3}n \times (4\sqrt{2 - 2\sqrt{1 - \frac{a^2}{r^2}}} - \frac{a}{r})$ nearly, Or

$$A = \frac{n}{105} \times (144\sqrt{2 - 2\sqrt{1 - \frac{a^2}{r^2}}} - 39\,\frac{a}{r} - \frac{a^3}{2r^3})\text{more nearly.}$$

2. $A = n \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9}$ &c.

3. $a = r \times : \frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7}$ &c.

4. $r = \dfrac{a}{\frac{A}{n} - \frac{A^3}{2.3.n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7}\ \&c}$.

5. $r = \dfrac{2abc}{\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}}$

$$= \frac{2abc}{\sqrt{(a+b+c) \times (a+b-c) \times (a-b+c) \times (-a+b+c)}}.$$

6. $C = \sqrt{a^2 + b^2 - 2ab\sqrt{1 - (\frac{C}{n} - \frac{C^3}{2.3n^3} + \frac{C^5}{2.3.4.5n^5} - \frac{C^7}{2.3.4.5.6.7n^7}\ \&c)^2}}$

Where

But if the hypotenuse be made radius; then each leg will represent the sine of its opposite angle; namely, the leg AB the sine of the arc AE or angle C, and the leg BC the sine of the arc CD or angle A.

And then the general rule for all these cases, is this, namely, that the sides bear to each other the same proportion, as the parts which they represent.

And this is called, making every side radius.

SECT. III.

OF HEIGHTS AND DISTANCES, &c.

BY the mensuration and protraction of lines and angles, we determine the lengths, heights, depths, or distances of bodies and objects. And this branch

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Where a, b, c, are the halves of the three sides of the triangle, and A the number of degrees in the angle opposite the side $2a$, and C the degrees in the angle opposite the side $2c$; also r is the radius of the circumscribed circle;

$$\text{and } n = \frac{180}{3{\cdot}14159} = 57{\cdot}2957795, \text{ or } \frac{n}{105} = {\cdot}54567409.$$

EXAMPLE.

Thus, if the three sides be given, as for example $a = 13$, $b = 14$, $c = 15$. Then is $r = 16\frac{1}{4}$, and the angles by these theorems come out as follows, viz.

Angles by the Theor.			The true Angles.
53° 7′	- -	angle A - -	53° 7′$\frac{4}{5}$
59 28	- -	angle B - -	59 29$\frac{2}{5}$
67 19	- -	angle C - -	67 22$\frac{4}{5}$
179 54		sum of all	180 00

branch more immediately reſpects navigation, ſurveying, and what is commonly called altimetry and longimetry, or heights and diſtances, if indeed this muſt be diſtinguiſhed from ſurveying.

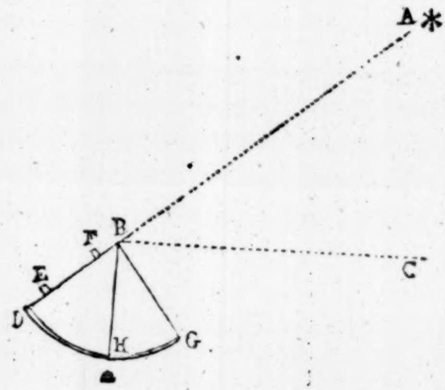
Acceſſible lines are meaſured by applying to them ſome certain meaſure, as an inch, a foot, &c. a number of times; but inacceſſible lines muſt be meaſured by taking angles, or by ſome ſuch-like method, drawn from the principles of geometry.

When inſtruments are uſed for taking the quantities of the angles in degrees, the lines are then calculated by trigonometry: in the other methods the lines are calculated from the principle of ſimilar triangles, without any regard to the quantities of the angles.

Angles of elevation, or of depreſſion, are uſually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furniſhed with a plummet ſuſpended from the center, and two ſights fixed perpendicularly upon one of the radii.

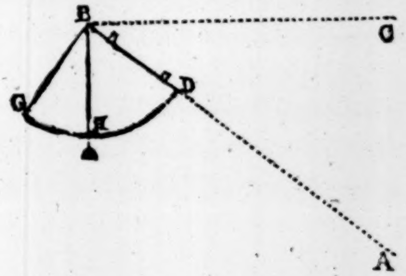
To take an Angle of Altitude and Depreſſion with the Quadrant.

Let A be any object, as the top of a tower, hill, or other eminence; or the ſun, moon, or a ſtar: and let it be required to find the meaſure of the angle ABC which a line drawn from the object makes with the horizontal line BC.



Fix the center of the quadrant in the angular point, and move it round there as a center till with one eye at D, the other being ſhut, you perceive the

the object A through the two sights E, F; then will the arc GH of the quadrant, cut off by the plumb line BH, be the measure of the angle ABC required.



The angle ABC of depression of any object A is taken in the same manner, except that here the eye is applied to the center, and the measure of the angle is the arc GH.

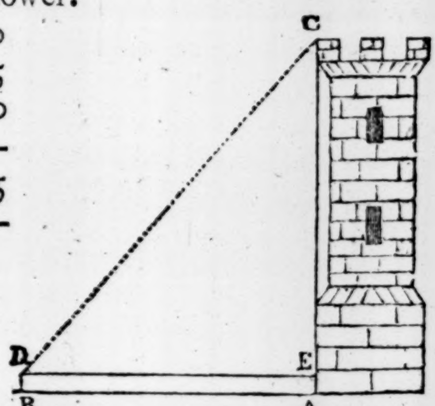
The observations with the quadrant, necessary to determine the heights and distances of objects, will be sufficiently apparent from the manner in which the following examples are proposed; and the solution may easily be given, by any one who understands plane trigonometry.

The construction of the figures to the following examples, are omitted; but they are to be constructed as in the problems of trigonometry.

EXAMPLE I.

Having measured AB equal to 100 feet from the bottom of a tower, in a direct line on a horizontal plane, I then took the angle CDE of elevation of the top, and found it to be 47° 30′, the center of the quadrant being fixed five feet above the ground: required the height of the tower.

As radius	—	10·0000000
Tot. ∠ D 47° 30′		10·0379475
So DE 100	—	2·0000000
To CE 109·13		2·0379475
Add AE 5		
AC 114·13		



Or,

Or, without the Logarithms.

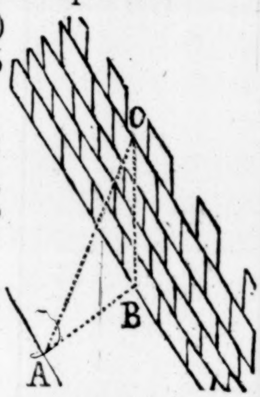
As 1 (rad.): 1·0913085 (tan. of 47° 30′) :: 100 : 100 × 1·0913085 = 109·13085 = CE.

By the calculation the height CE is found equal to 109·13, to which DB (equal to EA equal to 5 feet the height of the inſtrument) being added, gives AC equal to 114·13, the whole height.

Note. If you go off to ſuch a diſtance from the bottom, as that the angle of elevation ſhall be 45°, then will the height be equal to the diſtance with the height of the center of the inſtrument added.

EXAMPLE II.

From the edge of a ditch of 18 feet wide, ſurrounding a fort, I took the angle of elevation of the top of the wall, and found it 62° 40′: required the height of the wall, and the length of a ladder neceſſary to reach from my ſtation to the top of it.



Firſt, As 1 : 1·934702 (tan. 62° 40′) :: 18 : 1·934702 × 18 = 34·824636 = BC.

Then $\sqrt{18^2 + 34{\cdot}824636^2} = \sqrt{1536{\cdot}755272} = 39{\cdot}2014$ = AC.

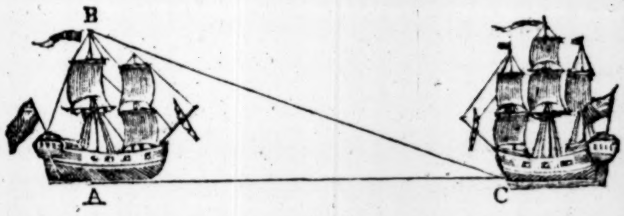
Or as 1 : 2·1778594 (ſecant of 62° 40′) :: 18 : 39·204692 = AC.

Anſ. The height BC = 34·82, and the length of the ladder AC = 39·2.

EXAMPLE III.

From the top of a ſhip's maſt, which was 80 feet above the water, the angle of depreſſion of another ſhip's hull, at a diſtance, upon the water, is 20°; what is their diſtance?

As

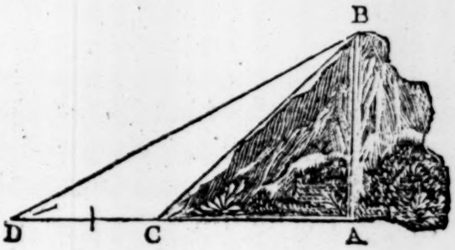


As 1 : 2·7474774 (tan. 70°) :: 80 : 2·7473774 × 80 = 219·790192 feet = AC the distance required.

EXAMPLE IV.

What is the perpendicular height of a hill whose angle of elevation, taken at the bottom of it, was 46°; and 100 yards farther off, on a level with the bottom of it, the angle was 31°;

∠C	46°	subtract
∠D	31	
∠DBC 15	9·4129962	
∠D 31	9·7118393	
DC 100	2·0000000	
BC - -	2·2988431	

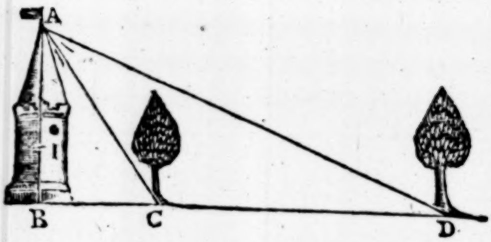


∠A 90°	— —	10·0000000
∠C 46	— —	9·8569341
CB —	— —	2·2988431
AB 143·14	—	2·1557772

EXAMPLE V.

From the top of a tower, whose height was 120 feet, I took the angle of depression of two trees which lay in a direct line, upon the same horizontal plane, with the bottom of the tower, viz. that of the nearer 57°, and that of the farther 25°½: what is the distance between the two trees, and the distance of each from the bottom of the tower?

2 As



As 1 : ·6494076 (tan. ∠ BAC = 33°) :: 120 : ·6494076 × 120 = 77·928912 feet = BC the distance from the bottom of the tower to the nearest tree.

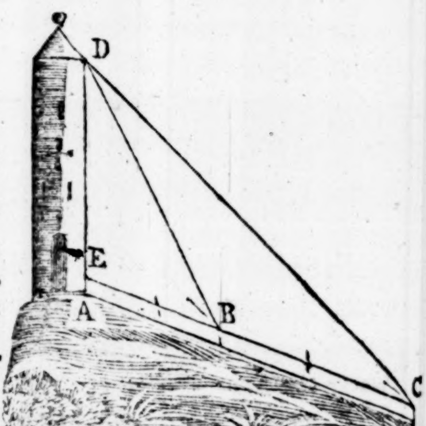
And as 1 · 2·0965436 (tan. ∠ BAD = 64°½) :: 120 : 2·0965436 × 120 = 251·585232 feet = BD the distance of the farther tree.

Therefore, BD — BC = 251·585232 — 77·928912 = 173·65632 feet = CD the distance between the two trees.

EXAMPLE VI.

An obelisk standing on the top of a declivity, I measured from its bottom a distance of 40 feet, and then took the angle formed by the plane and a line drawn to the top 41°; and going on in the same direction 60 feet farther, the same angle was 23° 45′, the height of the instrument being five feet: what was the height of the obelisk?

∠ B	41° 00′	subt.
∠ C	23 45	
∠ BDC	17 15	9·4720856
∠ C	23 45	9·6050320
BC	60 —	1·7781513
BD	81·488	1·9110977



BD

BD	81·488			
BE	40·			
Sum	121·488	—	—	2·0845333
Diff.	41·488	—	—	1·6179225
Tang. $\frac{E + EDB}{2}$	69° 30′		—	10·4272623
Tang. $\frac{E - EDB}{2}$	42 24½		—	9·9606516
Diff. = ∠EDB	27 05½			
∠EDB	27° 05½	—	—	9·6582842
∠B	41 00	—	—	9·8169429
BE	40 —	—	—	1·6020600
ED	57·623	—	—	1·7607187
add AE	5			
AD	62·623			

EXAMPLE VII.

Wanting to know the height of an inacceſſible object; at the leaſt diſtance from it, upon the ſame horizontal plane, I took its angle of elevation equal to 58°, and going 100 yards directly farther from it, found the angle there to be only 32°: required its height, and my diſtance from it at the firſt ſtation, the inſtrument being five feet above the ground at each obſervation.

∠C	58°	} ſubtract
∠D	32	}
∠DBC	26°	9·6418420
∠D	32	9·7242097
DC	100	2· - - - -
BC	—	2·0823677

∠A 90°

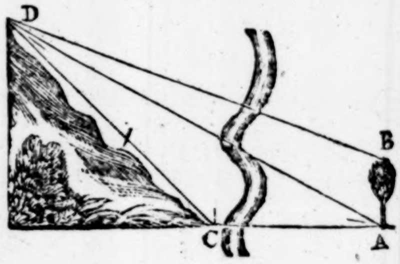
∠A 90°	—	10·........	∠A 90°		10·------
∠C 58	—	9·9284205	∠CBA 32°		9·7242097
BC	— —	2·0823677	BC	— —	2·0823677
AB 102·51 yds	-	2·0107882	AC 64·05 yds	-	1·8065774
1·66 &c yds = 5 feet					
104·17 whole height					

EXAMPLE VIII.

Wanting to know the height of, and my diſtance from, an object on the other ſide of a river, which ſeemed to be upon a level with the place where I ſtood cloſe by the ſide of the river; and not having room to go backwards, on the ſame plane, on account of the immediate riſe of the bank. I placed a mark where I ſtood, and meaſured, in a direct line from the object, up the hill, whoſe aſcent was ſo regular that I might account it for a right line, to the diſtance of 132 yards, where I perceived that I was above the level of the top of the object; I there took the angle of depreſſion of the mark by the river's ſide equal 42°, of the bottom of the object equal 27°, and of its top 19°: required the height of the object and the diſtance of the mark from its bottom.

Here 42° — 27° = 15° = ∠ADC. And 27° — 19° = 8° = ∠ADB. Alſo 90° + 19° = 109° = ∠B.

∠CAD	27°	9·6570468
∠CDA	15	9·4129962
CD	132	2·1205739
CA	75·25	1·8765233

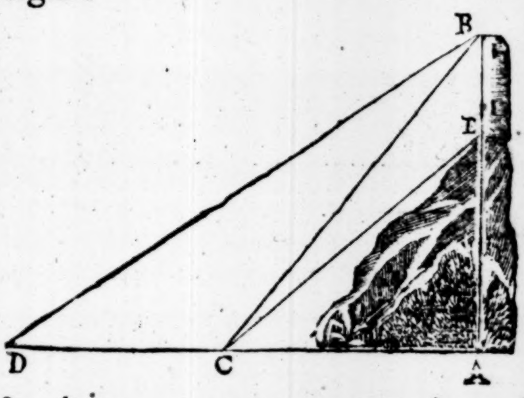


∠AB

∠CAD 27°	-	9·6570468	∠B 109°	-	9·9756701
∠C 138 or 42		9·8255109	∠ADB 8	-	9·1435553
CD 132	-	2·1205739	AD	- -	2·2890380
AD	- -	2·2890380	AB 28·63	-	1·4569232

EXAMPLE IX.

Being upon a horizontal plane, and wanting to know the height of an object on the top of an inaccessible hill; I took the angle of elevation of the top of the hill equal 40°, and of the top of the object equal 51°; going, then, in a direct line from it to the distance of 100 yards farther, I found the angle of the top of the object to be 33° 45′: what is the object's height?

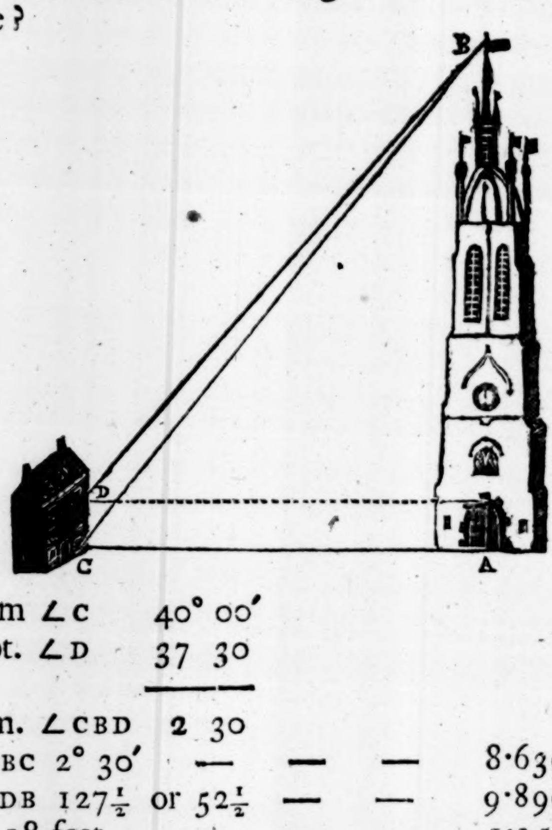


∠ACB	51° 00′	subtract	
∠D	33 45		
∠DBC	17 15	-	9·4720856
∠D	33 45	-	9·7447390
DC	100	-	2·0000000
CB		-	2·2726534

∠BEC	130°	-	9·8842549
∠BCE	11	-	9·2805988
CB	- -	-	2·2726534
BE	46·66574	-	1·6689982

EXAMPLE X.

From a window near the bottom of a houſe, which ſeemed to be upon a level with the bottom of a ſteeple, I took the angle of elevation of the top of the ſteeple equal 40°, and from another window 18 feet directly above the former, the ſame angle was 37° 30′: what then is the height and diſtance of the ſteeple?



From ∠C	40° 00′	
Subt. ∠D	37 30	
Rem. ∠CBD	2 30	
∠DBC 2° 30′	— — —	8·6396796
∠CDB 127½ or 52½	— —	9·8994667
DC 18 feet	— — —	1·2552725
CB	— — — —	2·5150596

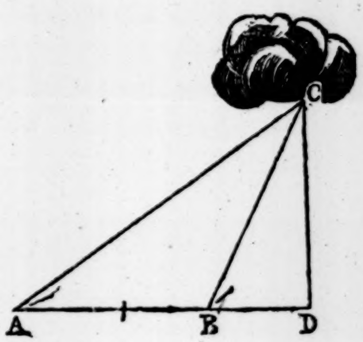
∠A 90° —	10·0000000	∠A 90° —	10·0000000
∠ACB 40 —	9·8080675	∠CBA 50 —	9·8842540
CB — —	2·5150596	CB — —	2·5150596
AB 210·44 f. -	2·3231271	CA 250·79	2·3993136

EX-

EXAMPLE XI.

What is the perpendicular height of a cloud whose angles of elevation are 35° and 64°, taken by two observers, at the same time, both on the same side of the cloud, and at the distance of 880 yards from one other, so placed that a vertical plane would pass through both their stations and the cloud; and what is its distance from the two places of observation?

∠D	90°	90°	∠D
∠B	64	35	∠A
∠BCD	26	55	∠ACD
		26	∠BCD
		29	∠ACB



∠ACB	29° —	9·6855712
∠A	35 —	9·7585913
AB	880 —	2·9444827
BC	1041·125 -	3·0175028

∠ACB	29° —	9·6855712	∠D 90°	—	10·0000000
∠ABC	116 —	9·9536602	∠B 64	—	9·9536602
AB	880 —	2·9444827	BC	—	3·0175028
AC	1631·442	3·2125717	DC 935·757	-	2·9711630

Note. In finding the distances of inaccessible objects, if they be of such a height as to admit of a pretty large angle of elevation, their distance will be found as in some of the foregoing examples.—If not, the theodolite, or some such instrument, is used to take the angles of the distance, or the horizontal angles, of objects, as in the following examples.

EXAMPLE XII.

Suppose I wanted to know the distance of the two places A, B, to whose ends there is free access, but not to the intermediate parts, because of a hill, precipice, or water, between A and B; and that therefore I measured from A and B, to any convenient place C, the distance AC, equal 7·35 chains, and BC equal 8·4 chains, and found the angle ACB equal 55° 40′. What is the distance of the places A, B?

	8·40		
	7·35		
Sum	15·75	—	1·1972806
Diff.	1·05	—	0·0211893
Tang. $\frac{A+B}{2}$	62° 10′	-	10·2773793
Tang. $\frac{A-B}{2}$	7 $11\frac{11}{14}$	-	9·1012880
$\angle$A	69 $21\frac{11}{14}$	sum	

$\angle$A	69° $21\frac{11}{14}$′	—	—	9·9711982
$\angle$C	55 40	—	—	9·9168593
CB	8·4 chains	—	—	0·9242793
AB	7·412 chains	—	—	0·8699404

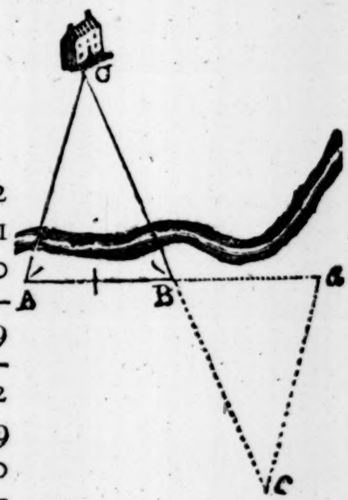
Note. If the lines AC, BC, be produced to *a* and *b*, till C*a*, C*b*, be equal to CA, CB, or equal to CB, CA; then the distance *ba*, will be equal to the distance AB, and therefore AB will be obtained, without any calculation, by only measuring *ba*.

EXAMPLE XIII.

Being on the side of a river, and wanting to know the distance to a house which stood on the other side, I measured 200 yards in a right line by the side of the river, and found that the two angles, at each end of this line, formed by the other end and the

the house, were 73° 15′ and 68° 02′: What was the distance between each station and the house?

∠ A	73° 15′		
∠ B	68 02		
sum	141 17		
from	180 00		
∠ C	38 43	—	9·7962062
∠ A	73 15	—	9·9811711
AB	200	—	2·3010300
BC	306·19	—	2·4859949
∠ C	38° 43′	—	9·7962062
∠ B	68 02	—	9·9672679
AB	200	—	2·3010300
AC	296·54	—	2·4720917



Note. If, in the right line AB*a*, you measure B*a* equal AB, and the line CB*c* be produced until the angle *a* be equal to the angle A; the distances B*c*, *ac*, will be equal to BC, AC.

EXAMPLE XIV.

Wanting to know the breadth of a river, I measured 100 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree close by the other side of the river, to be 53° and 79° 12′. What is the perpendicular breadth?

∠ A	53° 00′		
∠ B	79 12		
sum	132 12		
from	180		
∠ ATB	47 48	—	9·8697037
∠ B	79 12	—	9·9922385
AB	100	—	2·0000000
AT	— —	—	2·1225348

∠ P

∠P	90°	—	10·0000000
∠A	53	—	9·9023486
AT		—	2·1225348
TP	105·89	—	2·0248834

EXAMPLE XV.

Two ſhips of war intending to cannonade a fort, are, by the ſhallowneſs of the water, kept ſo far from it, that they ſuſpect their guns cannot reach it; in order therefore to meaſure the diſtance, they ſeparate from each other half a mile or 880 yards; then each ſhip obſerves the angles which the other and the fort ſubtends, and finds them to be 85° 15′ and 83° 45′: What is the diſtance between each ſhip and the fort?

∠A	83° 45′
∠B	85 15
ſum	169 00
from	180 00

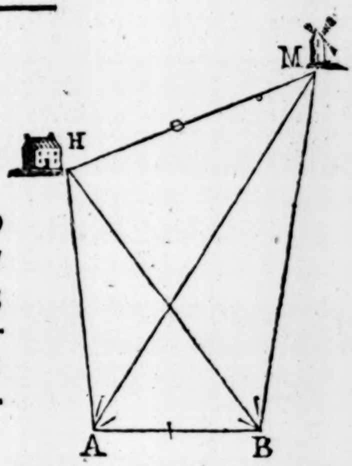
∠F	11 00	—	9·2805988
∠A	83 45	—	9·9974110
AB	880	—	2·9444827
BF	4584·5	—	3·6612949
∠F	11° 00′	—	9·2805988
∠B	85 15	—	9·9985058
AB	880	—	2·9444827
AF	4596·1	—	3·6623897

EXAMPLE XVI.

Wanting to know the diſtance between a houſe and a mill, which were ſeparated from me by a river, I took

I took another station B at the distance of 300 yards from the first station A: now from the first station A, the angle subtended by B and the mill was 58° 20′, and by the mill and the house 37°; from B, the angle subtended by A and the house was 53° 30′, and by the house and the mill 45° 15′. What is the distance of the house and mill?

37° 00′	58° 20′	53° 30′	
58 20	53 30	45 15	
53 30	45 15	98 45	∠ABM
148 50	157 05		
180 00	180 00		
∠AHB 31 10	22 55 AMB		



∠AHB	31 10	-	9·7139349
∠ABH	53 30	-	9·9051787
AB 300	-	-	2·4771213
AH 465·9776		-	2·6683651

∠AMB 22° 55′			—	9·5903869
∠ABM 98 45 or 81 15				9·9949158
AB 300		—	—	2·4771213
AM	761·4655	—	—	2·8816502
AH	465·9776			
Sum	1227·4431	—	—	3·0890013
Dif.	295·4879	—	—	2·4705397
Tan. $\frac{AHM + AMH}{2}$	71° 30′	—		10·4754801
Tan. $\frac{AHM - AMH}{2}$	35 44	—		9·8570185
AMH	35 46			

∠ AMH	35° 46′	—	—	9·7667739
∠ HAM	37 00	—	—	9·7794630
AH	—	—	—	2·6683651
HM	479·7933 yds		—	2·6810542

Note. Pretty much after the manner of these last examples many curious and useful problems may be resolved: for if we can determine our distance from one remote object, we can do the same for any number of objects; or if the distance between two remote objects can be determined, those between any number of objects may be determined likewise: so, we may determine the angles and sides of fields, or of very large tracts of land, and that whether we be within them, or any where without them, from whence the angles can be seen; hence also ships at sea may determine their distances from known visible ports; and plans may be taken of countries, towns, harbours, fleets, fortifications, &c.

EXAMPLE XVII.

Suppose I want to know the breadth of a river, or my distance from an inaccessible object O, and that I have no instrument for taking angles, but only a chain or chord for measuring distances; and suppose that from each of the two stations A, B, which are 500 yards asunder, I measure in a direct line from the object O 100 yards, viz. A*a* and B*b* each equal to 100 yards, and that the diagonal A*b* measures 550 yards, and the diagonal *a*B measures 560: What then is the distance of the object from each station A and B?

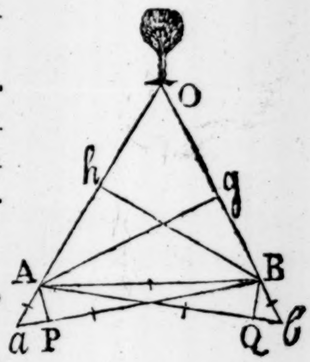
Let

Let fall the perpendiculars AP, BQ. Then, in the triangle ABa,

$$\text{B}a : \text{BA} + \text{A}a :: \text{BA} - \text{A}a : \text{BP} - \text{P}a,$$

that is $560 : 600 :: 400 : 428\frac{4}{7} = \text{BP} - \text{P}a,$

its half		$214\frac{2}{7}$
half sum		280
BP	-	$494\frac{2}{7}$
aP	-	$65\frac{5}{7}$



And

$$\text{BP} = \frac{560 + 428\frac{4}{7}}{2} = \frac{988\frac{4}{7}}{2} = 494\tfrac{2}{7},$$

$$\text{P}a = \frac{560 - 428\frac{4}{7}}{2} = \frac{131\frac{3}{7}}{2} = 65\tfrac{5}{7},$$

Then

$\text{A}a\ 100 : a\text{P}\ 65\frac{5}{7} :: 1 : \text{s.}\ \angle a\text{AP}\ 41^\circ\ 5'$

$\text{AB}\ 500 : \text{BP}\ 494\frac{2}{7} :: 1 : \text{s.}\ \angle\text{BAP}\ 81\ \ 20$

the sum	122	25
taken from	180	00
leaves ∠BAO	57	35

Again, in the triangle ABb,

$\text{A}b\ 550 : \text{AB} : \text{B}b\ 600 :: \text{AB} - \text{B}b\ 400 : \text{AQ} - \text{QB}\ 436\frac{4}{11}$

the half dif.		$218\frac{2}{11}$
half sum		275
AQ	-	$493\frac{2}{11}$
Qb	-	$56\frac{9}{11}$

Then

Then

$$\text{B}b\ 100 : b\text{Q}\ 56\tfrac{9}{11} :: 1 : \text{s.}\ \angle b\text{BQ}\ 34^\circ\ 37'$$
$$\text{BA}\ 500 : \text{AQ}\ 493\tfrac{2}{11} :: 1 : \text{s.}\ \angle \text{ABQ}\ 80\quad 32$$

the sum	115	9
taken from	180	0
leaves $\angle$ ABO	64	51
add BAO	57	35
the sum	122	26
taken from	180	00
*leaves $\angle$ O	57	34

Whence

$\angle$ O 57° 34′	9·9263507	$\angle$ O 57° 34′	9·9263507
$\angle$ A 57 35	9·9264310	$\angle$ B 64 51	9·9567437
AB 500	2·6989700	AB 500	2·6989700
BO 500·1	2·6990503	AO 536·25	2·7293630

EX-

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* Or *the angles* ABO *and* BAO *may be otherwise found thus:*

Draw the perpendiculars Ag, Bh: Then by Eucl. II. 12,

$$\text{B}g = \frac{\text{A}b^2 - \text{AB}^2 - b\text{B}^2}{2b\text{B}} = 212\tfrac{1}{2};$$

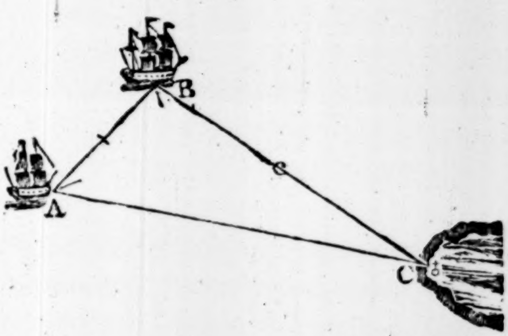
And AB : Bg :: 1 : ·425 = cosine of 64° 51′ the $\angle$ABO.

In like manner $\text{A}h = \dfrac{\text{B}a^2 - \text{AB}^2 - \text{A}a^2}{2a\text{A}} = 268;$

And AB : Ah :: 1 : ·536 the cosine of 57° 35′ the $\angle$BAO; both the same as above.

EXAMPLE XVIII.

From a ſhip at ſea I obſerved a point of land to bear *E.* by *S.* and after ſailing *N. E.* 12 miles, I ſet it again, and found its bearing to be *S. E.* by *E.* How far was the laſt obſervation made from the point of land?



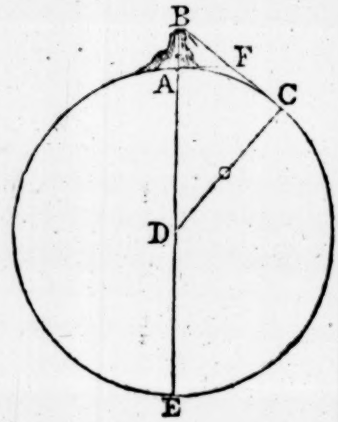
Here are given the angle A, at the place of the firſt obſervation, 5 points or 56° 15′; the angle B, 9 points or 101° 15′; and the angle C, 2 points or 22° 30′; alſo the ſide AB 12 miles: to find the ſide BC.

As s. ∠C 22° 30′	—	9·5828397
To s. ∠A 56 15	—	9·9198464
So is AB 12 miles	—	1·0791812
To BC 26·07281 miles		1·4161879

EXAMPLE XIX.

If the height AB of the mountain called the pike of Teneriff be 4 miles, and the angle ABC made by a plumb line and a line BC conceived to touch the earth in the fartheſt viſible point C, be 87° 25′ 55″; required the diameter AE of the earth, and the utmoſt diſtance BC that can be ſeen from the top of the mountain, ſuppoſing the earth to be a perfect ſphere.

Draw



Draw AF perpendicular to AB, then from the principles of geometry it is known that CF is = FA; draw, alſo, the radius CD, which will be perpendicular to CB.

Hence from	90°	0′	0″
take ∠B	87	25	55
∠D or AFB	2	34	5

Then as radius

1 : AB 4 :: 2·22960 tang. ∠B : 89·184 AF or FC
1 : AB 4 :: 2·23185 ſec. ∠B : 89·274 BF

their ſum 178·458 BC

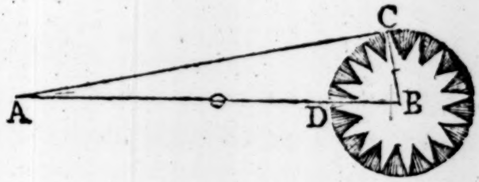
Laſtly, As radius	— —	—	10·0000000
To t. ∠B	— —	—	11·3482280
So BC	178·458	—	2·2515360
To CD	3978·909	—	3·5997640
	2		
	7957·818 diameter of the earth.		

Note. This method of determining the magnitude of the earth, is very eaſy and ſimple; but to do it with any tolerable degree of exactneſs, the height of the mountain muſt be very accurately aſcertained, and the angle at the top muſt be taken with a quadrant divided into minutes and ſeconds, and furniſhed with a teleſcope, inſtead of the common ſights.

EXAMPLE XX.

According to Sir Iſaac Newton, the diameter of the ſun, at a mean diſtance from the earth, ſubtends an

an angle of 32′ 15″: Then how many times his diameter in length is his mean distance from the earth equal to?



Here, in the triangle ABC, we have given CB = $\frac{1}{2}$ a diameter, the angle C = 90°, and the angle A = 16′ 7$\frac{1}{2}$″.

Hence, As s. ∠A	16′ 7$\frac{1}{2}$″	—	7·6711356
To s. ∠C	90°	—	10·0000000
So is CB	1 semi-diam	—	0·0000000
To BA	213·2379	—	2·3288644

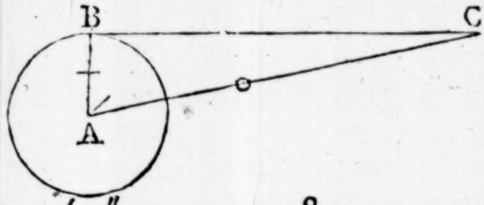
That is, the mean distance of the sun's center is 213·2379 semi-diameters, or 106·6189 of his whole diameters. And if from AB be taken BD or $\frac{1}{2}$, we shall have remaining nearly 106 diameters for the distance of the surface.

Note. After the same manner may be calculated the proportion of the diameter to the distance of any other celestial body, whose apparent diameter is large enough to be measured. And, in particular, the mean distance of the moon, whose mean apparent diameter is 31′ 16″, will be found to be 109·95 times her diameter.

EXAMPLE XXI.

If when the moon appears in the horizon to a spectator on the earth, at her mean distance from it, her zenith distance, as calculated from astronomical tables, be 89° 2′ 55″; it is required to find how many of the earth's semi-diameters the said mean distance of the moon is equal to.

Here



Here AB is = one ſemi-diameter of the earth, and the $\angle$A = 89° 2′ 55″.

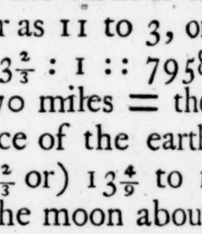
Hence, As s. $\angle$C 57′ 5″	—	8·2202155
To s. $\angle$B 90°	—	10·0000000
So is AB 1 ſemi-diam.		0·0000000
To AC 60·226	—	1·7797845

That is, the diſtance of their centers is 60·226 of the earth's ſemi-diameters, or $30\frac{1}{9}$ whole diameters.

Note. Hence, and from the note to the laſt example, it appears that the diameter of the earth is to that of the moon as 109·95 to $30\frac{1}{9}$, or as 11 to 3, or as $3\frac{2}{3}$ to 1 nearly.—Conſequently, as $3\frac{2}{3} : 1 :: 7958$ (the earth's diameter in miles) : 2170 miles = the moon's diameter.—Likewiſe the ſurface of the earth is to that of the moon as ($3\frac{2}{3} \times 3\frac{2}{3}$ or) $13\frac{4}{9}$ to 1 nearly; or, the earth reflects upon the moon about $13\frac{1}{2}$ times as much light as the moon does upon the earth.—Moreover, the bulk of the earth is to that of the moon as ($3\frac{2}{3} \times 3\frac{2}{3} \times 3\frac{2}{3}$ or) 48 to 1 nearly.

GENERAL SCHOLIUM.

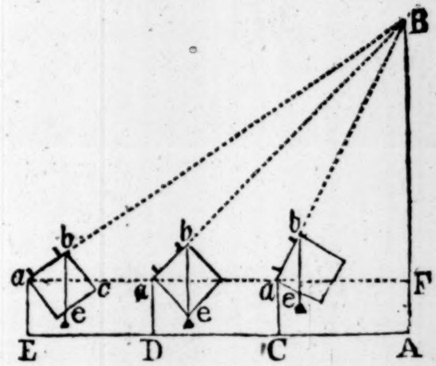
There are many other methods and inſtruments uſed to find the altitudes and diſtances of objects, ſome of which are here ſubjoined.



1. One very eaſy method is by a ſquare with a plummet AE ſuſpended from one corner A, and the two ſides BC, DC, meeting in the oppoſite angle C, divided into 10, or 100, or 1000 equal parts; and two ſights on the ſide AB.

It

It is evident that, in taking any altitude AB with the square, the plummet will always cut off from the square a triangle similar to that formed by the base line *a*F, the perpendicular FB, and B*a*.



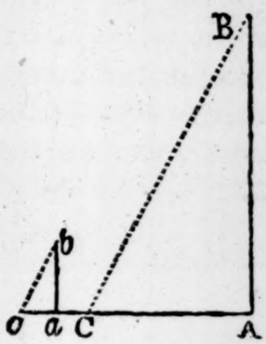
If the angle B*a*F be equal to 45°. Then the plumb line will pass through the opposite angle of the square, and the distance DA will be equal to the altitude BF: So if DA be 60, then BF will be 60 also.

If the angle be greater than 45°, as at the station C. Then the part of the side cut off *ae*, will be to the whole side *ab*, as *a*F, to FB: So if *ae* be 6 divisions of which *ab* is 10, and the distance CA be 36 feet; then 6 : 10 :: 36 : 60 feet = the altitude BF.

If the angle be less than 45°, as at the station E. Then the part is cut off the other decimated side, and *bc* : *ce* :: EA : BF: So if the parts cut off be 6, and the distance 100 feet, then 10 : 6 :: 100 : 60 feet = BF.

2. Another method is by shadows, from the property of similar triangles also. For any object, and a pole set up parallel to it, are in proportion to each other as the length of their shadows, formed by the sun, &c.

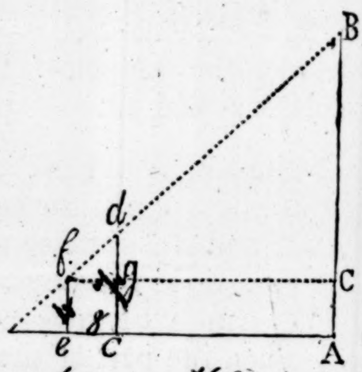
Let the height of the pole *ab* be 6 feet, the length of its shadow *ac* 4, and the shadow AC of the altitude 40 feet; then *ca* = 4 : *ab* = 6 :: CA = 40 : AB = 60.



 3. Another

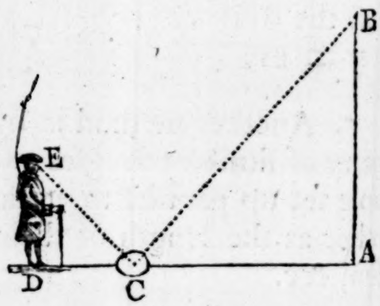
3. Another method is by two poles ſet up parallel to the object, the one longer than the other, ſo that the obſerver may ſee the top of the object over the tops of both the poles.

Let the pole ef be $= 4$ feet, $cd = 7$ feet, their diſtance aſunder $ec = fg = 8$ feet, and the diſtance eA of the ſhorter pole from the object $= 160$ feet.— Then, the triangles fgd, fCB being ſimilar, $fg : gd :: f\mathrm{C} : \mathrm{CB}$, that is $8 : 3 \; (= 7 - 4) :: 160 : 60$ feet $= \mathrm{BC}$; hence $\mathrm{BC} + \mathrm{CA} = \mathrm{BC} + fe = 60 + 4 = 64 = \mathrm{AB}$.

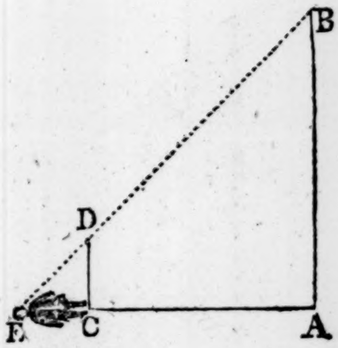


4. A fourth method is by viewing the image of the top of the object reflected from a ſmooth ſurface, as a mirror placed horizontally, or a veſſel of water.

Let C be the reflecting ſurface, at the diſtance of 84 feet from the bottom of the object AB; and let a perſon at D, 7 feet from C, with his eye 5 feet above the ground, view the image of the object at C; then becauſe the triangles CDE, CAB are ſimilar, agreeably to a principle of the opticians, we ſhall have $\mathrm{CD} : \mathrm{DE} :: \mathrm{CA} : \mathrm{AB}$, that is $7 : 5 :: 84 : 60$ feet $= \mathrm{AB}$.

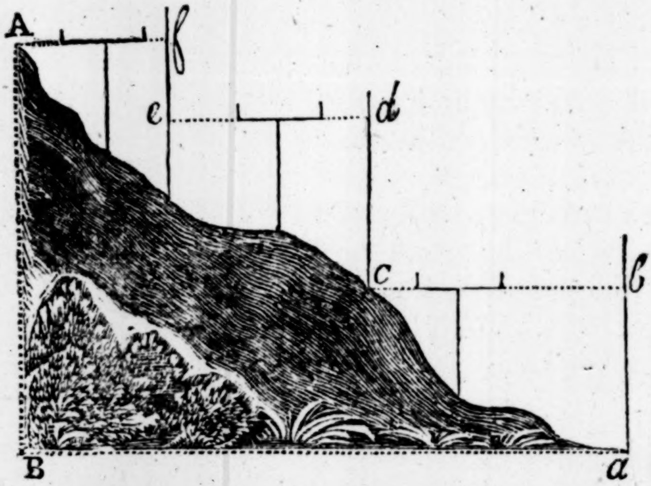


5. A



5. A fifth method is for the observer to fix a pole CD, equal, in length, to the height of his eye, perpendicularly at C, by trials, at such a distance from A, that having laid himself upon his back, with his feet against the bottom of the pole, he may see the tops D and B of the pole and object in the same line: for then EA will be equal to AB, because EC is equal to CD.—Or the pole may be of any length; for the distance from the foot to the eye of the observer will be in proportion to the height of the pole, as the whole distance is to the height of the object required.

6. When the perpendicular altitude of an irregular hill or ascent is to be ascertained; or when, in levelling, for conducting water, &c. it is required to find how much one assigned place is above another; a level and perpendicular poles, or objects, commonly are used.



So the level being fixed horizontally, and the places *b*, *c*, *d*, &c seen upon the poles through the sights of the level, being marked; the heights *ab*, *cd*, *ef*, added all together, will give the whole height AB of A above *a*; and the distances *bc*, *de*, *f*A, added together, will give the horizontal distance *a*B of A from *a*.

Another method of levelling is sometimes used, viz. The angles of elevation or depression are taken, from station to station, with an arc of a circle, having a plummet pendent from its center, and the several distances, being measured upon the ground, are the hypotenuses of the right-angled triangles, of which the angles at the bases are expressed by the aforesaid observed angles; and consequently, the perpendiculars of these triangles being calculated, their sum will be the difference of level between the extreme places, if the angles were all of elevation, or all of depression; but if the angles be some of elevation, and some of depression, the perpendiculars which belong to the angles of the same kind being added together, will give two sums, whose difference will be the difference of level required.

These

These two methods, when accurately performed, are both very just, at least when the stations are at no great distance from each other; yet the former is more to be depended on than the latter, in as much as the observations required by it, are less susceptible of error; for besides the inaccuracy of the measured hypotenuses, on account of the unevenness of the ground, the angles themselves can hardly be taken, with the generality of instruments for this purpose, to less than quarters of degrees.

But when the places, of which we would know whether of them is the higher, are far distant from each other, instruments of greater accuracy are to be used, such as very exact levels with telescopes; but in such cases an allowance must be made for the rotundity of the earth; for the true water-level course is determined by a line of which every part is at the same distance from the center of the earth, and which is, therefore, an arc of a great circle of the earth, considering it not as an oblate spheroid, but as a sphere, its difference from that form having, in this case, no sensible effect; and the allowance necessary to be made in the level, is at the rate of 8 inches nearly to a mile measured upon the earth; for the visual line, when the level is properly fixed, being a tangent to the earth, or at least parallel to one, that line at the distance of one mile from the station is 7·96 inches above the water-level line, or so much farther from the center of the earth than the instrument is. And for other distances, above or below a mile, the allowance for the level varies in proportion as the square of the distance.

It may be necessary farther to observe, that in taking, at several stations, the difference of level between two places, it is not necessary to go in a line from the one to the other, but every two successive stations may be taken in any direction that may seem most convenient.

PRACTICAL QUESTIONS IN TRIGONOMETRY, &c.

QUESTION I. A may-pole, 50 feet 11 inches high, at a certain time will caſt a ſhadow 98 feet 6 inches long; what then is the breadth of a river, which, running within 20 feet 6 inches of the foot of a ſteeple, 300 feet 8 inches high, will, at the ſame time, throw the extremity of its ſhadow 30 feet 9 inches beyond the ſtream? Anſ. 530 feet 5 inches.

QUEST. 2. Required the length of a ſhoar, which being to ſtrut 11 feet from the upright of a building, will ſupport a jamb 23 feet 10 inches from the ground. Anſ. 26 ft 3 inc.

QUEST. 3. A line 27 yards long will exactly reach from the top of a fort, to the oppoſite bank of a river, known to be 23 yards broad: what is the height of the wall? Anſ. 42 ft 5 inc.

QUEST. 4. Two ſhips ſet ſail from the ſame port, one of them goes 50 leagues due eaſt, and the other 84 leagues due north: how far are they then aſunder? Anſ. $97\frac{3}{4}$ leagues.

QUEST. 5. The height of an elm, growing in the center of a circular iſland, 30 feet in diameter, plumbs 53 feet; and a line of 112 feet long ſtretched from the top of the tree ſtraight to the nearer edge of the water: required the breadth of the moat, ſuppoſing the land on either ſide of the water, to be level. Anſ. $83\frac{2}{3}$ feet.

QUEST. 6. Suppoſe a light-houſe, built on the top of a rock, the diſtance between the place of obſervation, and that part of the rock level with the eye, and directly under the building, is given 310 fathoms; the diſtance from the top of the rock, to the place of obſervation, is 423 fathoms; and from the top of the building 425: required the height of the edifice. Anſ. 17 ft 7 inc.

QUEST.

QUEST. 7. A ladder, 40 feet long, may be so planted, that it shall reach a window 33 feet from the ground, on one side of the street; and by turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high, on the other side: required the breadth of the street.

Ans. 56·649 feet.

QUEST. 8. There are two columns left standing upright in the ruins of Persepolis; the one is 64 feet above the plane, and the other 50: In a right line between these stands an antient statue, the head of which is 97 feet from the summit of the higher, and 86 from that of the lower column; the base of which measures just 76 feet to the center of the figure's base: required the distance between the tops of the two columns. Ans. 157 feet.

QUEST. 9. A may-pole, whose top was broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole: what was the height of the whole may-pole, supposing the length of the broken piece to be 39 feet? Ans. 75 feet.

QUEST. 10. Suppose the breadth of a well at the top be 6 feet, and the angle formed by its side and a visual diagonal line from the edge at top to the opposite side at the bottom, 18° 30′: required the depth of the well. Ans. 17·89 feet.

QUEST. 11. At 85 feet distance from the bottom of a tower, the angle of its elevation was found to be 52° 30′: required the altitude of the tower.

Ans. $110\frac{1}{2}$ feet.

QUEST. 12. At a certain place the angle of elevation of an inaccessible tower was 26° 30′; then measuring 75 in a direct line towards it, the angle was then found to be 51° 30′: required the height of the tower, and its distance from the last station.

Ans. $\begin{cases} \text{height } 62, \\ \text{distance } 49. \end{cases}$

QUEST. 13. To find the diſtance of an inacceſſible caſtle gate, I meaſured a line of 73 yards, and at each end of it took the angle of poſition of the object and the other end, and found the one to be 90°, and the other 61° 45′: required the diſtance of the caſtle from each ſtation.

Anſ. { 135·8, 154·2.

QUEST. 14. From the top of a tower by the ſea ſide of 143 feet high, I obſerved that the angle of depreſſion of a ſhip's bottom, then at anchor, was 55°; what was its diſtance from the bottom of the wall?

Anſ. 204·56 feet.

QUEST. 15. How far at ſea can the pike of Teneriff be ſeen, its height being 4 miles, and the radius of the earth $3978\frac{7}{8}$ miles? Anſ. $178\frac{1}{2}$ miles.

QUEST. 16. If a ſhip, in the latitude of 50° north, ſail 52 miles in the direction S W by S: what latitude is ſhe arrived in, and how much farther to the weſt?

Anſ. { lat. 49° 16·8′, weſt 28·9 miles.

QUEST. 17. Sailing W S W, I ſaw, at ſome diſtance, a point of land, which I ſet, and found its bearing W by N; and after ſailing 6 leagues farther, I ſet it again, and found, its bearing N W by W. Required its diſtance. Anſ. 26·13 miles.

QUEST. 18. Obſerving three ſteeples, A, B, C, in a town at a diſtance, whoſe diſtances aſunder are known to be as follows, namely, AB $106\frac{1}{2}$, AC 202, and BC 131 fathoms, I took their angles of poſition from the place where I ſtood D, which was neareſt the ſteeple B, and found the angle ADB 13° 30′, and the angle CDB 29° 50′. Required my diſtance from each of the three ſteeples.

Anſ. { DA 302·8, DB 214·8, DC 262·0

QUEST. 19. Suppoſing my ſtation to be fartheſt from the ſteeple B, required to find the diſtances from it, when the diſtance AB is 9 furlongs, AC 12,

and BC 6 furlongs; also the angle ADB 33° 45′, and the angle CDB 22° 30′.

Ans. $\begin{cases} \text{DA} & 10\cdot64 \\ \text{DB} & 15\cdot64 \\ \text{DC} & 14\cdot01 \end{cases}$

QUEST. 20. Two ships sail from the same port; the one sails E N E 85 miles, the other sails E by S till the first ship bears N W by W: what is the distance of the second ship from the port, and also from the first ship?

Ans. $\begin{cases} \text{from the port} & 184\cdot7 \\ \text{from the 1st ship} & 123\cdot4 \end{cases}$

QUEST. 21. Two ports lie east and west of each other: a ship sails from each, namely, the ship from the west port sails N E 89 leagues, and the other sails 80 leagues, when she meets the former: required the latter ship's course, and the distance between the two ports.

Ans. $\begin{cases} \text{course} & 51° 52′ \\ \text{distance} & 112\cdot3 \end{cases}$

QUEST. 22. Two ships sail from a certain port; the one sails S by E 45 leagues, and the other S S W 64 leagues. What then are their bearings and distance asunder?

Ans. $\begin{cases} \text{bearing} & 43° 14′ \\ \text{distance} & 36\cdot5 \end{cases}$

QUEST. 23. A ship sailing N W, two islands appear in sight, of which the one bore N, and the other W N W; but after sailing 20 leagues, the former bore N E, and the latter W by S. What is the distance asunder of the two islands?

Ans. 32·38 leagues.

PART

PART II.

OF SUPERFICIAL MENSURATION, OR THE MENSURATION OF PLANE FIGURES.

SECTION I.

OF THE AREAS OF RIGHT-LINED AND CIRCULAR FIGURES.

THE meaſure of a plane figure is called its area. By the menſuration of plane figures is determined the extenſion of bodies as to length and breadth; ſuch as the quantities of lands, and the works of many artificers.

Plane figures, and the ſurfaces of bodies, are meaſured by ſquares; as ſquare inches, or ſquare feet, or ſquare yards, &c; that is, ſquares whoſe ſides are inches, or feet, or yards, &c. Our leaſt ſuperficial meaſure is the ſquare inch, other ſquares being taken from it according to the proportion in the following table.

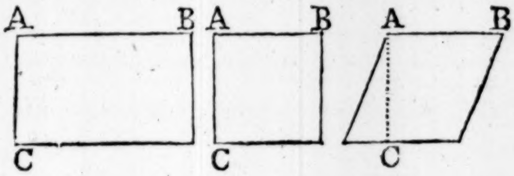
Table of Square Meaſure.

Square Inches	Sq. Feet	S. Yards	S. Poles	S. Cha.	Acres	S. Mile
144	1					
1296	9	1				
39204	$272\frac{1}{4}$	$30\frac{1}{4}$	1			
627264	4356	484	16	1		
6272640	43560	4840	160	10	1	
4014489600	27878400	3097600	102400	6400	640	1

PRO-

PROBLEM I.

To find the Area of a Parallelogram, whether it be a Square, a Rectangle, a Rhombus, or a Rhomboid.



RULE I.

* Multiply the length by the height or perpendicular breadth, and the product will be the area.

That is, AB × AC = the area.

Note. Because the length of a square is equal to its height, its area will be found by multiplying the side by itself.—That is AB × AB or AB^2 is the area of the square.

EX-

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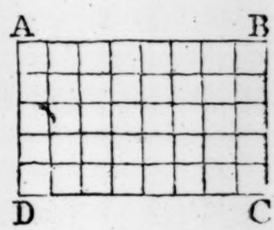
* DEMONSTRATION.



For, let ABCD be a rectangle; and let its length AB and CD, and its breadth AD and BC, be each divided into as many equal parts, as is expressed by the number of times they contain the lineal measuring unit; and let all the opposite points of division be connected by right lines.—Then, it is evident that, these lines divide the rectangle into a number of squares, each equal to the superficial measuring unit; and that the number of these squares is equal to the number of lineal measuring units in the length, as often repeated as there are lineal measuring units in the breadth, or height; that is, equal to the length drawn into the breadth. But the area is equal to the number of squares or superficial measuring units; and therefore the area of a rectangle is equal to the product of its length and breadth.

Again, a rectangle is equal to an oblique parallelogram of an equal length and perpendicular height, by Euclid I. 36.

Therefore the area of every parallelogram is equal to the product of its length and height. *Q. E. D.*

EXAMPLES.

1. What is the area of a parallelogram whose length is 12·25 chains, and its height 8·5 chains?

$$\begin{array}{rl} 12{\cdot}25 & \text{length} \\ 8{\cdot}5 & \text{breadth} \\ \hline 6125 & \\ 9800 & \\ \hline 10)\,104{\cdot}125 & \text{square chains} \\ \hline 10{\cdot}4125 & \text{acres} \\ 4 & \\ \hline 1{\cdot}6500 & \text{roods} \\ 40 & \\ \hline 26{\cdot}0000 & \text{perches} \\ \hline \end{array}$$

	a	r	p
Ans.	10	1	26

Note. Four roods are equal to an acre, and therefore 40 perches, or square poles, make a rood.

Ex. 2. What is the area of a square whose side is 35·25 chains? Ans. 124ac. 1r. 1p.

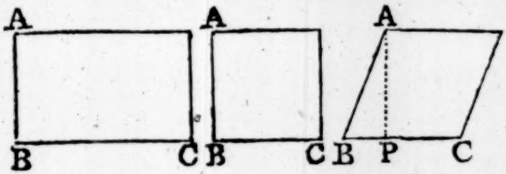
Ex. 3. What is the area of a rectangular board, whose length is 12·5 feet, and breadth 9 inches? Ans. 9·375 sq. feet.

Ex. 4. How many square yards of painting are in a rhombus, or a rhomboid, whose length is 37 feet, and perpendicular breadth 5·25 feet? Ans. $21{\cdot}58\frac{1}{3}$.

RULE II.

As radius (viz. sine of 90° or tang. of 45°) :
Is to the sine of any angle of a parallelogram ::
So is the product of the sides including the angle :
To the area of the parallelogram.

That



* That is, AB × BC × nat. sine of the angle B = the area.

Note. Because the angles of a square and rectangle are each 90°, whose sine is 1, this rule, for them, is the same as the former.

EXAMPLES.

1. What is the area of a rhomboides whose length is 36 feet, slope height 25·5 feet, and one of the less angles 58°?

(Rad.) 1 : ·8480481 (nat. sine of 58°) :: 918 (= 25·5 × 36) : 778·5081558 the area.

Or, to use the Logarithms.

Radius	—	—	—	10·0000000
Sine of 58°		—	—	9·9284205
918	—	—	—	2·9628427
778·5081		—	—	2·8912632

Ex. 2. What is the area of a parallelogram whose angle is 90°, and the including sides 20 and 12·25 chains? Ans. 245 acres.

Ex. 3. What is the area of a rhombus, each of whose sides is 21 feet 3 inches, and each of the less angles 53° 20′? Ans. 362·208757 feet.

Ex. 4. How many acres are in a rhomboides whose less angle is 30°, and the including sides 25·35 and 10·4 chains? Ans. 13 ac. 29·12 per.

RULE

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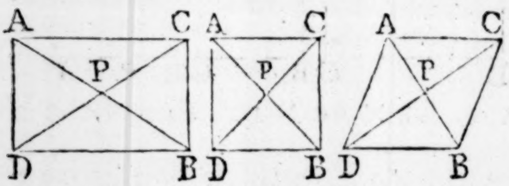
* DEMONSTRATION.

For, having drawn the perpendicular AP, the area, by the first rule, is AP × BC; but as rad. 1 (s. ∠P) : s. ∠B :: AB : AP = s. ∠B × AB; therefore AP × BC = BC × s. ∠B × AB, is the area; or 1 : s. ∠B :: AB × BC : s. ∠B × AB × BC = the area of the parallelogram. *Q.E.D.*

RULE III.



* As radius :
To the sine of the angle which the diagonals of a parallelogram make with each other ::
So is the product of the diagonals :
To double the area.

That is, $\frac{AB \times CD \times \text{nat. s.} \angle P}{2}$ = the area.

Note. Because the diagonals of a square and rhombus intersect at a right angle, whose sine is 1, therefore half the product of their diagonals is the area.

That is, $\frac{1}{2}AB^2$ in the square, and $\frac{1}{2}AB \times CD$ in the rhombus, is the area.

EXAMPLES.

1. How many square yards of pavement are in a square whose diagonal is 27 feet 6 inches?

```
   27·5
   27·5
   ----
   1375
  1925
  550
  ------
2)756·25
  ------
9)378·125 feet
  -------
   42·013 8/9 yards
```

Ex.

---

* This rule is common to all quadrilaterals, and is proved at case 2 of prob. 3.

Ex. 2. How many acres are in a piece of land, in the form of a rhombus, whose diagonals are 30 and 20 chains? Ans. 30 acres.

Ex. 3. How many yards of painting are in a rectangle whose diagonals, intersecting in an angle of 30°, are each 32 feet? Ans. $28\frac{4}{9}$.

Ex. 4. What is the area of a rhomboides whose diagonals, making an angle of 60°, are 30 and 25 feet? Ans. 324·7595 feet.

PROBLEM II.

To find the Area of a Triangle.

* RULE I.

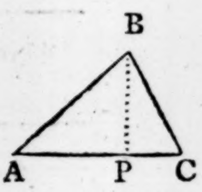
Multiply one of its sides by the perpendicular let fall upon it from its opposite angle, and half the product will be the area.

That is, $\frac{AC \times BP}{2}$ = the area.

EXAMPLES.

1. What is the area of a triangle whose base AC is 40, and the perpendicular BP is 14·52368 chains.

$$\begin{array}{r} 14{\cdot}52368 \\ 20 \\ \hline 29{\cdot}047360 \\ 4 \\ \hline 0{\cdot}189440 \\ 40 \\ \hline 7{\cdot}577600 \\ \hline \end{array}$$



Ans. 29 ac. 0 r 7 p.

Ex.

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* DEMONSTRATION.

This follows from rule 1 prob. 1, because a triangle is half a parallelogram of the same base and height.

Ex. 2. How many ſquare feet are in a right-angled triangle whoſe baſe is 40 and perpendicular 30 feet? Anſ. 600.

Ex. 3. How many ſquare yards are in a triangle whoſe baſe is 49 feet, and perpendicular $25\frac{1}{4}$ feet. Anſ. $68{\cdot}736\frac{1}{8}$.

* RULE II.

As radius :
To the ſine of any angle of a triangle ::
So is the product of the ſides including the angle :
To double the area of the triangle.

That is, $\frac{\text{AB} \times \text{AC} \times \text{nat. s. of} \angle \text{A}}{2} =$ the area.

EXAMPLES.

1. What is the area of a triangle whoſe two ſides are AB 30, and AC 40, and the included angle A 28° 57′ 18″?

·4841226	ſin. ∠A
30	AB
14·5236780	
20	$\frac{1}{2}$AC
290·4735600	anſ.

RULE

---

* DEMONSTRATION.

This follows from rule 2 prob. 1, becauſe a triangle is half a parallelogram of the ſame baſe and height.

Ex. 2. How many ſquare yards are in a triangle of which one angle is 45°, and its including ſides 25 and $21\frac{1}{4}$ feet? Anſ. 20·86947.

Ex. 3. How many acres are in a right-angled triangle whoſe baſe is 40 and perpendicular 30 chains? Anſ. 60.

* RULE III.

From half the ſum of the three ſides ſubtract each ſide ſeverally; multiply the half ſum and the three

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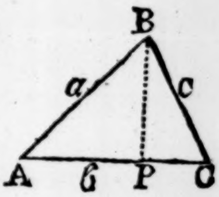
* DEMONSTRATION.



For, $b : a + c :: a - c : \frac{aa - cc}{b} = \text{AP} - \text{PC}$;

therefore $\frac{1}{2}b + \frac{aa - cc}{2b} = \frac{bb + aa - cc}{2b} = \text{AP}$:

hence $\sqrt{aa - (\frac{bb + aa - cc}{2b})^2} =$

$\sqrt{\frac{2a^2b^2 - b^4 + 2b^2c^2 - a^4 + 2a^2c^2 - c^4}{4bb}} = \text{BP}$: then $\text{BP} \times \frac{1}{2}\text{AC} =$

$\text{BP} \times \frac{1}{2}b = \sqrt{\frac{2a^2b^2 - b^4 + 2b^2c^2 - a^4 + 2a^2c^2 - c^4}{16}} =$

$\dagger\sqrt{\frac{-aa + bb + cc + 2bc}{4} \times \frac{aa - bb - cc + 2bc}{4}} =$

$\sqrt{\frac{a+b+c}{2} \times \frac{-a+b+c}{2} \times \frac{a-b+c}{2} \times \frac{a+b-c}{2}} =$

$\sqrt{\frac{a+b+c}{2} \times \frac{a+b+c}{2} - a \times \frac{a+b+c}{2} - b \times \frac{a+b+c}{2} - c} =$ the area. *Q. E. D.*

remainders continually together, and the ſquare root of the laſt product will be the area of the triangle.

That is,

$$\sqrt{\frac{a+b+c}{2} \times \overline{\frac{a+b+c}{2} - a} \times \overline{\frac{a+b+c}{2} - b} \times \overline{\frac{a+b+c}{2} - c}} \text{ or}$$

$$\sqrt{\frac{a+b+c}{2} \times \frac{-a+b+c}{2} \times \frac{a-b+c}{2} \times \frac{a+b-c}{2}} = \text{the area.}$$

Or, if s be half the ſum of the ſides, then

$$\sqrt{s \times \overline{s-a} \times \overline{s-b} \times \overline{s-c}} \text{ is the area.}$$

E X-

Cor. 1. The expreſſion † is alſo $= \frac{1}{4}\sqrt{(\overline{b+c}^2 - a^2) \times (a^2 - \overline{b \backsim c}^2)} =$ (putting $s = b + c$ the ſum, and $d = b \backsim c$ the difference of b and c) $\frac{1}{4}\sqrt{\overline{ss - aa} \times \overline{aa - dd}}$; which is another rule, very uſeful on many occaſions.

Cor. 2. If all the ſides be equal, the rule will become $\sqrt{\frac{3}{2}a \times \frac{1}{2}a \times \frac{1}{2}a \times \frac{1}{2}a} = \frac{1}{4}aa\sqrt{3}$, for the equilateral triangle whoſe ſide is a.

Cor. 3. If the triangle be right-angled, a being the hypotenuſe, the rule will become $\frac{a+b+c}{2} \times \frac{-a+b+c}{2}$ or $\frac{1}{2}p \times \overline{\frac{1}{2}p - a}$, putting p for the perimeter. For the two quantities $\frac{a+b+c}{2} \times \frac{-a+b+c}{2}$ and $\frac{a-b+c}{2} \times \frac{a+b-c}{2}$ or $\frac{-aa+bb+cc+2bc}{4}$ and $\frac{aa-bb-cc+2bc}{4}$ are then = one another, being each $= \frac{1}{2}bc$, becauſe aa is $= bb + cc$.

EXAMPLES.

1. What is the area of a triangle whose three sides are 20, 30, 40 chains?

40	45	45
30	30	20
20	—	—
—	15 sec. dif.	25 third dif.
2)90	—	15 second dif.
—		——
45 half sum		375
40		5 first dif.
—		——
5 first dif.		1875
—		45 half sum
		——
		9375
		7500
		———

84375 (290·4737 sq. chains
4   or 29·04737 acres

49	443	4
9	441	———
		·18948
5804	27500	40
4	23216	———
		7·5792
5808	4284	
	4066	
	——	
	218	
	174	
	——	
	44	

Ans. 29 ac. 7 per.

Ex. 2. How many square yards of plastering are in a triangle whose sides are 30, 40, 50 feet?

Ans. $66\frac{2}{3}$.

Ex. 3. How many acres are in a triangle whose sides are 49, 50·25, 25·69 chains?

Ans. 61 ac. 1 r. 39·68 p.

PROBLEM III.

To find the Area of a Trapezium.

GENERAL RULE.

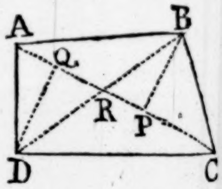
Divide it into triangles according to the manner which you judge most convenient; then the sum of the areas of the triangles, calculated by the last problem, will be the area of the trapezium.

Observe, also, the following particular rules.

* RULE I.

If the trapezium be divided into two triangles by a diagonal joining two opposite angles, and perpendiculars be drawn to it from the other angles: then the sum of the perpendiculars being multiplied by the diagonal, half the product will be the area.

That is, $\frac{BP + DQ}{2} \times AC =$ the area.



EXAMPLE.

If the diagonal AC be 20 feet, and the perpendicular BP equal 4·2 feet, and DQ equal 3·8 feet.

Then $\frac{4{\cdot}2 + 3{\cdot}8}{2} \times 20 = 8 \times 10 = 80$ square feet, the area.

RULE

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* DEMONSTRATION.

For the trapezium $= \triangle ABC + \triangle ADC = \frac{AC \times BP}{2} + \frac{AC \times DQ}{2} = \frac{BP + DQ}{2} \times AC$. *Q. E. D.*

* RULE II.

If there be drawn two diagonals cutting each other: the product of the diagonals multiplied by the natural sine of the angle of intersection of the diagonals, will be double the area. And this rule, as well as the former, is common to all quadrilaterals.

That is, $\frac{AC \times BD \times \text{nat. s.} \angle R}{2}$ = the area.

Or, as radius : s. $\angle R$: : $\frac{1}{2} AC \times BD$: the area.

EXAMPLE.

Suppose the two diagonals be 40 and 30 chains, and that at their intersection one of the less angles is 48°.

Then, since the nat. sine of 48° is ·7431448, the area $= \frac{40 \times 30 \times \cdot 7431448}{2} = 600 \times \cdot 7431448 = 445 \cdot 88688$ sq. chains = 44 ac. 2 r. 14·19008 p.

By the Logarithms.

Rad.	—	—	10·0000000
Sine of 48°	—	—	9·8710735
$\frac{40 \times 30}{2} = 600$	—	—	2·7781513
Area 445·8869 sq. chains		—	2·6492248

---

* DEMONSTRATION.

For the trapez. = the four $\triangle$s ARB, BRC, CRD, DRA, = $(AR \times RB + BR \times RC + CR \times RD + DR \times RA) \times \frac{1}{2} s. \angle R = (\overline{AR + RC} \times BR + \overline{CR + RA} \times DR) \times \frac{1}{2} s. \angle R = \overline{AR + RC} \times \overline{BR + RD} \times \frac{1}{2} s. \angle R = AC \times BD \times \frac{1}{2} s. \angle R$. *Q.E.D.*

* RULE III.

Square each ſide of the trapezium; add the ſquares of each pair of oppoſite ſides together; ſubtract the leſs ſum from the greater, and multiply the difference by the tangent of the angle formed by the diagonals, and $\frac{1}{4}$ of the product will be the area.

That is, $\overline{AB^2 + CD^2} \backsim \overline{DA^2 + BC^2} \times \frac{1}{4}$ tan. $\angle R =$ area,

Or As radius : $\frac{1}{4}$ tan. $\angle R :: \overline{AB^2 + CD^2} \backsim \overline{DA^2 + BC^2}$: area.

Note. This rule fails when the diagonals interſect at right angles: for then the tangent is infinite, and the difference of the aggregates of the ſquares is nothing.

EXAMPLE.

If the ſides be $AB = 10$, $BC = 9$, $CD = 8$, $DA = 7$ feet; and the leſs angle made by the diagonals equal to 80 degrees; what is the area?

The

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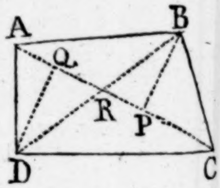
* DEMONSTRATION.

For call AR, m; RE, n; CR, p; RD, q; s, c, t, the ſine, coſine, and tan. $\angle R$.



Then, by 12. and 13. II. Eucl.

$$BA^2 = mm + nn + 2mnc$$
$$AD^2 = mm + qq - 2mqc$$
$$BC^2 = nn + pp - 2npc$$
$$CD^2 = qq + pp + 2pqc$$

And by add. and ſubt. $AB^2 + CD^2 - DA^2 - BC^2 = \overline{mn + pq + mq + np} \times 2c = \overline{m+p} \times \overline{n+q} \times 2c = AC \times BD \times 2c$.

But $AC \times BD \times \frac{1}{2}s =$ the area by the laſt rule.

Theref. $AB^2 + CD^2 - DA^2 - BC^2 = \frac{4c}{s} \times \text{area} = \frac{4}{t} \times \text{area}$

And $(\overline{AB^2 + CD^2} - \overline{DA^2 + BC^2}) \times \frac{1}{4}t =$ area. *Q. E. D.*

Corol. It appears from the demonſtration, that the ſum of the ſquares is greater in that pair of oppoſite ſides which ſubtend the greater, or obtuſe, angles at the interſection R; and, of conſequence, when all the angles at R are equal to each other, that is, when the diagonals interſect at right angles, the ſums of the ſquares of both pairs of oppoſite ſides will be equal to each other.

The tangent of 80° is 5·6712818, therefore $(\overline{AB^2 + CD^2} - \overline{DA^2 + BC^2}) \times \frac{1}{4}$ tan. 80 is $= \overline{164 - 130} \times \frac{1}{4} \times 5{\cdot}6712818 = 34 \times 1{\cdot}417782045 = 48{\cdot}2058953$ square feet, the area.

Or, by the Logarithms thus.

Radius	— —	10·0000000
Tan. of 100°	—	10·7536812
$\frac{34}{4} = 8{\cdot}5$	— —	0·9294189
Area = 48·20589		1·6831001

* RULE IV.

If the trapezium can be inscribed in a circle, that is, if the sum of any two opposite angles be equal to two right angles, or 180°; then multiply any two adjacent sides together, and the other two sides together; and multiply the sum of these two products by the sine of the angle included by either of the pairs of sides which are multiplied together; so shall half the last product be the area.

That is, $\frac{(AD \times DC + AB \times BC) \times s.\ \angle D \text{ or } s.\ \angle B}{2} =$ the area.

Or As radius : s. ∠D or s. ∠B :: $\frac{AD \times DC + AB \times BC}{2}$: area.

EXAMPLE.

If the sides be AB equal to 7·5, BC equal to 5·5, CD equal to 6, DA equal to 4, and the angle B equal

 to

* DEMONSTRATION.

For this product will be equal to the two triangles ADC, ABC, found by rule 2, prob. 2, since the sines of the opposite angles D, B, of a trapezium inscribed in a circle, are equal to each other.

to $74^\circ\ 40\frac{1}{4}'$, and, therefore, the angle D equal to $105^\circ\ 19\frac{3}{4}'$; what is the area?

Here the ſine of $74^\circ\ 40\frac{1}{4}'$ is $\cdot9644229$.

Then $\frac{4 \times 6 + 7\cdot5 \times 5\cdot5}{2} \times \cdot9644229 = 31\cdot46429$ ſquare feet, the area.

Or,	Radius	— —	$10\cdot0000000$
	Sine $\angle$B $74^\circ\ 40\frac{1}{4}'$	—	$9\cdot9842675$
	$\frac{4 \times 6 + 7\cdot \times 55\cdot5}{2} = 32\cdot625$		$1\cdot5135505$
Area $= 31\cdot4643$		— —	$1\cdot4978180$

* RULE V.

Or, when the trapezium can be inſcribed in a circle, the area may be, otherwiſe, found thus:

From

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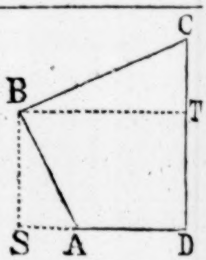
* DEMONSTRATION.



For, upon the ſides AD, DC, let fall the perpendiculars BS, BT: put $s =$ the ſine of the angle A, or of the angle C; and a, b, c, $d =$ the four ſides AB, BC, CD, DA reſpectively.

Then $1 : s :: \begin{cases} a : as = \text{BS}, \\ b : bs = \text{BT}, \end{cases}$ and $1 : \sqrt{1-ss} :: \begin{cases} a : a\sqrt{1-ss} = \text{AS}, \\ b : b\sqrt{1-ss} = \text{CT}; \end{cases}$

hence $\text{SD} = d + a\sqrt{1-ss}$, and $\text{DT} = c - b\sqrt{1-ss}$. But by right-angled triangles, $\text{BS}^2 + \text{SD}^2 = \text{DT}^2 + \text{TB}^2$, that is, $dd + 2ad\sqrt{1-ss} + aa = cc - 2bc\sqrt{1-ss} + bb$, and hence $\sqrt{1-ss} = \frac{bb + cc - aa - dd}{2ad + 2bc}$, and $s = \frac{\sqrt{(2ad + 2bc)^2 - (bb + cc - aa - dd)^2}}{2ad + 2bc}$; then, by the laſt rule, the area will be $\frac{1}{4}\sqrt{(2ad + 2bc)^2 - (bb + cc - aa - dd)^2} =$
$\frac{1}{4}\sqrt{(\overline{b+c}^2 - \overline{a-d}^2) \times (\overline{a+d}^2 - \overline{b-c}^2)} =$
$\frac{1}{4}\sqrt{(a+b+c-d) \times (a+b-c+d) \times (a-b+c+d) \times (-a+b+c+d)}$
$= \sqrt{\overline{s-a} \times \overline{s-b} \times \overline{s-c} \times \overline{s-d}}$, if s be half the ſum of the four ſides, *Q. E. D.*

From half the sum of the four sides subtract each side severally; multiply the four remainders continually together, and the square root of the last product will be the area.

That is,

$$\sqrt{\frac{a+b+c-d}{2} \times \frac{a+b-c+d}{2} \times \frac{a-b+c+d}{2} \times \frac{-a+b+c+d}{2}}$$

$=$ the area. Or if s be half the sum of the sides, then $\sqrt{(s-a) \times (s-b) \times (s-c) \times (s-d)}$ is the area.

The letters a, b, c, d, denoting the four sides of the trapezium.

EXAMPLE.

The four sides of a trapezium inscribed in a circle, are 6, 5·5, 7·5, 4 feet; required the area.

$$\sqrt{\frac{6+5\cdot5+7\cdot5-4}{2} \times \frac{6+5\cdot5-7\cdot5+4}{2} \times \frac{6-5\cdot5+7\cdot5+4}{2} \times \frac{-6+5\cdot5+7\cdot5+4}{2}}$$

$= \frac{1}{4}\sqrt{15 \times 8 \times 12 \times 11} = \frac{1}{4}\sqrt{15840} = \frac{125\cdot85708}{4} =$ 31·46427 square feet the area.

RULE

Corol. 1. The expression marked † is a very pretty theorem, and may be useful on many occasions.

Corol. 2. Hence may be deduced rule 3 for the triangle; for, if we here suppose one of the sides, as d, to be nothing, or to decrease till it vanish, the rule will become $\sqrt{(s-a) \times (s-b) \times (s-c) \times s}$, the same as in the triangle.

Corol. 3. If, in the trapezium, $a = d$, and $b = c$, the rule will be barely ab.

Corol. 4. When all the four sides are equal, the rule becomes $\sqrt{a \times a \times a \times a} = aa$.

Corol. 5. If the sides be in arithmetical proportion, either continued or discontinued; the square root of the continual product of all the four sides will be the area. Or, take a mean proportional between any two sides, and also a mean proportional between the other two sides; then the figure will be equal to the rectangle of these mean proportionals. For, half the sum of the sides is equal to the sum of the extremes, or equal to the sum of the means; and each extreme and mean being taken from it, will leave the other extreme and mean; and consequently the four remainders will be equal to the four given sides. Wherefore, &c. That is, $\sqrt{abcd} =$ the area.

RULE VI.

In a trapezoid, multiply the ſum of the parallel ſides by the diſtance between them, and half the product will be the area.*

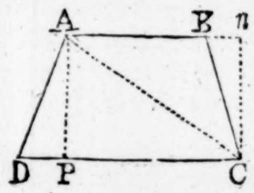
That is, if AB, DC be the parallel ſides, and AP perpendicular to DC; then $\frac{AB + CD}{2} \times AP$ will be the area.

Note. This caſe is very uſeful in ſurveying. For many irregular pieces of land may be, very conveniently, divided into a number of trapezoids, of that kind in which the parallel ſides are perpendicular to one of the other ſides, which, in that caſe, is the diſtance of the parallel ſides.

EXAMPLE.

If AB be 7·5, CD 12·25, and AP 15·4 chains; what is the area?

```
   12·25
    7·50
  ------
   19·75
    15·4
  ------
    7900
   9875
  1975
  ------
2)304·150
  ------
 15,2075
       4
  ------
   ·8300
      40
  ------
 33·2000
```



Anſ. 15 ac. 33·2 per.

SOME

* DEMONSTRATION.

For, the diagonal AC divides the trapezoid into two triangles, whoſe baſes are the two parallel ſides, and heights each equal to the diſtance AP; therefore, &c.

SOME PROMISCUOUS EXAMPLES FOR PRACTICE.

EXAMPLE I.

How many square yards of paving are in the trapezium ABCD, whose diagonal AC is 65 feet, and the perpendiculars BP equal to 33.5, and DQ equal to 28 feet? By rule 1.

$$\begin{array}{r} 33\cdot5 \\ 28\cdot0 \\ \hline 61\cdot5 \\ 65 \\ \hline 3075 \\ 3690 \\ \hline 2)3997\cdot5 \\ \hline 9)1998\cdot75 \text{ feet} \\ \hline \text{Ans. } 222\cdot08\tfrac{1}{3} \text{ yards.} \end{array}$$

EXAMPLE II.

What is the area of a trapezium whose south side is 27·4 chains, east side 35·75 chains, north side 37·55 chains, and west side 41·05 chains; also the diagonal from south-west to north-east 48·35 chains?

By rule 3 of prob. 2,

$\frac{27\cdot4 + 35\cdot75 + 48\cdot35}{2} = \frac{111\cdot5}{2} = 55\cdot75 =$ half the sum of the sides of the south-east triangle. And

$\frac{37\cdot55 + 41\cdot05 + 48\cdot35}{2} = \frac{126\cdot95}{2} = 63\cdot475 =$ half the sum

of

of the sides of the north-west triangle. Whence $\sqrt{55{\cdot}75 \times \overline{55{\cdot}75-27{\cdot}4} \times \overline{55{\cdot}75-35{\cdot}75} \times \overline{55{\cdot}75-48{\cdot}35}} =$ $\sqrt{55{\cdot}75 \times 28{\cdot}35 \times 20 \times 7{\cdot}4} = \sqrt{233915{\cdot}85} = 483{\cdot}6485$, the area of the former triangle. And $\sqrt{63{\cdot}475 \times \overline{63{\cdot}475-37{\cdot}55} \times \overline{63{\cdot}475-41{\cdot}05} \times \overline{63{\cdot}475-48{\cdot}35}}$ $= \sqrt{63{\cdot}475 \times 25{\cdot}925 \times 22{\cdot}425 \times 15{\cdot}125} = \sqrt{558148{\cdot}1}$ &c $= 747{\cdot}0932$ the area of the latter. And their sum is $1230{\cdot}7417$ square chains $= 123$ ac. $11{\cdot}8672$ per. the area required.

EXAMPLE III.

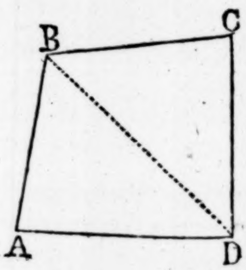
How many square feet are in a trapezium whose sides are AB equal to 12, BC equal to 13, CD equal to 14, DA equal to 15 feet, and the angle A equal to 83° 30′?

First, by rule 2, prob. 2, $\frac{{\cdot}9935719 \times 12 \times 15}{2} =$ $\cdot 9935719 \times 90 = 89{\cdot}421471 =$ the area of the triangle ABD.

Again, by trigonometry,

As $27 = \text{DA} + \text{AB} : 3 = \text{DA} - \text{AB} ::$ $1{\cdot}1204053$ (tang. $\frac{\text{DBA}+\text{ADB}}{2} = 48°\ 15'$) : $\frac{1{\cdot}1204053}{9} =$ $\cdot 1244895 =$ tang. $\frac{\text{DBA}-\text{ADB}}{2} = 7°\ 06'$. Therefore $48°\ 15' - 7°\ 06' = 41°\ 09' =$ ADB. And,

∠ADB	41° 09′	9·8182474
∠A	83 30	9·9971993
AB	12	1·0791812
BD	18·119	1·2581331



Then,

Then, by rule 3, prob. 2,

$$\sqrt{\frac{18{\cdot}119+14+13}{2}\times\frac{18{\cdot}119+14-13}{2}\times\frac{18{\cdot}119-14+13}{2}\times\frac{-18{\cdot}119+14+13}{2}}$$

$=\sqrt{22{\cdot}5595\times9{\cdot}5595\times8{\cdot}5595\times4{\cdot}4405}=\sqrt{8196{\cdot}812}$
$=90{\cdot}53625=$ the area of the triangle DCB.

The sum of these two is $179{\cdot}957721=$ the area of the trapezium required.

EXAMPLE IV.

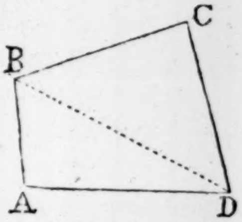
In a quadrangular field the south side is 23·4, the east side 19·75, and the north side 20·5 chains; also the south-east and north-east angles are 73° and 87° 30′: what is the area?

First, by rule 2 prob. 2, $\frac{{\cdot}9990482\times19{\cdot}75\times20{\cdot}5}{2}=$ $\cdot4995241\times19{\cdot}75\times20{\cdot}5=202{\cdot}24482$ the area of the north-east triangle BCD.

Again, $40{\cdot}25=$ BC + CD : ·75 = BC − CD :: 1·0446136 (tang. $\frac{\text{BDC}+\text{CBD}}{2}=46°\ 15'$) : $\frac{1{\cdot}0446136\times3}{161}={\cdot}01946485=$ the tang. of $\frac{\text{BDC}-\text{CBD}}{2}=$ 1° 07′. Wherefore ∠ BDC $=46°\ 15'+1°\ 07'=47°\ 22'$.

And ∠ ADB = (ADC − CDB $=73°-47°\ 22'=$) 25° 38′. But,

∠ BDC	47° 22′	9·8667026
∠ C	87 30	9·9995865
CB	20·5	1·3117539
BD	—	1·4446378



Whence,

Whence, by rule 2, prob. 2, alſo,

$\frac{1}{2}$ AD	—	11·7	—	1·0681859
BD	—		—	1·4446378
S. ADB	—	25° 38′	—	9·6360969
140·9031 the area of the △ ABD			-	2·1489206

Their ſum is 343·1479 ſquare chains = 34 acres 1 rood and 10·3664 perches, the area required.

EXAMPLE V.

What is the area of a trapezium, inſcribed in a circle, whoſe four ſides are 24, 26, 28, 30 yards?

Here $\frac{24+26+28+30}{2} = \frac{108}{2} = 54$ the half ſum of the ſides.

Then, by rule 5, $\sqrt{\overline{54-24}\times\overline{54-26}\times\overline{54-28}\times\overline{54-30}} = \sqrt{30\times28\times26\times24} = 24\sqrt{5\times7\times13\times2} = 24\sqrt{910} = 24\times30\cdot1662062 = 723\cdot9889488$ ſquare yards, the area required.

EXAMPLE VI.

How many ſquare feet are in a board whoſe length is $12\frac{1}{2}$ feet, the breadth of the greater end $1\frac{1}{4}$ feet, and of the leſs 11 inches?

By rule 6, $\frac{1\frac{1}{4}+\frac{11}{12}}{2}\times12\frac{1}{2} = \frac{13}{12}\times12\frac{1}{2} = 13\frac{6\frac{1}{2}}{12} = 13\cdot541\frac{2}{3}$ ſquare feet, the area required.

EXAMPLE VII.

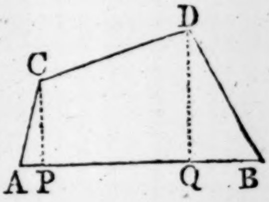
In a trapezoid, the length of the two parallel ſides are 30 and 46 chains, and the length of one of the other ſides, which is perpendicular to the parallel ſides, is 60·37 chains: what is the area?

By rule 6, $\frac{30+46}{2} \times 60\cdot37 = 38 \times 60\cdot37 = 2294\cdot06$ square chains = 229 acres 1 rood 24·96 perches, the area required.

EXAMPLE VIII.

In measuring along one side AB of a quadrangular field, that side and the two perpendiculars upon it from the opposite corners, measured as in the field book below: required the area.

FIELD BOOK.			
	chains		chains
AP =	1·10	PC =	3·52
AQ =	7·45	QD =	5·95
AB =	11·10		



$$AP \times PC \qquad = 1\cdot10 \times 3\cdot52 = 3\cdot872 = 2APC$$
$$PQ \times \overline{PC + QD} = 6\cdot35 \times 9\cdot47 = 60\cdot1345 = 2CPQD$$
$$DQ \times QB \qquad = 5\cdot95 \times 3\cdot65 = 21\cdot7175 = 2DQB$$

the sum = 85·724 = double the whole figure, whose half = 42·862 squ. chains = 4 acres 1 rood 5·792 perches is the area required.

PROBLEM IV.

To find the Area of any Irregular Figure.

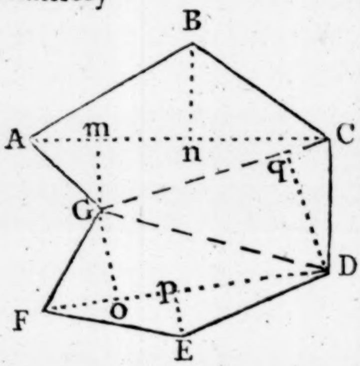
Draw diagonals dividing the figure into trapeziums and triangles. Then find the areas of all these separately, and their sum will be the content of the whole irregular figure.

EXAMPLES.

1. To find the content of the irregular figure ABCDEFGA,

ABCDEFGA, in which are given the following diagonals and perpendiculars: namely

AC	5·5
FD	5·2
GC	4·4
Gm	1·3
Bn	1·8
GO	1·2
Ep	0·8
Dq	2·3



$$AC \times \overline{Gm + Bn} = 5{\cdot}5 \times 3{\cdot}1 = 17{\cdot}05 = 2ABCG$$
$$FD \times \overline{GO + Ep} = 5{\cdot}2 \times 2{\cdot}0 = 10{\cdot}40 = 2GFED$$
$$GC \times Dq = 4{\cdot}4 \times 2{\cdot}3 = 10{\cdot}12 = 2GCD$$

2) 37·57
answer 18·785

PROBLEM V.

To find the Areas of Regular Figures or Polygons.

RULE I.

Multiply the perimeter of the figure by the radius of its inscribed circle, or by the perpendicular demitted from its center to one of the sides, and half the product will be the area.*

Note

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* DEMONSTRATION.

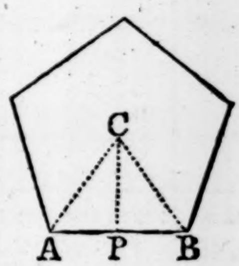
For, the polygon consisting of as many triangles as it hath sides, whose heights and bases are each equal to the perpendicular, and the side of the polygon respectively; if s be a side of the polygon, p the perpendicular, and n the number of sides; then $\frac{1}{2}ps$ will be one triangle, and $\frac{1}{2}nps$ all the triangles, or the whole polygon. *Q. E. D.*

Note 1. The perimeter of a figure is the ſum of its ſides, and in a regular figure is equal to the length of one ſide multiplied by the number of ſides.

2. The center of the figure is the ſame as the center of the inſcribed or circumſcribed circle.

EXAMPLE.

Required the area of a regular pentagon whoſe ſide is 25 yards.



Since $\frac{360^\circ}{5} = 72^\circ = \angle ACB$;

hence $\frac{72}{2} = 36^\circ = \angle ACP$;

and $90 - 36 = 54^\circ = \angle CAP$;

Then as $1 = \text{rad.}$: $1{\cdot}3763819 = \text{tang. } 54^\circ :: 12\frac{1}{2} = AP : 17{\cdot}2047737 = PC$.

Whence $\frac{17{\cdot}2047737 \times 25 \times 5}{2} = 1075{\cdot}298356$, the area required.

* RULE II.

Multiply the ſquare of the ſide, of any regular figure, by the multiplier ſtanding oppoſite its name,

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* DEMONSTRATION.

This rule is founded upon the property, that like polygons, as ſimilar figures, are to one another as the ſquares of their like ſides; which has been proved.

Now, the multipliers in the table, are the areas of the polygons to the ſide 1; whence the rule is manifeſt.

SCHOLIUM.

The table is formed thus: As radius $= 1$: tang. $\angle CAP = t :: AP : PC = t \times AP = \frac{1}{2}t$, ſuppoſing $AB = 1$; then $\frac{1}{4}t$ $(= CP \times PA)$ = the $\triangle$ ACB, and $\frac{1}{4}nt$ = the polygon; where n is the number of ſides: So that, by finding the tangent of the $\angle CAP$, by

in the following table; and the product will be the area.

N° of sides	Names.	Multipliers, or area when the side is 1.
3	Trigon or equi. △	$0{\cdot}4330127 = \frac{3}{4}\tan.30^\circ = \frac{1}{4}\sqrt{3}$
4	Tetragon or □	$1{\cdot}0000000 = \frac{4}{4}\tan.45^\circ = 1 \times 1$
5	Pentagon	$1{\cdot}7204774 = \frac{5}{4}\tan.54^\circ = \frac{5}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$
6	Hexagon	$2{\cdot}5980762 = \frac{6}{4}\tan.60^\circ = \frac{3}{2}\sqrt{3}$
7	Heptagon	$3{\cdot}6339124 = \frac{7}{4}\tan.64\frac{2}{7}^\circ$
8	Octagon	$4{\cdot}8284271 = \frac{8}{4}\tan.67\frac{1}{2}^\circ = 2 \times \overline{1+\sqrt{2}}$
9	Nonagon	$6{\cdot}1818242 = \frac{9}{4}\tan.70^\circ$
10	Decagon	$7{\cdot}6942088 = \frac{10}{4}\tan.72^\circ = \frac{5}{2}\sqrt{5+2\sqrt{5}}$
11	Undecagon	$9{\cdot}3656399 = \frac{11}{4}\tan.73\frac{7}{11}^\circ$
12	Dodecagon	$11{\cdot}1961524 = \frac{12}{4}\tan.75^\circ = 3 \times \overline{2+\sqrt{3}}$

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the table of tangents, and multiplying it by the number of sides, $\frac{1}{4}$ of the product will be the multiplier in the table.

All these multipliers may be derived from other methods, and, indeed, many of them from such as are very simple.

Thus, in the trigon, or equilateral triangle, $\sqrt{1^2 - \frac{1}{2}^2} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$ = the perpendicular, and $\frac{1}{4}\sqrt{3}$ is the area.—In the trigon the angle CAP is 30°, and therefore the tangent of 30° is $= \sqrt{\frac{1}{3}} = \frac{1}{3}\sqrt{3}$.

In the tetragon, or square, the side drawn into itself $1 \times 1 = 1$, is the area.

In the pentagon, the ∠CAP is 54°, whose sine is $\frac{\sqrt{5}+1}{4}$, and its cosine $\frac{\sqrt{10-2\sqrt{5}}}{4}$; and, because the tangent is equal to the sine divided by the cosine, we shall have the tangent of 54° or $t = \frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}} = \sqrt{\frac{6+2\sqrt{5}}{10-2\sqrt{5}}} = \sqrt{1+\frac{2}{5}\sqrt{5}}$; therefore the area of the pentagon is $\frac{5}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$.

In the hexagon, all the triangles are equilateral, and therefore it will be 6 times the trigon, or 6 times $\frac{1}{4}\sqrt{3} = \frac{6}{4}\sqrt{3} = \frac{3}{2}\sqrt{3}$.—And, since, in it, the ∠CAP is = 60°, the tangent of 60° will be $= \sqrt{3}$ = triple the tangent of 30°.

In the octagon, the ∠CAP is $67\frac{1}{2}$°, whose sine is $\frac{1}{2}\sqrt{2+\sqrt{2}}$, and its cosine $\frac{1}{2}\sqrt{2-\sqrt{2}}$; whence the tangent of $67\frac{1}{2}$° or $t = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = 1+\sqrt{2}$; and therefore the area of the octagon is $\frac{8}{4} \times (1+\sqrt{2}) = 2 \times (1+\sqrt{2})$.

In the decagon, the angle CAP is 72°, whose sine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$, and its cosine $\frac{1}{4} \times (\sqrt{5}-1)$; whence t the tangent of 72° is

$\sqrt{}$

EXAMPLE I.

What is the area of a pentagon whose side is 25 feet?

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$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1} = \sqrt{\frac{10+2\sqrt{5}}{6-2\sqrt{5}}} = \sqrt{5+2\sqrt{5}}$; and therefore the area of the decagon is $\frac{10}{4}\sqrt{5+2\sqrt{5}} = \frac{5}{2}\sqrt{5+2\sqrt{5}}$.

In the dodecagon, the angle CAP is 75°, whose sine is $\frac{1}{2}\sqrt{2+\sqrt{3}}$, and its cosine $\frac{1}{2}\sqrt{2-\sqrt{3}}$; whence the tangent of 75° is $\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} = 2+\sqrt{3}$; and therefore the area of the dodecagon is $\frac{12}{4}\times(2+\sqrt{3}) = 3\times(2+\sqrt{3})$.

The tangents of the angle CAP in the heptagon, nonagon, and undecagon, are not so easily found, independent of a table of tangents; but we may find equations whose roots shall be the required tangent, in the following manner.

Put y and x for the sine and cosine of the angle ACB at the center, subtended by the side of any polygon, and $n =$ the number of sides: then it is known that the sine of n times that angle, or of 360°, the whole circumference of a circle, is

$$nx^{n-1}y - \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}x^{n-3}y^3 + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4.5}x^{n-5}y^5 \&c.$$

But the sine of the whole circumference is nothing, and therefore this series, or the same divided by ny, will be equal to nothing also, viz.

$$x^{n-1} - \frac{\overline{n-1}.\overline{n-2}}{2.3}x^{n-3}y^2 + \frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{2.3.4.5}x^{n-5}y^4 \&c = 0.$$

And since n is an integer number, it is evident that the series will always terminate. Now exterminating y by its value $\sqrt{1-xx}$, the series will become

$$x^{n-1} - \frac{\overline{n-1}.\overline{n-2}}{2.3}x^{n-3}(1-xx) + \frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{2.3.4.5}x^{n-5}(1-xx)^2 \&c = 0,$$

$$\text{or } 1 - \frac{\overline{n-1}.\overline{n-2}}{2.3}.\frac{1-xx}{xx} + \frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{2.3.4.5}\left(\frac{1-xx}{xx}\right)^2 \&c = 0.$$

Or exterminating x by its value $\sqrt{1-y^2}$ will give

$$(1-y^2)^{\frac{n-1}{2}} - \frac{\overline{n-1}.\overline{n-2}}{2.3}(1-y^2)^{\frac{n-3}{2}}y^2 + \frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{2.3.4.5}(1-y^2)^{\frac{n-5}{2}}y^4 \&c = 0.$$

$$\text{or } 1 - \frac{\overline{n-1}.\overline{n-2}}{2.3}(1-y^2)y^2 + \frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{2.3.4.5}(1-y^2)^2y^4 \&c = 0.$$

And by writing any particular value for n, the root of the equation

The multiplier for the pentagon being 1·7204774, we ſhall have $1·7204774 \times 25 \times 25 = 1075·298375$ the area required.

E X-

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equation reſulting will be the coſine, or ſine, of the angle ACB.

Thus, if n be $= 7$, the equation will be

$$\left\{\begin{array}{l} x^6 - \frac{5}{4}x^4 + \frac{3}{8}x^2 - \frac{1}{64} = 0 \\ y^6 - \frac{7}{4}y^4 + \frac{7}{8}y^2 - \frac{7}{64} = 0 \end{array}\right\} \text{whoſe root} \left\{\begin{array}{l} x \\ y \end{array}\right\} \text{is the} \left\{\begin{array}{l} \text{coſine} \\ \text{ſine} \end{array}\right\}$$

of double the complement of the ∠CAP $(64\frac{2}{7}°)$ in the heptagon.

If $n = 9$, the ſeries will become

$$\left\{\begin{array}{l} x^8 - \frac{7}{4}x^6 + \frac{15}{16}x^4 - \frac{5}{32}x^2 + \frac{1}{256} = 0 \\ y^8 - \frac{9}{4}y^6 + \frac{27}{16}y^4 - \frac{15}{32}y^2 + \frac{9}{256} = 0 \end{array}\right\} \text{whoſe root} \left\{\begin{array}{l} x \\ y \end{array}\right\} \text{is the} \left\{\begin{array}{l} \text{coſine} \\ \text{ſine} \end{array}\right.$$

of double the complement of the angle CAP (70°) in the nonagon.

If $n = 11$, there will come out the equation

$$\left\{\begin{array}{l} x^{10} - \frac{9}{4}x^8 + \frac{7}{4}x^6 - \frac{35}{64}x^4 + \frac{15}{256}x^2 - \frac{1}{1024} = 0 \\ y^{10} - \frac{11}{4}y^8 + \frac{11}{4}y^6 - \frac{77}{64}y^4 + \frac{55}{256}y^2 - \frac{11}{1024} = 0 \end{array}\right\} \begin{array}{l} \text{whoſe} \\ \text{root is} \end{array} \left\{\begin{array}{l} \text{coſine} \\ \text{ſine} \end{array}\right\}$$

of double the complement of the ∠CAP $(73\frac{7}{11}°)$ in the undecagon.

When the root x of the equation is extracted, the tangent of the ∠CAP may be eaſily found thus. Call that tangent t; then t being the cotangent of an angle of the double of which

$$\left\{\begin{array}{l} x \\ y \end{array}\right\} \begin{array}{l} \text{is} \\ \text{the} \end{array} \left\{\begin{array}{l} \text{coſine} \\ \text{ſine} \end{array}\right\} \text{theref.}\ t = \sqrt{\frac{1+x}{1-x}} = \frac{1+\sqrt{1-y^2}}{y} = \frac{y}{1-\sqrt{1-y^2}}$$

Or, by writing $\frac{tt-1}{tt+1}$ inſtead of x, or $\frac{2t}{1+tt}$ inſtead of y, in the three equations above, we ſhall have,

Firſt, $t^{12} - 26t^{10} + 143t^8 - 245\frac{1}{7}t^6 + 143t^4 - 26t^2 + 1 = 0$, an equation whoſe root is the tangent of the ∠CAP $(64\frac{2}{7}°)$ in the heptagon.

Secondly, $t^{16} - 45\frac{1}{3}t^{14} + 476t^{12} - 1768t^{10} + 2701\frac{1}{9}t^8 - 1768t^6 + 476t^4 - 45\frac{1}{3}t^2 + 1 = 0$, an equation whoſe root is the tangent of the ∠CAP (70°) in the nonagon.

And laſtly, $t^{20} - 70t^{18} + 1197t^{16} - 7752t^{14} + 22610t^{12} - 32065\frac{1}{11}t^{10} + 22610t^8 - 7752t^6 + 1197t^4 - 70t^2 + 1 = 0$, the root of which equation is the tangent of the angle CAP $(73\frac{7}{11}°)$ in the undecagon.

MOREOVER,

EXAMPLE II.

What is the area of a hexagon whose side is 20?

Here $2{\cdot}598076 \times 20 \times 20 = 1039{\cdot}2304 =$ the area.

MOREOVER, If in a square be inscribed a circle, and in the circle the several regular polygons; then

The square will be to 4 as the circle to 3·14159265 &c, and as the inscribed

Trigon	to	$\frac{3}{2} \times$ sine of 120°	$= \frac{3}{4}\sqrt{3}$
Tetragon	to	$\frac{4}{2} \times$ sine of 90	$= 2$
Pentagon	to	$\frac{5}{2} \times$ sine of 72	$= \frac{5}{8}\sqrt{10+2\sqrt{5}}$
Hexagon	to	$\frac{6}{2} \times$ sine of 60	$= \frac{3}{2}\sqrt{3}$
Heptagon	to	$\frac{7}{2} \times$ sine of $51\frac{3}{7}$	$= \frac{7}{2}x$
Octagon	to	$\frac{8}{2} \times$ sine of 45	$= 2\sqrt{2}$
Nonagon	to	$\frac{9}{2} \times$ sine of 40	$= \frac{9}{2}y$
Decagon	to	$\frac{10}{2} \times$ sine of 36	$= \frac{5}{4}\sqrt{10-2\sqrt{5}}$
Undecagon	to	$\frac{11}{2} \times$ sine of $32\frac{8}{11}$	$= \frac{11}{2}z$
Dodecagon	to	$\frac{12}{2} \times$ sine of 30	$= 3$
&c.		&c.	

Where $\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\}$ is the root of the equation $\left\{\begin{matrix} x^6 - \frac{7}{4}x^4 + \frac{7}{8}x^2 - \frac{7}{64} = 0 \\ y^8 - \frac{9}{4}y^6 + \frac{27}{16}y^4 - \frac{15}{32}y^2 + \frac{9}{256} = 0 \\ z^{10} - \frac{11}{4}z^8 + \frac{11}{4}z^6 - \frac{77}{64}z^4 + \frac{55}{256}z^2 - \frac{11}{1024} = 0 \end{matrix}\right.$

as appears from page 116.

For, if the side of the square, or diameter of the circle, be 2, their areas will be 4 and 3·14159 &c. And, if radii be drawn to each angular point of the polygons, they will divide the polygons into as many equal triangles, as they have sides; and if, in one of these triangles, a perpendicular be demitted from the extremity of one radius to the other, half this perpendicular will express the area of the triangle, because the base upon which it falls is equal to 1, and consequently, half the number of sides drawn into the perpendicular, will denote the area of the polygon: but the perpendiculars are the sines of the angles above expressed; and consequently the truth of the proportions is manifest.

As to the values of the sines, they are sufficiently well known, excepting those denoted by x, y, z, whose values are as above specified.

EXAMPLE III.

What is the area of a trigon whose side is 20?

Here $\cdot 433013 \times 20 \times 20 = 173\cdot 2052 =$ the area required.

PROBLEM VI.

*To find the Diameter and Circumference of a Circle, the one from the other.**

RULE I.

As 7 is to 22, so is the diameter to the circumference.

As 22 is to 7, so is the circumference to the diameter.

RULE

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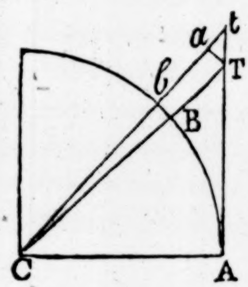
* DEMONSTRATION.

The proportion of the diameter of a circle to the circumference, is best found from the tangent of some arc, as follows.

Let AT be the tangent of the arc AB, and draw Ct indefinitely near CT, and Ta perpendicular to Ct. Put $r =$ CA, $t =$ AT, and $a =$ the arc AB. Then T$t = \dot{t}$, B$b = \dot{a}$, and CT $= \sqrt{rr + tt}$.



In the similar $\triangle$s CAT, Tat, we have CT : CA :: Tt : T$a = \frac{r\dot{t}}{\text{CT}}$; and in the similar $\triangle$s CBb, CTa, we have CT : Ta :: CB : Bb, or $\dot{a} = \frac{\text{CE} \times \text{T}a}{\text{CT}} = \frac{rr\dot{t}}{\text{CT}^2} = \frac{rr\dot{t}}{rr + tt} = \dot{t} \times (1 - \frac{t^2}{r^2} + \frac{t^4}{r^4} - \frac{t^6}{r^6} + \frac{t^8}{r^8}$ &c.) And the fluent is $a = t \times (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8}$ &c.) $=$ the length of the arc AB. Then, by taking AB $=$ any given arc, whose tangent can be found in terms of the radius, the series will then become known; and being repeated as often as AB is contained in the whole circumference, we shall have the length of the circumference in terms of the diameter.

This

RULE II.

As 113 is to 355, so is the diam. to the circumf.
As 355 is to 113, so is the circumf. to the diameter.

RULE III.

As 1 is to 3·1416, so is the diam. to the circumf.
As 3·1416 is to 1, so is the circumf. to the diam.

This series was first given, for this purpose, by Mr. JAMES GREGORY, in a letter of the 15th of February 1671, sent to Mr. COLLINS, and inserted in the *Commerc. Epistol.*—If the arc be assumed greater than half a quadrant, the series will diverge, and become of no use here; because the tangent will, in that case, be greater than the radius: but in every other case the series will converge, and become a proper expression for the arc.

Thus, if AB be the $\frac{1}{8}$ part of the circumference, or 45°, its tangent will be equal to the radius, and the series will become $r \times (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$ &c) = the arc of 45°; and therefore $8r \times (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$ &c) will be the whole circumference.

Henc the circumferences of circles are to one another as their radii, or as their diameters; and, consequently, the circumference of a circle whose diameter is 1, being multiplied by the diameter of any other circle, will produce the circumference of this circle; which is our rule.

The last series is the simplest form that the general series will admit of, although it does not converge so quickly as may be required; and to find others converging quicker we must take the arc AB less. Thus,

Suppose AB = an arc of 30°, or the $\frac{1}{12}$ part of the circumference; then its tangent t is $= r\sqrt{\frac{1}{3}}$. Or, to make the forms still simpler, let us suppose $r = 1$; then $t = \sqrt{\frac{1}{3}}$, and the series will become $a = \sqrt{\frac{1}{3}} \times (1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4}$, &c) = the arc of 30°; which converges so fast, that it may be very quickly summed; and being multiplied by 12, will give the whole circumference. And this is Doctor HALLEY's method.

And by taking still smaller arcs, series may be found, in the same manner, converging much quicker; but then they will be more compounded with surd quantities.

As the famous quadrature of the late Mr. JOHN MACHIN, *Professor of Astronomy in Gresham College*, is extremely expeditious, and but little known, I shall take this opportunity of explaining

EXAMPLE I.

If the diameter of a circle be 7, what is the circumference?

$3{\cdot}1416 \times 7 = 21{\cdot}9912$ the circumference, $= 22$ nearly.

Therefore the diameter is to the circumference, nearly as 7 to 22.

EX-

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plaining it as follows.—Since the chief advantage conſiſts in taking ſmall arcs, whoſe tangents ſhall be numbers eaſy to manage, Mr. Machin very properly conſidered, that ſince the tangent of 45° is 1; and that the tangent of any arc being given, the tangent of double that arc can eaſily be had; if there be aſſumed ſome ſmall ſimple number for the tangent of an arc, and then the tangent of the double arc be continually taken, until a tangent be found nearly equal to 1, the tangent of 45°; by taking the tangent anſwering to the ſmall difference between 45° and this multiple, there would be had two very ſmall tangents, viz. the tangent firſt aſſumed, and the tangent of the difference between 45° and the multiple arc; and that, therefore, the lengths of the arcs correſponding to theſe two tangents being calculated, and the arc belonging to the tangent firſt aſſumed being as often doubled as the multiple directs, the reſult increaſed or diminiſhed by the other arc, would be the arc of 45°, according as the multiple arc ſhould be below or above it.

Having thus thought of his method, by a few trials he was lucky enough to find a number, and perhaps the only one, proper for this purpoſe, viz. knowing that the tangent of $\frac{1}{4}$ of 45° is nearly $= \frac{1}{5}$, he aſſumed $\frac{1}{5}$ as the tangent of an arc: then, ſince if t be the tangent of an arc, the tangent of the double arc will be $\frac{2t}{1 - tt}$, the radius being 1; the tangent of an arc double to that of which $\frac{1}{5}$ is the tangent, will be $\frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{10}{24} = \frac{5}{12}$, and the tangent of the double of this laſt is $\frac{\frac{10}{12}}{1 - \frac{25}{144}} = \frac{10 \times 12}{144 - 25} = \frac{120}{119}$; which, being very nearly equal to 1, ſhews that the arc which is equal to 4 times the firſt, is very near 45°. Then, ſince the tangent of the difference between 45° and an arc whoſe tangent is T, is $\frac{T - 1}{T + 1}$, we ſhall have the tangent of the difference between

45°

EXAMPLE II.

What is the diameter of the circle whose circumference is 355?

$355 \div 3{\cdot}1416 = 112{\cdot}9997 +$, the diameter, $= 113$ nearly.

And therefore the diameter is to the circumference nearly as 113 to 355.

EX-

45° and the arc whose tangent is

$$\frac{120}{119}, \text{ equal to } \frac{\frac{120}{119} - 1}{\frac{120}{119} + 1} = \frac{120 - 119}{120 + 119} = \frac{1}{239}.$$

Now, by calculating, from the general series, the arcs whose tangents are $\frac{1}{5}$ and $\frac{1}{239}$, which may be very quickly done, by reason of the smallness and simplicity of the numbers, and taking the latter arc from 4 times the former, the remainder will be the arc of 45°. And this is Mr. Machin's famous quadrature of the circle.

But it was by means of Dr. Halley's method that he found the circumference of a circle, whose diameter is 1, to be 3·1415926535, 8979323846, 2643383279, 5028841971, 6939937510, 5820974944, 5923071864, 0628620899, 8628034825, 3421170679 +, true to above 100 places of figures.

Or, by substituting the above numbers in the general series, we get the series $(\frac{16}{5} - \frac{4}{239}) - \frac{1}{3}(\frac{16}{5^3} - \frac{4}{239^3}) + \frac{1}{5}(\frac{16}{5^5} - \frac{4}{239^5})$ &c. equal to the semicircumference whose radius is 1, or the whole circumference whose diameter is 1. And this is the series published by Mr. JONES, and which he acknowledges he received from Mr. MACHIN.

But because the arc whose tangent is $\frac{1}{5}$, is $= 2$ times the arc whose tangent is $\frac{1}{10}$, minus the arc to tangent $\frac{1}{515}$; (for $\frac{\frac{2}{10}}{1 - \frac{1}{100}} = \frac{20}{99} =$ tang. of twice arc to tang. $\frac{1}{10}$, and $\frac{\frac{20}{99} - \frac{1}{5}}{1 + \frac{4}{99}} = \frac{1}{515}$. = tang. of dif. between the arcs whose tangents are $\frac{20}{99}$ and $\frac{1}{5}$); therefore 8 times arc to tang. $\frac{1}{10}$ − 4 times arc to tang. $\frac{1}{515}$ − arc to tang. $\frac{1}{239}$ = arc of 45°, or whose tang. is 1. Which is much easier than Machin's way. And various other ways may easily be discovered from the same principles.

If instead of t, in the series $t(1 \times - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6}$ &c) be substituted its value $\frac{rr}{\tau}$, we shall have $\frac{rr}{\tau} \times (1 - \frac{r^2}{3\tau^2} + \frac{r^4}{5\tau^4} - \frac{r^6}{7\tau^6}$ &c) for the length of the arc whose co-tangent is τ, and radius r.

Or,

EXAMPLE III.

What is the circumference of the earth, supposing it to be perfectly round, and that its diameter is 7958 miles? Ans. 25000·8528 miles.

EX-

Or, if we write $\frac{rs}{c}$ instead of t, we shall have

$$\frac{rs}{c}\times(1-\frac{s^2}{3c^2}+\frac{s^4}{5c^4}-\frac{s^6}{7c^6}\ \&c)$$

for the length of the arc whose sine is s and cosine c.

And, farther, by writing $rr-ss$ for cc in this last series, we obtain $s\times(1+\frac{s^2}{2.3r^2}+\frac{3s^4}{2.4.5r^4}+\frac{3.5s^6}{2.4.6.7r^6}+\frac{3.5.7s^8}{2.4.6.8.9r^8}\ \&c)$ for the arc whose sine is s.

Again, by substituting $\sqrt{2rv-vv}$, the value of s, instead of it, in the last series, we get

$$\sqrt{2rv}\times(1+\frac{v}{2.3.2r}+\frac{3v^2}{2.4.5.2^2r^2}+\frac{3.5v^3}{2.4.6.7.2^3r^3}\ \&c)$$

for the length of the arc whose versed sine is v.—Or, by writing d, the diameter, instead of $2r$, the value of the arc will be expressed by $\sqrt{dv}\times(1+\frac{v}{2.3d}+\frac{3v^2}{2.4.5d^2}+\frac{3.5v^3}{2.4.6.7d^3}\ \&c)$.

Much after the same manner the arc may be expressed by the secants, co-secants, or co-sines. And if for any of the letters, in the general series, be substituted any of their particular values, as was done for the tangent, we may, by those means, obtain several different particular forms of infinite series, whose sums will be expressed by means of the circumferences of any given circles: Thus, if, in the last series but one, s be taken $=r$, or the arc be supposed to be a quadrant, we shall obtain the infinite series $r+\frac{r^3}{2.3r^2}+\frac{3r^5}{2.4.5r^4}+\frac{3.5r^7}{2.4.6.7r^6}\ \&c$, or $r\times(1+\frac{1}{2.3}+\frac{3}{2.4.5}+\frac{3.5}{2.4.6.7}\ \&c)$, whose sum will be equal to $\frac{1}{4}$ of the circumference of the circle whose radius is r.

And, if from this series be taken the former series, expressed by s, there will result

$$\frac{r-s}{1}+\frac{1}{2.3}\cdot\frac{r^3-s^3}{r^2}+\frac{3}{2.4.5}\cdot\frac{r^5-s^5}{r^4}+\frac{3.5}{2.4.6.7}\cdot\frac{r^7-s^7}{r^6}\ \&c,$$

for the length of the arc whose cosine is s.

EXAMPLE IV.

If the circumference of the earth be 25000 miles, what is the diameter? Ans. 7957·744 miles.

PROBLEM VII.

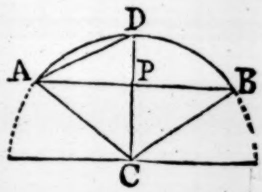
To find the Length of any Arc of a Circle

RULE I.*

As 180 is to the number of degrees in the arc,
So is 3·1416 times the radius, to its length.
Or as 9 is to the number of degrees in the arc,
So is ·0174533 times the radius, to its length.

EXAMPLE.

Required the length of the arc ADB, whose chord AB is 6, the radius being 9.



By trigonometry, $9 = AC : 3 = AP :: 1 =$ rad. of the tables : $\frac{3}{9} =$ ·3333333 the sine of the $\angle$ ACP, or of the arc AD, to the radius 1; and the degrees, in the table of sines, answering to this sine, are 19·4712206, the double of which is 38·9424412, the degrees in the whole arc ADB.

Then, by the rule, $38·9424412 \times ·0174533 \times 9 =$ 6·117063, the length of the arc required.

RULE

* DEMONSTRATION.

For, when the radius is 1, half the circumference is 3·14159265 &c, and therefore $\frac{3·14159265 \text{ \&c}}{180 \text{ degrees}} =$ ·0174532925199 &c is the length of an arc of 1 degree; hence $r \times$ ·01745 &c = the length of 1 degree to the radius r, and, consequently, ·01745 &c $\times r \times$ number of degrees in any arc, will be the length of that arc. *Q. E. D.*

RULE II.

* If d denote the diameter of the circle, and v the versed sine of half the arc; the arc will be expressed by $2\sqrt{dv} \times (1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3}$ &c), or by $2d\sqrt{q} + \frac{q}{2.3}\text{A} + \frac{3.3q}{4.5}\text{B} + \frac{5.5q}{6.7}\text{C}$ &c. putting $q = \frac{v}{d}$, and A, B, C, &c, for the first, second, third, &c, terms.

EXAMPLE.

Let there be taken here the same example as before.

Then $\text{PC} = \sqrt{\text{CA}^2 - \text{AP}^2} = \sqrt{9^2 - 3^2} = \sqrt{72} = 6\sqrt{2}$, and $\text{PD} = \text{DC} - \text{CP} = 9 - 6\sqrt{2} = \cdot 51471862 = v$; also $\cdot 51471862 \div 18 = \cdot 02859548 = \frac{v}{d} = q$.

$$
\begin{array}{llll}
\text{Hence A} = 2d\sqrt{q} & = & 6\cdot 087672 \\
\text{B} = \frac{q}{2.3}\text{A} & = & 29013 \\
\text{C} = \frac{3.3q}{4.5}\text{B} & = & 373 \\
\text{D} = \frac{5.5q}{6.7}\text{C} & = & 6
\end{array}
$$

the sum $6\cdot 117064$ = the arc ABD as before, nearly.

RULE III.

† If r be the radius, and s the sine or half chord AP of the arc, then the length of the arc ADB will be

$$2s \times (1 + \frac{s^2}{3.2r^2} + \frac{3s^4}{5.2.4r^4} + \frac{3.5s^6}{7.2.4.6r^6} \text{ \&c,})$$

$$\text{or } 2s + \frac{q}{2.3}\text{A} + \frac{3.3q}{4.5}\text{B} + \frac{5.5q}{6.7}\text{C} + \frac{7.7q}{8.9}\text{D} \text{ \&c,}$$

where q is $= \frac{ss}{rr}$; and A, B, C, &c, are the preceding terms.

EX-

* This rule is proved in page 122. † This likewise is proved in page 122.

EXAMPLE.

Taking the same example as in the first rule,

$$\begin{array}{lcl} 2s & = \text{A} = & 6{\cdot}000000 \\ \frac{ss}{2.3rr}\text{A} & = \text{B} = & 0{\cdot}111111 \\ \frac{3.3ss}{4.5rr}\text{B} & = \text{C} = & 5555 \\ \frac{5.5ss}{6.7rr}\text{C} & = \text{D} = & 367 \\ \frac{7.7ss}{8.9rr}\text{D} & = \text{E} = & 28 \\ \frac{9.9ss}{10.11rr}\text{E} & = \text{F} = & 2 \end{array}$$

the sum 6·117063 is the arc ADB.

RULE IV.

From 8 times the chord of half the arc, subtract the chord of the whole arc, and $\frac{1}{3}$ of the remainder will be the length of the arc, nearly.

That is, $\frac{8\text{AD} - \text{AB}}{3} = \text{ADB}$ nearly.*

EX-

* DEMONSTRATION.

For since, by the last rule, the arc $\text{AD} = \text{A} = s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4}$ &c.

And $c =$ the chord $\text{AD} = \sqrt{ss + xx} = \sqrt{s^2 + (r - \text{CP})^2} =$

$$\sqrt{s^2 + (r - \sqrt{rr - ss})^2} = s + \frac{s^3}{8r^2} + \frac{7s^5}{128r^4} \text{ \&c.}$$

Therefore $\text{A} - c = \frac{s^3}{24r^2} + \frac{13s^5}{640r^4}$ &c.

But $\frac{c - s}{3} = \frac{s^3}{24r^2} + \frac{7s^5}{384r^4}$ &c.

Therefore $\text{A} - c - \frac{c - s}{3} = \text{A} - \frac{4c - s}{3} = \frac{s^5}{480r^4}$ &c.

And $2\text{A} = \frac{8c - 2s}{3}$ nearly, = the arc ADB, as in the rule. And this is Huygen's theorem.

EXAMPLE.

Taking, again, the ſame example; we ſhall have, as in the example to rule 2, PD $= 9 - 6\sqrt{2}$, and hence DA $= \sqrt{\text{AP}^2 + \text{PD}^2} = \sqrt{3^2 + (9 - 6\sqrt{2})^2} = 3\sqrt{18 - 12\sqrt{2}} = 3{\cdot}043836$.

Then, by the rule, $\frac{3{\cdot}043836 \times 8 - 6}{3} = 6{\cdot}116896$ the length of the arc, nearly the ſame as before.

RULE V.

* Divide 3 times the verſed ſine of the half arc by the difference between the ſaid verſed ſine and 3 times the diameter; multiply the ſquare root of the quotient

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* DEMONSTRATION.

By the ſecond rule, half the arc is $\sqrt{dv} \times (1 + \frac{v}{2.3d} + \frac{3vv}{2.4.5dd}$ &c).

Let there be aſſumed an expreſſion which, when expanded into a ſeries, ſhall be of the ſame form with the ſeries above, and affected with indeterminate coefficients; then from the compariſon of as many of the firſt terms as there are indeterminate quantities aſſumed, will be found the values of the ſaid quantities in given numbers; which being written for them in the primary aſſumed expreſſion, will make that expreſſion an approximate value of the arc.

Thus, aſſume $d\sqrt{\frac{v}{gd - hv}}$

This in a ſeries is $\sqrt{dv} \times (\frac{1}{\sqrt{g}} + \frac{hv}{2dg\sqrt{g}} + \frac{3hhvv}{2.4ddgg\sqrt{g}}$ &c); the two firſt terms of which compared with thoſe of the ſeries above, give theſe two equations, $\frac{1}{\sqrt{g}} = 1$, and $\frac{h}{g\sqrt{g}} = \frac{1}{3}$; hence $g = 1$, and $h = \frac{1}{3}$; and therefore the aſſumed expreſſion $d\sqrt{\frac{v}{gd - hv}}$ becomes $d\sqrt{\frac{v}{d - \frac{1}{3}v}} = d\sqrt{\frac{3v}{3d - v}}$ = half the arc nearly; the double of which will be the whole arc, as in rule 5.

tient by 2 times the diameter; and the product will be the arc nearly.

That is, $4\text{CD}\sqrt{\frac{3\text{DP}}{6\text{CD}-\text{DP}}} = \text{ADB}$ nearly.

EXAMPLE.

Taking, still, the same example; we find, as in the example to the last rule, $\text{DP} = 9 - 6\sqrt{2} = \cdot5147186$.

Then, by the rule, $36\sqrt{\frac{\cdot5147186 \times 3}{54 - \cdot5147186}} = 36 \times \cdot1699137$ $= 6\cdot116893$ the arc nearly the same as before.

RULE VI.

† Divide 5 times the versed sine of the half arc, by the difference between 5 times the diameter and 3 times the said versed sine; multiply the square-root of the quotient by 5 times the diameter; and to the product add 4 times the root of the product of the said versed sine and diameter; and $\frac{2}{9}$ of the sum will be the arc very nearly.

That is, if d be the diameter, and v the versed sine PD, then $5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv} \times \frac{2}{9}$ will be the arc ADB very near.

EX-

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† Again, to find another, still nearer, approximation, let there be introduced another indeterminate quantity; thus, suppose $d\sqrt{\frac{v}{gd-hv}} + k\sqrt{dv} = \sqrt{dv} \times (k + \frac{1}{\sqrt{g}} + \frac{hv}{2dg\sqrt{g}} + \frac{3hhvv}{2.4ddgg\sqrt{g}})$ = the semi-arc; then, comparing the first three terms with those of the given series, we have these three equations, $k + \frac{1}{\sqrt{g}} = 1$, $\frac{h}{g\sqrt{g}} = \frac{1}{3}$, and $\frac{hh}{gg\sqrt{g}} = \frac{1}{5}$; whence $g = \frac{81}{25}$, $h = \frac{241}{125}$, and $k = \frac{4}{9}$; and by writing these values in the assumed quantity, we obtain

$$\tfrac{5}{9}d\sqrt{\frac{v}{d-\frac{3}{5}v}} + \tfrac{4}{9}\sqrt{dv} = \tfrac{1}{9} \times (5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv})$$

for half the arc, the double of which will be rule 6.

EXAMPLE.

Taking, here, the ſame example as in the former rules, we ſhall have $5d\sqrt{\frac{5v}{5d-3v}}+4\sqrt{dv}\times\frac{2}{9}=6\cdot11706396$ $=$ the arc very nearly, it being true to the ſixth or ſeventh place of decimals.

PROBLEM VIII.

To find the Area of a Circle.

* RULE I.

Multiply half the circumference by half the diameter, and the product will be the area.

Note. This rule will, alſo, ſerve for any ſector of a circle.

RULE

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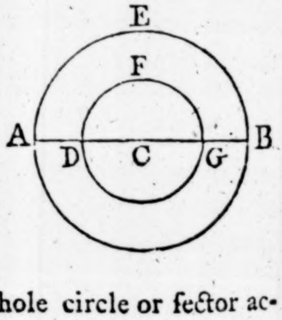
* DEMONSTRATION OF RULE I.

For, a circle may be conſidered as a polygon of an infinite number of ſides, the circumference being the perimeter, and the radius the perpendicular.

Otherwiſe.



Put $r =$ the radius AC, $c =$ the whole circumference AEBA, or any part of it, and $x =$ the radius CD of a circle continually expanded. Then $\frac{c}{r}x\dot{x}$ will expreſs the fluxion of the whole circle or ſector whoſe circumference is c; and, conſequently $\frac{cxx}{2r} =$ the area CDF; and $\frac{1}{2}cr =$ the area CAE of the whole circle or ſector accordingly. *Q. E. D.*

* RULE II.

Multiply the ſquare of the diameter by ·7854, and the product will be the area.

EXAMPLES.

1. What is the area of a circle whoſe diameter is 1, and circumference 3·1416?

$$\frac{3·1416 \times 1}{4} = ·7854 \text{ the area.}$$

Ex. 2. What is the area of a circle whoſe diameter is 7?

By the ſecond rule. $7 \times 7 \times ·7854 = 38·4846$ the area.

By the firſt rule. $7 \times 3·1416 =$ the circumference.

K Then

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* DEMONSTRATION.

To prove the ſecond rule, let it be firſt obſerved, that all circles, being ſimilar figures, are as the ſquares of their diameters or circumferences: which, alſo, appears thus; by rule 1, circles are as the rectangles of their diameters and circumferences, but the circumference is as the diameter, by the notes to the laſt problem; therefore the circles are as the ſquares of either of them. Then, by rule 1, the area of a circle whoſe diameter is 1, is $\frac{1 \times 3·14159 \text{ \&c}}{2 \times 2 = 4} = ·78539$ &c, whence $1^2 : d^2$ (the ſquare of any diameter) :: ·78539 &c : $·78539d^2$, the circle whoſe diameter is d. *Q.E.D.*

COROLLARY.

If D be the diameter, C the circumference, and A the area of any circle; and $p = 3·14159$ &c; from this prob. and prob. 6, we may deduce theſe equations:

1. $\text{D} = \frac{\text{C}}{p} = \frac{4\text{A}}{\text{C}} = 2\sqrt{\frac{\text{A}}{p}}$.

2. $\text{C} = p\text{D} = \frac{4\text{A}}{\text{D}} = 2\sqrt{p\text{A}}$.

3. $\text{A} = \frac{p\text{D}^2}{4} = \frac{\text{C}^2}{4p} = \frac{\text{DC}}{4}$.

4. $p = \frac{\text{C}}{\text{D}} = \frac{4\text{A}}{\text{DD}} = \frac{\text{CC}}{4\text{A}}$.

Then $\frac{7 \times 3 \cdot 1416 \times 7}{4} = 7 \times 7 \times \cdot 7854$ the area, the ſame as before.

Ex. 3. What is the area of the circle whoſe diameter is 1·13 chains, and circumference 3·550008 chains? Anſ. 1·00287726 ſquare chains.

Ex. 4. How many ſquare yards are in a circle whoſe diameter is $3\frac{1}{2}$ feet? Anſ. $1 \cdot 06901\frac{2}{3}$.

Ex. 5. How many ſquare feet are in a circle whoſe circumference is 10·9956 yards?

Anſ. 86·19266.

PROBLEM IX.

To find the Area of any Sector of a Circle.

RULE I.

Multiply the radius, or half the diameter, by half the arc of the ſector, and the product will be the area, as in the whole circle.*

EXAMPLE I.

What is the area of a ſector whoſe radius is 10 and arc 20?

By the rule, $\frac{20 \times 10}{2} = 100$ the area.

EX-

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* The demonſtration of this rule is contained in that of the laſt problem.

Putting $r =$ the radius of a circle, $d =$ the diameter, $A =$ the area of a ſector of it, $a =$ the length of the arc of the ſector, $b =$ the degrees in $\frac{1}{2}a$, $s =$ half the chord of the arc a, or the ſine of $\frac{1}{2}a$, and $v =$ the verſed ſine of $\frac{1}{2}a$; then, by multiplying the radius by half the arc, as found by the ſeveral rules of the laſt problem but one, we ſhall obtain the following values of the area of the ſector.

1. A

EXAMPLE II.

To find the area of the ſector whoſe radius is 9, and the chord of its arc 6.

By the 7th problem the length of the arc is 6·117063.

Therefore $\frac{6{\cdot}117063 \times 9}{2} = 27{\cdot}52667835$ the area required.

RULE II.

As 360 is to the degrees in the arc of the ſector, ſo is the whole area of the circle, to the area of the ſector.

Note. For a ſemicircle take one half, for a quadrant one quarter, &c, of the whole circle.*

EXAMPLE I.

What is the area of a ſemicircle, and of a quadrant, the diameter being 10?

 Firſt,

---

1. $A = \frac{1}{2}ar = {\cdot}01745329\,brr.$

2. $A = r\sqrt{dv} \times (1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3}$ &c).

3. $A = rs \times (1 + \frac{s^2}{2.3r^2} + \frac{3s^4}{2.4.5r^4} + \frac{3.5s^6}{2.4.6.7r^6}$ &c).

4. $A = \frac{4\sqrt{ss+vv}-s}{3}r = \frac{4\sqrt{2rv}-s}{3}r = \frac{4\sqrt{ss+vv}-s}{3} \times \frac{ss+vv}{2v}$, nearly.

5. $A = rd\sqrt{\frac{3v}{3d-v}}$, nearly.

6. $A = \frac{r}{9} \times (5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv})$ nearly.

It is evident that the area of the ſector might be expreſſed in ſeveral other manners; ſuch as by the tangent, coſine, &c, of its ſemi-arc; but the forms above given, are the moſt uſeful ones.

* This rule is too evident to need any formal proof.

First, $\cdot7854 \times 10 \times 10 = 78\cdot54 =$ the whole circle.
Then $78\cdot54 \div 2 = 39\cdot27 =$ the semicircle.
And $78\cdot54 \div 4 = 19\cdot635 =$ the quadrant.

EXAMPLE II.

To find the area of a sector whose arc or angle contains 18 degrees, to a diameter of 3 feet?

First, $\cdot7854 \times 3 \times 3 = 7\cdot0686 =$ the whole circle.

Then $360 : 18 :: 7\cdot0686 : \frac{7\cdot0686 \times 18}{360} = \frac{7\cdot0686}{20} =$ $\cdot35343$ the area of the sector required.

EXAMPLE III.

To find the area of a sector whose radius is 9, and the chord of its arc 6.

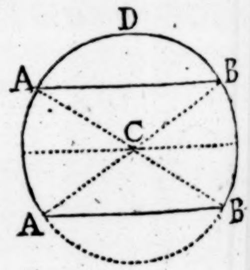
By the example to rule 1 of prob. 7, the degrees in the arc were found to be $38\cdot9424412$. But $\cdot785398 \times 9 \times 9 \times 4 =$ the area of the circle.

Therefore $360 : 38\cdot9424412 :: \cdot785398 \times 9 \times 9 \times 4 :$ $\frac{38\cdot9424412 \times \cdot785398 \times 9 \times 9 \times 4}{360} = 38\cdot9424412 \times \cdot0785398$ $\times 9 = 27\cdot526678388$ the area of the sector required.

PROBLEM X.

To find the Area of the Segment of a Circle.

RULE I.



By the last problem, find the area of the sector ACB having the same arc with the segment ADB required.

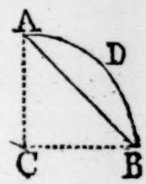
Find also the area of the triangle ABC, formed by the chord of the segment and the two radii of the sector.

Then the sum or difference of these areas will be that

that of the segment, according as it is greater or less than a semicircle.

EXAMPLE I.

What is the area of a segment whose arc is a quadrant, or contains 90 degrees, the diameter being 18 feet?



First, $\cdot 7854 \times 18 \times 18$ is the whole circle.

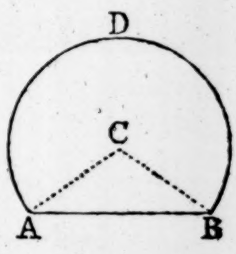
And $\frac{\cdot 7854 \times 18 \times 18}{4} = \cdot 7854 \times 9 \times 9 =$ the quadrant or sector CADB.

But $\frac{AC \times CB}{2} = \frac{9 \times 9}{2} = \cdot 5 \times 9 \times 9 =$ the triangle ACB.

Therefore $\cdot 7854 \times 9 \times 9 - \cdot 5 \times 9 \times 9 = \cdot 2854 \times 9 \times 9 = 23\cdot 1174 =$ the area of the segment required.

EXAMPLE II.

What is the area of a segment of a circle, whose arc contains 280°, the diameter being 50?



First, $\cdot 7854 \times 50 \times 50 = 1963\cdot 5 =$ the whole circle.

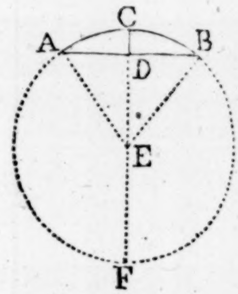
And $360 : 280 :: 1963\cdot 5 : \frac{1963\cdot 5 \times 280}{360} = \frac{1963\cdot 5 \times 7}{9} = 1527\cdot 16666 =$ the sector CADBC.

Again, the angle $ACB = 360 - 280 = 80°$. And the triangle $ACB = AC \times CB \times \frac{1}{2}$ sine $\angle ACB = 25 \times 25 \times \frac{1}{2}$ s. of $80° = 25 \times 25 \times \cdot 4924039 = 307\cdot 7524375$.

Wherefore the sum $1834\cdot 9191 =$ the segment ADBA required.

EXAMPLE III.

Required the area of the ſegment whoſe chord is 20, and height, or verſed ſine of half its arc, 2.



Since $AD^2 = DC \times DF$, therefore $DF = \frac{AD^2}{DC} = \frac{100}{2} = 50$. And the diameter $CF = FD + DC = 50 + 2 = 52$. Alſo $DE = EC - CD = \frac{1}{2}CF - CD = 26 - 2 = 24$.

Therefore the triangle $ABE = AD \times DE = 10 \times 24 = 240$.

Again, by trigonometry, $26 = AE : 10 = AD :: 1 = s. \angle D : \frac{10}{26} = \frac{5}{13} = \cdot 38461538 = s. \angle AED$, or of the arc $AC = 22\cdot6198614$ degrees; the double of which, or $45\cdot2397228$, are the degrees in the arc ACB.

But $\cdot785398 \times 52 \times 52 =$ the whole circle.

Therefore $360 : 45\cdot2397228 :: \cdot785398 \times 52 \times 52 : \frac{45\cdot2397228 \times \cdot785398 \times 26 \times 26}{90} = 266\cdot8786995 =$ the ſector AEB.

Whence $266\cdot8786995 - 240 = 26\cdot8786995 =$ the ſegment ACBA required.

Otherwiſe.

Having found the triangle $= 240$, as above; we ſhall have, by rule 6 of the laſt problem, the ſector $= \left(\frac{5 \times 52}{4}\sqrt{\frac{5 \times 2}{5 \times 52 - 3 \times 2}} + \sqrt{52 \times 2}\right) \times \frac{2 \times 52}{9} = \left(65\sqrt{\frac{5}{127}} + 2\sqrt{26}\right) \times \frac{104}{9} = \left(\frac{65}{127}\sqrt{635} + 2\sqrt{26}\right) \times \frac{104}{9} = 266\cdot87868$.

Then the difference is $26\cdot87868 =$ the ſegment, nearly the ſame as before.

RULE

RULE II.

Let d be the diameter, and v the versed sine, or the height of the segment;

Then $2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2} - \frac{v^3}{72d^3}$ &c), or $2v\sqrt{dv} \times (\frac{2}{3} - \frac{3v}{5.2d}\text{A} - \frac{5v}{7.4d}\text{B} - \frac{7.3v}{9.6d}\text{C} - \frac{9.5v}{11.8d}\text{D}$ &c) = the segment; A, B, C, &c, being the first, second, third, &c, terms*.

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* DEMONSTRATION.

By the last problem, the value of the sector CADB (fig. in page 132) is $\frac{1}{2}d\sqrt{dv} \times (1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3}$ &c). But $\pm \frac{1}{2}d \mp v =$ the altitude of the triangle ABC, and its base $\text{AB} = 2\sqrt{dv - vv} = 2\sqrt{dv} \times (1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6.d^3}$ &c); and therefore the area of the triangle ABC $= (\pm \frac{1}{2}d \mp v) \times \sqrt{dv} \times (1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6d^3}$ &c); which being added to, or subtracted from, the sector, gives

$$2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{4.7d^2} - \frac{3v^3}{4.6.9d^3} - \frac{3.5v^4}{4.6.8.11d^4} \text{ \&c})$$

for the value of the segment.

Otherwise.

Putting $y =$ the semi-chord of the segment, and the other letters as before. We have the fluxion of the segment $y\dot{v} = \dot{v}\sqrt{dv - vv} = \dot{v}\sqrt{dv} \times (1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6d^3}$ &c); the fluent of which gives $v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{4.7d^2} - \frac{3v^3}{4.6.9d^3}$ &c) for the value of half the segment, as before. *Q. E. D.*

Corol.

EXAMPLE.

Take here the last example, in which $d = 52$, and $v = 2$; then $\frac{v}{d} = \frac{1}{26}$; and the process will be thus:

$$+ \text{A} = \frac{4v\sqrt{dv}}{3} = \tfrac{8}{3}\sqrt{104} = + \quad 27{\cdot}1947707$$

$$- \text{B} = \frac{3v}{5.2d}\text{A} = \frac{3}{10.26}\text{A} = - \quad {\cdot}3137858$$

$$- \text{C} = \frac{5v}{7.4d}\text{B} = \frac{5}{7.4.26}\text{B} = - \quad {\cdot}0021551$$

$$- \text{D} = \frac{7.3v}{9.6d}\text{C} = \frac{7}{3.6.26}\text{C} = - \quad {\cdot}0000322$$

$$- \text{E} = \frac{9.5v}{11.8d}\text{D} = \frac{9.5}{11.8.26}\text{D} = - \quad {\cdot}0000007$$

the sum of the negative terms $-$ ·3159738

taken from the first term, leave 26·8787969 $=$ the area required.

RULE

Corol. 1. Let the chord of half the arc of the segment be denoted by c; then $d = \frac{cc}{v}$, by which exterminating d out of the above series, we have $cv \times (\frac{2}{3} - \frac{v^2}{5c^2} - \frac{v^4}{4.7c^4} - \frac{3v^6}{4.6.9c^6} - \frac{3.5v^8}{4.6.8.11c^8}$ &c) for the value of the semi-segment.

Corol. 2. Let the supplemental versed sine be denoted by v, viz. $\text{v} = d - v$; then $d = \text{v} + v$, by which exterminating d out of the same series, we have

$$2v\sqrt{\text{v}v} \times \left(\frac{1}{1.1.3} + \frac{v}{1.3.5\text{v}} - \frac{v^2}{3.5.7\text{v}^2} + \frac{v^3}{5.7.9\text{v}^3} - \frac{v^4}{7.9.11\text{v}^4} \text{ \&c}\right)$$

for the value of the said semi-segment. And this series, which is rule 3, converges very quickly in small segments.

Corol. 3. By supposing the several indeterminate quantities in these series to be expressed by any given numbers, we may obtain an endless variety of infinite series, whose sums will be given. And, in particular, if the diameter be 1, and supposing the versed sines to be equal to the radius, or the diameter, we shall obtain the following series; in which n denotes ·785398 &c, the

RULE III.

If v denote the versed sine, as before, and v the versed sine of the supplement, or $\mathrm{v} = d - v$; then $\frac{4}{3}v\sqrt{v\mathrm{v}} + \frac{v}{5\mathrm{v}}\mathrm{A} - \frac{1v}{7\mathrm{v}}\mathrm{B} + \frac{3v}{9\mathrm{v}}\mathrm{C} - \frac{5v}{11\mathrm{v}}\mathrm{D}$ &c, will be = the segment; where A, B, C, &c, denote the preceding terms.

EXAMPLE.

Taking, again, the same example; we have $v = 2$, and $\mathrm{v} = 52 - 2 = 50$. Hence $\frac{v}{\mathrm{v}} = \frac{2}{50} = \cdot04$, and

$$\begin{array}{llllr}
\mathrm{A} & = & \frac{4}{3}v\sqrt{v\mathrm{v}} & = + & 26\cdot66666666\frac{2}{3} \\
\mathrm{B} & = & \frac{v}{5\mathrm{v}}\mathrm{A} & = + & 21333333\frac{1}{3} \\
\mathrm{C} & = & \frac{v}{7\mathrm{v}}\mathrm{B} & = - & 121904\frac{16}{21} \\
\mathrm{D} & = & \frac{3v}{9\mathrm{v}}\mathrm{C} & = + & 1625\frac{8}{21} \\
\mathrm{E} & = & \frac{5v}{11\mathrm{v}}\mathrm{D} & = - & 29\frac{12}{21} \\
\mathrm{F} & = & \frac{7v}{13\mathrm{v}}\mathrm{E} & = + & \frac{13}{21}
\end{array}$$

sum of the affir. terms $+\ 26\cdot88001626$
sum of the negat. terms $-\ 0\cdot00121934\frac{1}{3}$

the difference is $26\cdot87879691\frac{2}{3}$ the area.

RULE

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the area of the whole circle whose diameter is 1, or $\frac{1}{4}$ of the circumference of the same circle.

1. $n = \sqrt{2} \times (\frac{2}{3} - \frac{1}{5\cdot2} - \frac{1}{4\cdot7\cdot2^2} - \frac{1\cdot3}{4\cdot6\cdot9\cdot2^3} - \frac{1\cdot3\cdot5}{4\cdot6\cdot8\cdot11\cdot2^4}$ &c).

2. $n = 2 \times (\frac{2}{3} - \frac{1}{5} - \frac{1}{4\cdot7} - \frac{1\cdot3}{4\cdot6\cdot9} - \frac{1\cdot3\cdot5}{4\cdot6\cdot8\cdot11}$ &c).

3. $n = 2 \times (\frac{1}{1\cdot1\cdot3} + \frac{1}{1\cdot3\cdot5} - \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} - \frac{1}{7\cdot9\cdot11} +$ &c).

Several other values of n may be assigned from the corollaries in pages 53 and 54, and from pages 118, &c.

RULE IV.*

If c denote the cosine of half the arc of the segment, or the difference between the radius and versed sine, the radius being r, and $q = \frac{c}{r}$; then the area of the segment will be equal to the difference between the semicircle and this series, viz.

$$2rc \times (1 - \frac{1}{2.3}q^2 - \frac{1}{2.4.5}q^4 - \frac{1.3}{2.4.6.7}q^6 - \frac{1.3.5}{2.4.6.8.9}q^8 \text{ \&c});$$

$$\text{or } 2rc - \frac{1}{2.3}q^2 A - \frac{1.3}{4.5}q^2 B - \frac{3.5}{6.7}q^2 C - \frac{5.7}{8.9}q^2 D \text{ \&c}:$$

where A, B, C, &c, are the 1st, 2d, 3d, &c, terms.

And this rule will be proper to use when the segment is large.

EX-

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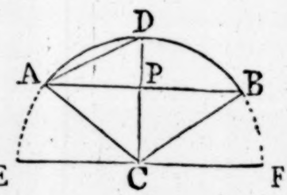
* DEMONSTRATION.



Put $x = CP$, and $r = CD$ or CA, also $AP = y$. Then $r^2 = x^2 + y^2$, and $y = \sqrt{r^2 - x^2}$; and the fluxion of the area ABFE, or $2y\dot{x}$, is $2\dot{x}\sqrt{r^2 - x^2} =$

$$2r\dot{x} \times (1 - \frac{1}{2} . \frac{x^2}{r^2} - \frac{1}{2.4} . \frac{x^4}{r^4} - \frac{1.3}{2.4.6} . \frac{x^6}{r^6} \text{ \&c});$$ and the fluent

is $2rx \times (1 - \frac{1}{2.3} . \frac{x^2}{r^2} - \frac{1}{2.4.5} . \frac{x^4}{r^4} - \frac{1.3}{2.4.6.7} . \frac{x^6}{r^6}$ &c); which is the area of the zone ABFE; and this being taken from the semicircle EADBF, will leave the area of the segment ADB, as in the rule. And this will converge very fast when x is small, or the segment large.

Corol. When x becomes equal to r, the zone becomes a semicircle, and then $2r^2 \times (1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7}$ &c) is the area of the semicircle, to the radius r; and consequently the sum of the series is

$$n = 1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7} - \frac{1.3.5}{2.4.6.8.9} - \text{\&c.}$$

where $n = .78539$ &c, the area of the circle whose diameter is 1; being one more value of n, to be added to those found in page 137.

EXAMPLE.

Suppose the versed sine be 24, or the cosine 2; the radius being 26, as before.

Here $r = 26$, and $c = 2$; then $q = \frac{c}{r} = \frac{1}{13}$, & $q^2 = \frac{1}{169}$.

Hence $+ \text{A} = 2rc \quad = + 104{\cdot}0000000$

$- \text{B} = \frac{1}{2.3} q^2 \text{A} = - \quad 0{\cdot}1025642$

$- \text{C} = \frac{1.3}{4.5} q^2 \text{B} = - \quad 910$

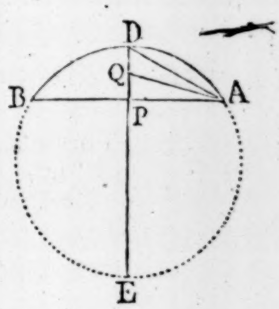
$- \text{D} = \frac{3.5}{4.7} q^2 \text{C} = - \quad 2$

the sum of the neg. terms — 0·1026554
taken from the first, leaves 103·0973446 for the series;
this taken from the semicir. 1061·8494892

leaves the area of the seg. 958·7521446

RULE V.

To the chord of the whole arc, add $\frac{4}{3}$ of the chord of half the arc, multiply the sum by the versed sine, or height of the segment, and $\frac{4}{10}$ of the product will be the area of the segment nearly.



That is, $(\text{c} + \frac{4}{3}c) \times \frac{4}{10}v$, or $(\text{c} + \frac{4}{3}\sqrt{dv}) \times \frac{4}{10}v$, or $(\text{c} + \frac{4}{3}\sqrt{\frac{1}{4}\text{c}^2 + v^2}) \times \frac{4}{10}v$, or $(\sqrt{dv - vv} + \frac{2}{3}\sqrt{dv}) \times \frac{4}{5}v =$ the area nearly: where $d = \text{DE}$ the diameter, $\text{c} = \text{BA}$ the chord of the whole arc, $c = \text{AD}$ the chord of the half arc, and $v = \text{DP}$ the height of the segment.*

EX-

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* DEMONSTRATION.

The segment is $= 2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v}{28d}$ &c) by rule 3.

Suppose

EXAMPLE.

Taking the ſame example as in the firſt three rules, in which $\mathrm{c} = 20$, and $v = 2$.

Then $(\mathrm{c} + \frac{4}{3}\sqrt{\frac{1}{4}\mathrm{cc} + vv}) \times \frac{4}{10}v = 26{\cdot}877908 =$ the area, nearly the ſame as before.

RULE VI.

Draw AQ ſo, that DP : PQ :: $5 : \sqrt{10}$, then the ſegment ADBA will be nearly equal to $\frac{4}{3}\mathrm{DP} \times \mathrm{AQ} = \frac{4}{3}v\sqrt{dv - \frac{3}{5}vv} = \frac{4}{3}v\sqrt{cc - \frac{3}{5}vv} = \frac{4}{3}v\sqrt{\frac{1}{4}\mathrm{cc} + \frac{2}{5}vv}$. Where $d =$ the diameter, $v =$ DP, $c =$ DA, and $\mathrm{c} =$ AB, as in the laſt rule.*

EX-

Suppoſe it $= 2\mathrm{DP} \times (m \times \mathrm{PA} + n \times \mathrm{AD}) = 2v \times (m\sqrt{dv - vv} + n\sqrt{dv})$ $= 2v\sqrt{dv} \times (m\sqrt{1 - \frac{v}{d}} + n) = 2v\sqrt{dv} \times (\begin{smallmatrix}+m\\+n\end{smallmatrix} - \frac{mv}{2d} - \frac{mv^2}{8d^2}$ &c)

Then let the coefficients of the correſponding terms be equated, and we ſhall have $m + n = \frac{2}{3}$, and $\frac{m}{2} = \frac{1}{5}$; whence $m = \frac{2}{5}$, and $n = \frac{2}{3} - m = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$.

Then, by ſubſtituting theſe values of m and n in the aſſumed quantity, we ſhall have $2\mathrm{DP} \times (m \times \mathrm{PA} + n \times \mathrm{AD}) = 2\mathrm{DP} \times (\frac{2}{5}\mathrm{PA} + \frac{4}{15}\mathrm{AD}) = \frac{4}{10}\mathrm{DP} \times (2\mathrm{PA} + \frac{4}{3}\mathrm{AD})$, which is the rule. Or it is $= \frac{4}{15}\mathrm{DP} \times (3\mathrm{PA} + 2\mathrm{AD})$. And this is one of Sir I. NEWTON's rules, publiſhed by JONES, WALLIS, and COLSON.

* DEMONSTRATION OF RULE VI.

By rule 2 the ſegment is $= 2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2}$ &c).

Suppoſe it equal to $2v\sqrt{mdv - nvv} = 2v\sqrt{dv} \times \sqrt{m - \frac{nv}{d}} =$ $2v\sqrt{dv} \times (m^{\frac{1}{2}} - \frac{nv}{2m^{\frac{1}{2}}d} - \frac{n^2v^2}{8m^{\frac{3}{2}}d^2}$ &c). Then, by equating the

coefficients,

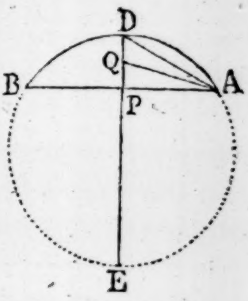
EXAMPLE.

Taking, again, the same example;

We have $\frac{4}{3}v\sqrt{\frac{1}{4}cc+\frac{2}{5}vv}=\frac{8}{3}\sqrt{101\frac{3}{5}}=26\cdot87915$ = the area nearly.

RULE VII.

If AQ bisect DP, the segment ADBA will be very nearly equal to $(4QA+AD)\times\frac{4}{15}DP=(2\sqrt{cc+vv}+c)\times\frac{4}{15}v=(2\sqrt{4dv-3vv}+\sqrt{dv})\times\frac{4}{15}v=(2\sqrt{cc+vv}+\sqrt{\frac{1}{4}cc+vv})\times\frac{4}{15}v$; where the letters denote the same quantities as in the last rule.*



EX-

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coefficients, we have $\sqrt{m}=\frac{2}{3}$, and $\frac{n}{\sqrt{m}}=\frac{2}{5}$; whence $m=\frac{4}{9}$, and $n=\frac{4}{15}$.

Then, by substituting these values, we get $2v\sqrt{mdv-nvv}=2v\sqrt{\frac{4}{9}dv-\frac{4}{15}vv}=\frac{4}{3}v\sqrt{dv-\frac{3}{5}vv}$, as in the rule.

Corol. But, now, to find what line the quantity $\sqrt{dv-\frac{3}{5}vv}$, represents; I consider that it is greater than AP, which is $=\sqrt{dv-vv}$, and therefore I suppose it to be some line AQ meeting DP between D and P.

Then $PQ=\sqrt{QA^2-AP^2}=\sqrt{dv-\frac{3}{5}vv-dv+vv}=v\sqrt{\frac{2}{5}}=\frac{v\sqrt{10}}{5}$.

Wherefore if there be made DP : PQ :: 5 : $\sqrt{10}$, and QA be drawn; the segment will be nearly $\frac{4}{3}$ DP × AQ. And this is another of Sir I. NEWTON's Rules.

* DEMONSTRATION OF RULE VII.

Suppose the segment $2v\sqrt{dv}\times(\frac{2}{3}-\frac{v}{5d}$&c) to be = 2 PD ×

(m

EXAMPLE.

Taking, ſtill, the ſame example;

We ſhall have $(2\sqrt{cc+vv}+\sqrt{\frac{1}{4}cc+vv}) \times \frac{4}{15}v = 26\cdot8786888$ = the area very nearly.

RULE VIII.

From the ſquare of the chord of half the arc, ſubtract $\frac{5}{7}$ of the ſquare of the verſed ſine of the ſaid half arc; to 7 times the root of the difference, add $\frac{4}{3}$ of the ſaid chord; multiply the ſum by the ſaid verſed ſine, and $\frac{4}{25}$ of the product will be the area very near.

That is, $(7\sqrt{cc-\frac{5}{7}vv}+\frac{4}{3}c)\times\frac{4}{25}v = (7\sqrt{dv-\frac{5}{7}vv}+\frac{4}{3}\sqrt{dv})\times\frac{4}{25}v = (7\sqrt{\frac{1}{4}cc+\frac{2}{7}vv}+\frac{4}{3}\sqrt{\frac{1}{4}cc+vv})\times\frac{4}{25}v$ = the area very near: where the letters are the ſame as in the laſt rules.*

EX-

$(m\times QA+n\times AD) = 2v\times(m\sqrt{dv-\frac{3}{4}vv}+n\sqrt{dv}) = 2v\sqrt{dv}\times(m\sqrt{1-\frac{3v}{4d}}+n) = 2v\sqrt{dv}\times(\begin{smallmatrix}+m\\+n\end{smallmatrix}-\frac{3mv}{8d}$ &c).

Then, equating the coefficients, we get $m+n=\frac{2}{3}$, and $\frac{3m}{8}=\frac{1}{5}$; whence $m=\frac{8}{15}$, and $n=\frac{2}{15}$.

Therefore $2PD\times(m\times QA+n\times AD) = 2PD\times(\frac{8}{15}QA+\frac{2}{15}AD) = \frac{4}{15}PD\times(4QA+AD)$, as in the rule. And this likewiſe is a rule given by Sir I. NEWTON.

* DEMONSTRATION.

The ſegment is $= 2v\sqrt{dv}\times(\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}$ &c).

Suppoſe it $= 2v\times(m\sqrt{dv-nvv}+p\sqrt{dv}) = 2v\sqrt{dv}\times(p+m\sqrt{1-\frac{nv}{d}}) = 2v\sqrt{dv}\times(\begin{smallmatrix}+p\\+m\end{smallmatrix}-\frac{mnv}{2d}-\frac{mn^2v^2}{8d^2}$ &c).

Then, equating the like coefficients, we have $m+p=\frac{2}{3}$, $\frac{mn}{2}=\frac{1}{5}$, and $\frac{mnn}{8}=\frac{1}{28}$; whence $n=\frac{5}{7}$, $m=\frac{14}{25}$, and $p=\frac{8}{75}$.

Wherefore

EXAMPLE.

Taking, again, the same example as before, where $c = 20$, and $v = 2$;

We have $(7\sqrt{\frac{1}{4}cc + \frac{2}{7}vv} + \frac{4}{3}\sqrt{\frac{1}{4}cc + vv}) \times \frac{4}{25}v$ $= 26{\cdot}8787998 =$ the area very exact.

RULE IX.

1. Divide the versed sine, or height, by the diameter.

2. Find the quotient in the column of versed sines in the table of circular segments, at the end of the volume.

3. Multiply the number immediately on the right of the versed sine, in the table, by the square of the diameter, and the product will be the area.*

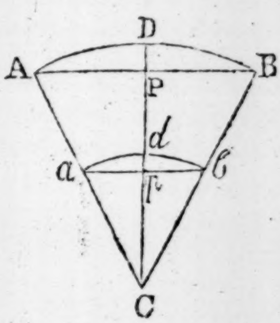
Note.

Wherefore $2v \times (m\sqrt{dv - nvv} + p\sqrt{dv}) = \frac{4}{25}v \times 7\sqrt{dv - \frac{5}{7}vv} + \frac{4}{3}\sqrt{dv})$, as in the rule.

Corol. Suppose $\sqrt{dv - \frac{5}{7}vv} =$ AQ [*see last fig.*]; then PQ $= \sqrt{QA^2 - AP^2} = \sqrt{dv - \frac{5}{7}vv - dv + vv} = v\sqrt{\frac{2}{7}} = \frac{v\sqrt{14}}{7}$.

Wherefore if AQ divide DP in Q, so that $7 : \sqrt{14} ::$ DP : PQ: then (7 QA + $\frac{4}{3}$ AD) $\times \frac{4}{25}$ DP will be nearly equal to the segment ADBA. And this rule is not only new and very simple, but also the most exact of any approximation that I know of for this purpose.

* DEMONSTRATION.



The rule is founded on this property, That segments whose versed sines are as the diameters, will be to each other as the squares of the diameters; which is thus proved.

Let ADBA, *adba*, be two similar segments, cut off from the similar sectors ADBCA, *adbca*, by the chords AB, *ab*. And draw CD to bisect them.

Then, by similar triangles, CA : C*a* :: CA − DP (CP) : C*a* − *dp* (C*p*) :: DP : *dp*.

4 That

Note. When the quotient, arifing from the verfed fine divided by the diameter, has a remainder, or fraction, after the fourth place of decimals; having taken the area anfwering to the faid four firft figures, fubtract it from the next following area, multiply the remainder by the faid fraction, and add the product to the firft area; and the fum will be the area for the whole quotient.—This method is ufed when accuracy is required; but for common ufe, the fraction may be omitted.

EXAMPLE.

If the diameter be 52, and the verfed fine 2; what is the area?

Firft $\frac{2}{52} = \frac{1}{26} = \cdot 0384\frac{8}{13}$, the tabular verfed fine.

Then againft ·0384 ftands ·00991672, and the difference between this area and the next ·00995517 is ·00003845, which multiplied by $\frac{8}{13}$ produces ·00002366, which added to ·00991672 gives ·00994038 = the area belonging to the verfed fine $\cdot 0384\frac{8}{13}$.

Wherefore $\cdot 00994038 \times 52 \times 52 = 26\cdot 87878752$ = the area, nearly the fame as before.

PRO-

That is, fimilar fegments have their verfed fines as their radii or diameters.

Again, fince fimilar fectors are as the fquares of the diameters, as well as fimilar triangles as the fquares of their like fides, we fhall have $CA^2 : ca^2$ :: fector CADB : fector *cadb* :: $\triangle$ CAB : $\triangle$ *cab* :: fegment ADBA (fector CADB $-$ $\triangle$CAB) : fegment *adba* (fector *cadb* $-$ $\triangle$ *cab*).

Now, putting $d =$ any diameter, and $v =$ verfed fine. We fhall have $d : v :: 1$ (diameter in the table) : $\frac{v}{d} =$ the verfed fine of a fimilar fegment in the table, whofe area let be called a.

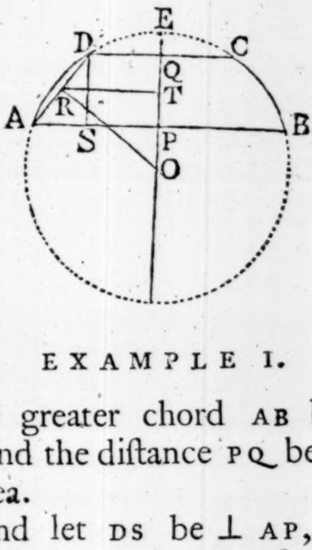
Then $1^2 : dd :: a : add =$ the area of the fegment whofe height is v and diameter d, as in the rule.

PROBLEM XI.

To find the Area of the Zone or Space ABCDA *included between two parallel Chords* AB, CD, *and the two Arcs* AD, CB.

RULE I.

Find the area of each segment AEBA, DECD, and their difference will be that of the zone required.



EXAMPLE I.

Suppose the greater chord AB be 96; the less chord DC 60, and the distance PQ between them 26; required the area.

Draw AD, and let DS be $\perp$ AP, OR $\perp$ AD, and RT $\perp$ PQ; then QT = TP, because AR = RD; and TR $= \frac{1}{2}$AP $+ \frac{1}{2}$DQ. But, in the similar triangles DAS, ROT, it will be DS : AS :: RT : TO $= \frac{RT \times AS}{DS} = \frac{AP + DQ \times AP - DQ}{2PQ} = \frac{78 \times 18}{52} = 27$. And hence OP = OT − TP = 27 − 13 = 14.

Then OE = OA $= \sqrt{AP^2 + PO^2} = \sqrt{48^2 + 14^2} =$ 50 = the radius.

Whence PE = OE − OP = 50 − 14 = 36
QE = OE − OQ = 50 − 40 = 10 the two versed sines.

 And

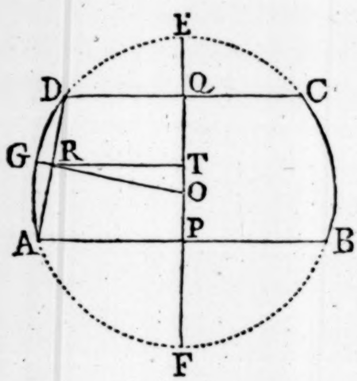
And $\frac{36}{100} = \cdot 36$
$\frac{10}{100} = \cdot 10$ the two tab. verſed ſines. Againſt which, in the table, ſtand the areas ·25455055, and ·04087528.

$$\text{Then } \cdot 25455055 \times 100 \times 100 = 2545\cdot 5055$$
$$\cdot 04087528 \times 100 \times 100 = 408\cdot 7528$$

their difference 2136·7527 is the area of the zone required.

EXAMPLE II.

Suppoſe the greater chord AB to be 20, the leſs DC 15, and their diſtance PQ $17\frac{1}{2}$. Required the area of the zone ABCD.



Here, as in the laſt example,

$$TO = \frac{\overline{PA+DQ} \times \overline{AP-DQ}}{2PQ} = \frac{35 \times 5}{2 \times 2 \times 35} = \frac{5}{4}.$$

Which being leſs than TP ($8\frac{3}{4}$) half the diſtance of the chords, it ſhews that the center falls within the zone; and then it is evident that the whole circle diminiſhed by the ſum of the two ſegments AFB, DEC, will give the zone.

Now, $OP = TP - TO = 8\frac{3}{4} - \frac{5}{4} = 7\frac{1}{2}$. Hence $AO = \sqrt{OP^2 + PA^2} = \sqrt{\left(\frac{15}{2}\right)^2 + 10^2} = \frac{25}{2} =$ the radius; therefore the diameter is 25. And $OQ = QT + TO = 8\frac{3}{4} + 1\frac{1}{4} = 10$.

Whence

Whence $QE = OE - OQ = 12\frac{1}{2} - 10 = 2\frac{1}{2}$,
$PF = OF - OP = 12\frac{1}{2} - 7\frac{1}{2} = 5$.

Then $\frac{2\frac{1}{2}}{25} = \frac{5}{50} = \cdot 1$,

and $\frac{5}{25} = \frac{1}{5} = \cdot 2$ the tab. verſed ſines.

Againſt which ſtand the areas $\cdot 04087528$,
and $\cdot 11182380$.

Then $\cdot 04087528 \times 25 \times 25 =$ the ſegment DEC,
$\cdot 11182380 \times 25 \times 25 =$ the ſegment AFB,

whoſe ſum is $\cdot 15269908 \times 25^2$.
But $\cdot 78539816 \times 25^2 =$ the whole circle.
Their dif. is $\cdot 62369908 \times 25^2 = 395\cdot 436925$ the zone ABCD required.

RULE II.

To the area of the trapezoid ADQP, add that of the ſmall ſegment AGDKA, and double the ſum will be the area of the zone ABCD required.—And this rule is eaſier than the other, as only one ſegment is to be found.

EXAMPLE.

Suppoſe the ſame dimenſions as in the laſt example.

Then, drawing the radius ORG, we ſhall have $OR = \sqrt{OT^2 + TR^2} = \sqrt{(\frac{1}{2}DQ + \frac{1}{2}PA)^2 + OT^2} = \sqrt{(\frac{35}{4})^2 - (\frac{5}{4})^2}$
$= \frac{5}{4}\sqrt{1^2 + 7^2} = \frac{5}{4}\sqrt{50} = \frac{25}{4}\sqrt{2} = 8\cdot 838834765$.
Hence $GR = OG - OR = 12\cdot 5 - 8\cdot 838834765 = 3\cdot 661165$ the verſed ſine.

Therefore $\frac{3\cdot 661165}{25} = \cdot 1464466$ the tabular verſed ſine; the tabular area for which is $\cdot 07134954$.

 Then

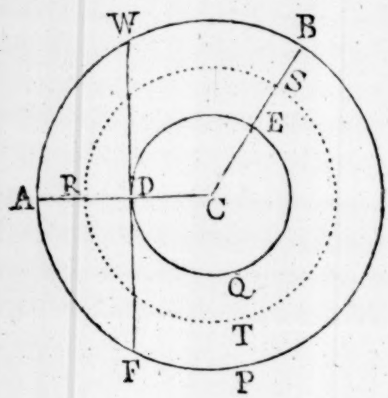
Then $\cdot 07134954 \times 25 \times 25 = 44\cdot 5934625$, the segment AGDRA. But

$(\frac{1}{2}DQ + \frac{1}{2}AP) \times PQ = \frac{35}{4} \times \frac{35}{2} = \frac{1225}{8} = 153\cdot 125$, the trapezoid APQD.

Their sum is . - - - - - - - 197·7184625, the semi-seg. APQDGA. The double 395·4369250, is the zone ABCD, the same as before.

PROBLEM XII.

To find the Area of the Ring included between the Circumferences ABPA, DEQD *of two concentric Circles.*



RULE I.

Multiply the sum of the diameters by the difference of the diameters, and the product again by ·7854, for the area of the ring required.*

EX-

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* DEMONSTRATION.

For, since the ring is evidently equal to the difference of the two circles, if the diameters be called D, d, and ·785398 &c. $= a$; the ring will be $= aD^2 - ad^2 = a \times (D + d) \times (D - d)$. *Q. E. D.*

Corol. 1. Whence, if there be made DW perpendicular to the radius CDA, the ring will be equal to the circle whose radius is DW;

EXAMPLE.

If the diameters of two concentric circles be 10 and 6, what will be the area of the ring included between their circumferences?

$\cdot 7854 \times (10+6) \times (10-6) = \cdot 7854 \times 16 \times 4 =$ $50\cdot 2656$, the area required.

RULE II.

Multiply half the ſum of the circumferences by half the difference of the diameters, and the product will be the area.*

EXAMPLE.

Taking the ſame example as before, we ſhall have

First $3\cdot 1416 \times \begin{matrix} 10 = 31\cdot 4160 \\ 6 = 18\cdot 8496 \end{matrix}$ the circumferences.

Then $\frac{31\cdot 416 + 18\cdot 8496}{2} \times \frac{10-6}{2} = \frac{50\cdot 2656}{2} \times \frac{4}{2} =$ $50\cdot 2656$, the area the ſame as before.

Note. This rule will ſerve, alſo, for any part ABEDA of the ring included between the parts AD, BE, of two radii, by uſing half the ſum of their intercepted arcs for half the ſum of the circumferences.

 Thus,

DW; or equal to the circle whoſe diameter is FW, a tangent to the leſs circle, and terminated by the greater.

Corol. 2. Or the ring is equal to an ellipſe whoſe axes are $D + d$ and $D - d$. As will appear by comparing the above rule with the rule for an ellipſe to be hereafter given.

* DEMONSTRATION OF RULE II.

For the circumferences C, c, are equal to $4aD$, $4ad$; therefore $a \times (D + d) = \frac{1}{4}C + \frac{1}{4}c$; which, ſubſtituted in the laſt rule, gives $a \times (D+d) \times (D-d) = (\frac{1}{4}C + \frac{1}{4}c) \times (D-d) = (\frac{1}{2}C + \frac{1}{2}c) \times (\frac{1}{2}D - \frac{1}{2}d)$, as in the rule. *Q. E. D.*

And it is evident that the ſame rule will ſerve for a part of the ring cut off by two radii, uſing the lengths of the included curves for C and c.

Thus, if the length of the arc AB be 15; then, by reason of the similarity of the arcs AB, DE, it will be $10 : 6 :: 15 : 3 \times 3 = 9$, the arc DE.

Whence $\frac{1}{2}(15+9) \times \frac{1}{2}(10-6) = 12 \times 2 = 24$, the area of the part ABEDA.

RULE III.

Multiply the perpendicular breadth of the ring, that is, the difference of the radii, by the circumference RST (or part RS for the part ABEDA) having the same center with, and equally distant from the bounding arcs.*

EXAMPLE.

Taking, still, the same example; it will be

First $AD = CA - CD = 5 - 3 = 2$, the distance of the circumferences.

And $CR = CD + DR = CD + \frac{1}{2}AD = 3 + 1 = 4$, the radius of the middle arc.

Hence $3{\cdot}1416 \times 8 = 25{\cdot}1328 =$ the middle circumference.

Therefore $25{\cdot}1328 \times 2 = 50{\cdot}2656$, the area the same as before.

And for the part ABEDA, we have CA : CR :: AB : $SR = \frac{15 \times 4}{5} = 3 \times 4 = 12$.

Then $AD \times RS = 2 \times 12 = 24$, the area.

PRO-

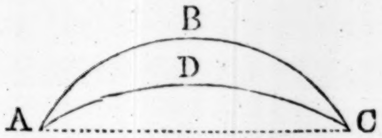
* DEMONSTRATION.

For this circumference, being equally distant from the other two, will be equal to half their sum. Wherefore &c.

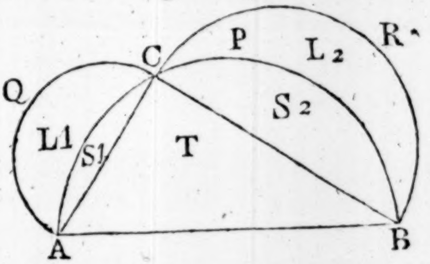
Corol. Hence the whole ring, or any part of it ABEDA, included between two radii, is equal to a parallelogram on the same base AD, and whose altitude is equal to RS the middle circumference.

PROBLEM XIII.

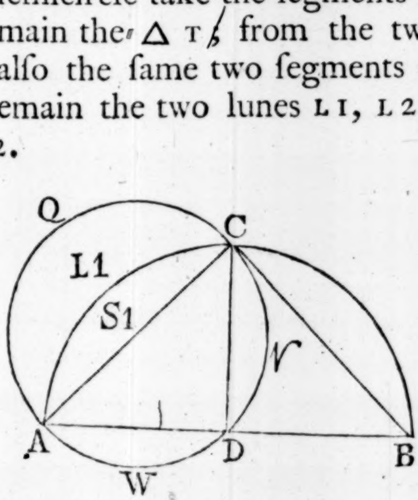
To find the Areas of Lunes.



Lunes, *lunulæ*, or little moons, are ſpaces included between the interſecting arcs of two excentric circles, as the lune ABCDA; which is, evidently, equal to the difference of the ſegments ABCA, ADCA.



If ABC be a triangle right-angled at C, and if ſemicircles be deſcribed on the three ſides as diameters; then the triangle T (ABC) will be equal to the ſum of the two lunes L1, L2.—For the greateſt ſemicircle is equal to the ſum of the other two; from the greateſt ſemicircle take the ſegments S1, S2, and there will remain the $\triangle$ T; from the two leſs ſemicircles take alſo the ſame two ſegments S1, S2, and there will remain the two lunes L1, L2; wherefore T = L1 + L2.



Whence,

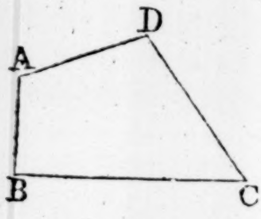
Whence, if the two ſides AC, CB, of the triangle be equal to each other; the two lunes will, alſo, be equal, and each lune LI equal to the $\triangle$ ACD; and therefore the ſegment SI = ſemicircle AQCA — $\triangle$ ACD = double the ſeg. AWDA, or double the ſeg. DVCD.

* SECT. II.

A PROMISCUOUS COLLECTION OF QUESTIONS CONCERNING AREAS.

QUESTION I.

IN the trapezium ABCD are given AB $= 6\frac{1}{2}$, BC $= 15\frac{3}{5}$, CD $= 12$, and DA $= 9$, alſo B a right angle; to find the area of the trapezium.



Firſt $\sqrt{AB^2 + BC^2} = \sqrt{(\frac{13}{2})^2 + (\frac{78}{5})^2} = 16{\cdot}9 =$ AC; and $\sqrt{\frac{37{\cdot}9}{2} \times \frac{4{\cdot}1}{2} \times \frac{13{\cdot}9}{2} \times \frac{19{\cdot}9}{2}} = \frac{1}{4}\sqrt{37{\cdot}9 \times 4{\cdot}1 \times 13{\cdot}9 \times 19{\cdot}9} = \frac{1}{4}\sqrt{42982{\cdot}4279} = 51{\cdot}8305098 =$ the area of the triangle ADC.

And

* Some of the queſtions in this ſection are taken from other books; but the methods of ſolution are, generally, different from thoſe uſed in the books from which they were taken, they being there moſtly ſolved by an analytical proceſs. And I have conſtructed thoſe of which the conſtructions do not appear to be ſelf-evident.

And $\frac{6\frac{1}{2} \times 15\frac{3}{5}}{2} = \frac{13}{4} \times \frac{78}{4} \times \frac{1014}{20} = 50{\cdot}7$, the area of the triangle ABC.

The ſum is 102·5305098, the area of the trapezium required.

Quest. 2. To find the area of a trapezium, the length of its ſides being as in the margin, and the ſum of the two oppoſite angles B and D equal to 180 degrees.

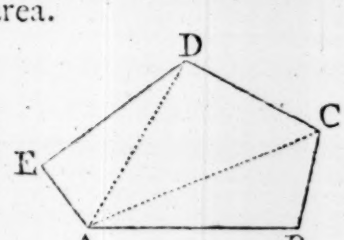
	Chains
AB =	15·6
BC =	13·2
CD =	10·0
DA =	26·0

By rule 5 of prob. 3 of ſect. 1. (32·4 being half the perimeter)

$\sqrt{(32{\cdot}4-15{\cdot}6)\times(32{\cdot}4-13{\cdot}2)\times(32{\cdot}4-10)\times(32{\cdot}4-26)} = \sqrt{16{\cdot}8\times19{\cdot}2\times22{\cdot}4\times6{\cdot}4} = 215{\cdot}04$ ſquare chains $= 21{\cdot}504$ acres $=$ 21 a. 2 r. 0·64 perches, the area required.

Note. A conſtruction of this problem may be ſeen in Simpſon's Select Exerciſes, page 135.

Quest. 3. In the pentangular field ABCDE are given AB = 14, BC = 7, CD = 10, DE = 12, EA = 5, and the diagonal AC = 17 chains, alſo E a right angle; required the area.



Firſt $AD = \sqrt{DE^2 + EA^2} = \sqrt{12^2 + 5^2} = 13$.

Then $AE \times \frac{1}{2}ED = 5 \times 6 = 30 = \triangle ADE$.

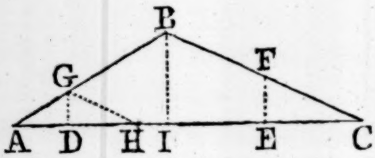
And $\sqrt{20\times3\times7\times10} = 10\sqrt{42} = 64{\cdot}807407 = \triangle ADC$.

Alſo $\sqrt{19\times2\times5\times12} = 2\sqrt{570} = 47{\cdot}749345 = \triangle ABC$.

The ſum of all three is 142·556752 ſquare chains = 14 a. 1·0227 r. the area required.

Quest.

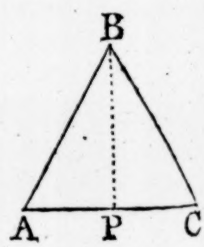
QUEST. 4. Given the baſe AC $=32$, AD $=5$, EC $=9$, the perpendicular EF $=4$, and the perpendicular DG $=3$; required the area of the triangle ABC.



Draw GH parallel to BC. Then, by the ſimilar triangles GDH, FEC, we have FE 4 : EC 9 :: GD 3 : DH $\frac{27}{4}$; and hence AH $= \frac{47}{4}$. Again, from the ſimilar triangles AGH, ABC, we have AH $\frac{47}{4}$: AC 32 :: DG 3 : the perpendicular IB $= \frac{384}{47}$.

Hence $\frac{1}{2}$AC×IB $= 16 \times \frac{384}{47} = 130\frac{34}{47} = 130{\cdot}7234\frac{2}{47}$ is the area of the triangle required.

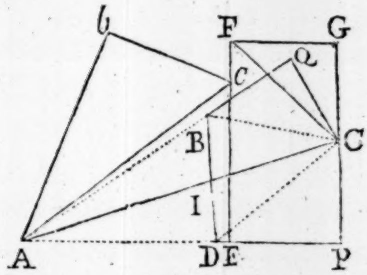
QUEST. 5. What is the ſide of that equilateral triangle, whoſe area coſt as much paving at 8d a foot, as the palliſading the three ſides did at a guinea a yard?



The ſides are 7s. a foot, and the area $\frac{2}{3}$s. a foot. And that the produces may be equal, the quantities muſt be inverſely as the prices; but, by rule 2, page 113, $\frac{1}{4}$BC$^2\sqrt{3}$ is the area; therefore $\frac{2}{3} : 7 :: 3$BC : $\frac{1}{4}$BC$^2\sqrt{3}$ :: $\frac{3\times4}{\sqrt{3}}$: BC $= \frac{3\times4\times7\times3}{2\sqrt{3}} = 42\sqrt{3}$ $= 72{\cdot}7461339$, the ſide required.

QUEST. 6. Surveying a quadrangular field, I found the four ſides to be 10, 9, 7, and 6 chains, in a ſucceſſive order: I likewiſe, at the two extremes of the longeſt ſide, took the bearings of the oppoſite angles, which were N. E. by E. and N. W. Hence the content of the field is required.

I From



From the given bearings of the angles C and D, from A and B, it appears that the diagonals AC, BD, make the angle AID at their intersection of 7 points or $78\frac{3}{4}$ degrees; therefore, by rule 3, problem 3, section 1, we have $\frac{1}{4}(10^2+7^2-9^2-6^2)\times\frac{1}{4}$ tang. $78\frac{3}{4}° = 32\times\frac{1}{4}\times 5{\cdot}0273395 = 8\times 5{\cdot}0273395 = 40{\cdot}218716$ square chains $=$ 4a. 3·499456 perches, the content required.*

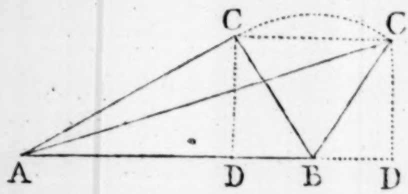
QUEST.

* CONSTRUCTION.

With two of the given lines Ab 10, and bc 6, make a right angle b; and with the other two complete the trapezium AbcD. In the perpendicular Ec produced, take EF $= 8{\cdot}937492 =$ the double of the area 40·218716 divided by AD 9; and with the center F and radius equal to $6\frac{2}{3}$ a fourth proportional to DA, Ab, bc, describe an arc meeting, in C, another arc described with the center D and radius Dc; then join D, C, and with the other two given lines, $= cb$ and bA, complete the trapezium ABCD, and it is done.

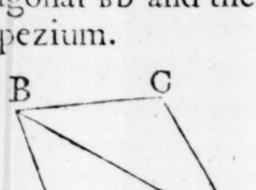
For, having drawn Ac, AC, and CF, and let fall the perpendiculars, CP upon AD, CQ upon AB, and FG upon PCG; since $AD^2 + DC^2 + 2AD \times DP = AC^2 = AB^2 + BC^2 + 2AB \times BQ$, and $AD^2 + Dc^2 + 2AD \times DE = Ac^2 = Ab^2 + bc^2$, by taking these latter quantities from the former, it follows that $2AD \times EP = 2AD \times DP - 2AD \times DE = 2AB \times BQ$, and consequently BQ : FG (EP) :: DA : AB :: BC : CF by the construction; whence the triangles CQB, CGF are similar, and therefore CQ : CG :: CB : CF :: (by the construction) AD : AB; and hence $CQ \times AB = AD \times CG$; and, by adding $CP \times AD$ to each we have $CP \times AD + CQ \times AB =$ double the area ABCD $= CP \times AD + CG \times AD = EF \times AD =$ (by the construction) double the given area.

QUEST. 7. Given two ſides of an obtuſe-angled triangle, which are 20 and 40 poles; required the third ſide, that the triangle may contain juſt an acre of land.



Now the area $160 \div 20\ (\frac{1}{2}AB) = 8 =$ the perpendicular CD; but $\sqrt{CB^2 - CD^2} = \sqrt{20^2 - 8^2} = DB$; hence $AD = AB \pm BD = 40 \pm \sqrt{20^2 - 8^2}$ and $AC = \sqrt{AD^2 + DC^2} = \sqrt{8^2 + 40^2 + 20^2 - 8^2 \pm 80\sqrt{(20^2 - 8^2)}}$ $= 4\sqrt{125 \pm 20\sqrt{21}} = 4 \times \sqrt{105 \pm 2\sqrt{5}} = 58{\cdot}87634686$ and $23{\cdot}09925922$, either of which may be the ſide required.

QUEST. 8. Given the four ſides $AB = 9$, $BC = 10$, $CD = 11$, and $DA = 12$, and the angle $BDC = 30°$, formed by the diagonal BD and the ſide DC; required the area of the trapezium.



Firſt, by trigonometry, $BC : CD :: \frac{1}{2} =$ s. $\angle BDC$ $= \frac{11}{20} = {\cdot}55$ the ſine of the angle $DBC = 33° \; 22'$; whence $180° - 30° - 33° \; 22' = 180° - 63° \; 22' = 116° \; 38' = \angle C$; and s. $\angle BDC$: s. $\angle C$ :: $BC : BD$ $= 10 \times 8938936 \times 2 = 17{\cdot}877872$.

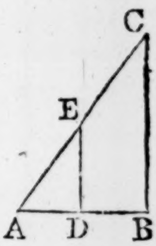
Then $BC \times CD \times \frac{1}{2}$s. $\angle C = 10 \times 11 \times {\cdot}4469468 = 49{\cdot}164148 =$ the area of the triangle BCD.

And

And $\sqrt{19{\cdot}438936 \times 10{\cdot}438936 \times 7{\cdot}438936 \times 1{\cdot}561064} = 48{\cdot}54337 =$ the area of the triangle ABD.

And their sum $= 97{\cdot}707518$, is the area of the whole trapezium.

QUEST. 9. If from the right-angled triangle ABC, whose base is 12, and perpendicular 16 feet, be cut off, by a line DE parallel to the perpendicular, a triangle whose area is 24 square feet; what are the sides of this triangle?



The triangles ABC, ADE, are similar, and the areas of similar triangles are as the squares of their like sides; but $12 \times 8 = 96$ is the area of the triangle ABC, therefore

$$\sqrt{96} : \sqrt{24} :: \begin{cases} 16 : 16\sqrt{\frac{24}{96}} = 16\sqrt{\frac{1}{4}} = 16 \times \frac{1}{2} = 8 = \text{ED} \\ 12 : 12\sqrt{\frac{24}{96}} = 12\sqrt{\frac{1}{4}} = 12 \times \frac{1}{2} = 6 = \text{DA}, \end{cases}$$

and $\sqrt{\text{AD}^2 + \text{DE}^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10 = \text{AE}$.

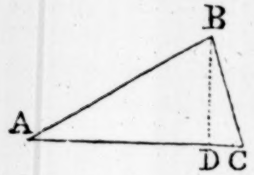
QUEST. 10. If from a triangle whose sides are 13, 14, 15, be cut off by a line drawn parallel to the longest side, an area of 24; what are the lengths of the sides including that area?

First, $\frac{13+14+15}{2} = \frac{42}{2} = 21 =$ half the sum of the sides; from which each side being taken, leaves 6, 7, 8; therefore $\sqrt{21 \times 6 \times 7 \times 8} = \sqrt{3^2 \times 7^2 \times 4^2} = 3 \times 4 \times 7 = 84 =$ the area of the given triangle. Then, as in the last,

$$\sqrt{84} : \sqrt{24} :: \begin{cases} 13 : 13\sqrt{\frac{2}{7}} = \frac{13}{7}\sqrt{14} \\ 14 : 14\sqrt{\frac{2}{7}} = 2\sqrt{14} \\ 15 : 15\sqrt{\frac{2}{7}} = \frac{15}{7}\sqrt{14} \end{cases}$$ the 3 sides inclosing the area cut off.

QUEST. 11. Given the area $= 144$, the base AC $= 24$, and one angle BAC $= 30°$, at the base; to find the sides AB, BC.

First,



First, $144 \div 12\,(\frac{1}{2}\text{AC}) = 12 =$ the perpendicular BD.

Then, as $\frac{1}{2} = $ s. $\angle$A $30^\circ : 1 =$ s. $\angle$D :: $12 =$ DB : $24 =$ AB.

But $\text{AD} = \sqrt{\text{AB}^2 - \text{BD}^2} = \sqrt{24^2 - 12^2}$, and $\text{DC} = \text{AC} - \text{AD} = 24 - \sqrt{24^2 - 12^2}$.

Therefore $\text{BC} = \sqrt{\text{BD}^2 + \text{DC}^2} = 24\sqrt{2 - \sqrt{3}} = 12 \times (\sqrt{6} - \sqrt{2}) = 12{\cdot}423314184$.

Or, since AB is = AC, the $\angle$C will be = the $\angle$B $= \frac{1}{2}(180 - 30) = 75^\circ$; hence $\angle$DBC $= 15^\circ$; and therefore as radius 1 : sec. $15^\circ = 1{\cdot}0352762$ :: BD 12 : $12 \times 1{\cdot}0352762 = 12{\cdot}4233144 =$ BC.

Note. It is pretty evident, that this triangle will be constructed by drawing AB to make with the given base AC an angle of 30°, and meeting with a line drawn parallel to, and at the distance of, the perpendicular BD from AC in B the vertex of the triangle.

QUEST. 12. Given the area 1012, and ratio of the sides of a triangle, viz. AC to CB, as 4 to 3, and AB to CB as 3 to 2; to find the sides.

It is evident that the sides AB, AC, BC, are in the ratio of the three numbers 9, 8, 6, respectively; and therefore a triangle whose sides are 9, 8, 6, will be similar to the triangle proposed; but similar triangles are as the squares of their like sides, and $\sqrt{\frac{23}{2} \times \frac{5}{2} \times \frac{7}{2} \times \frac{11}{2}} = \frac{1}{4}\sqrt{8855}$ is the area of the triangle

angle whose sides are 9, 8, 6; therefore $\frac{1}{2}\sqrt[4]{8855}$: $\sqrt{1012}$ ::

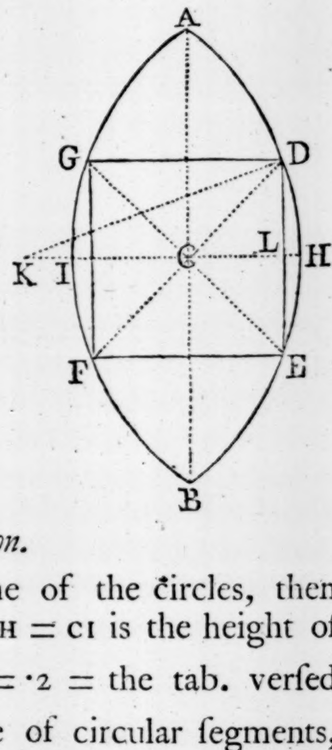
$$8 : 32\sqrt[4]{\frac{23\times 11}{35}} = \frac{32}{35}\sqrt[4]{10847375} = 52{\cdot}47024 = \text{AC},$$

$$6 : 24\sqrt[4]{\frac{23\times 11}{35}} = \frac{24}{35}\sqrt[4]{10847375} = 39{\cdot}35268 = \text{CB},$$

$$9 : 36\sqrt[4]{\frac{23\times 11}{35}} = \frac{36}{35}\sqrt[4]{10847375} = 59{\cdot}02902 = \text{BA}.$$

QUEST. 13. The distance of the centers of two circles, whose diameters are each 50, being given equal to 30; it is required to find the area of the space inclosed by their circumferences, and that of a square inscribed in the said space?

Construction.



Having described the circumferences including the common space AHBI, IH being the common part of both the diameters, bisect IH in C, and through C draw FCD, GCE, each making half a right angle with IH, and meeting the arcs in D, E, F, G, the four angular points of the square.

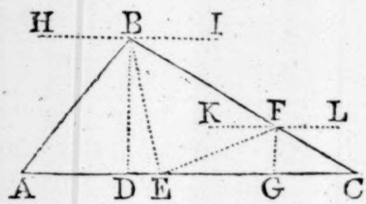
Calculation.

If K be the center of one of the circles, then $\text{KH} - \text{KC} = 25 - 15 = 10 = \text{CH} = \text{CI}$ is the height of each segment; and $\frac{\text{CH}}{2\text{KH}} = \frac{10}{50} = {\cdot}2 =$ the tab. versed sine; to which, in the table of circular segments, corresponds the area $\cdot 111823 80$; therefore $\cdot 1118238 \times 50 \times 50 \times 2 = 559{\cdot}119 =$ the two segments, or common space AHBIA.

Again,

Again, drawing KD, we shall have, as KD 25 : KC 15 :: s. $\angle$C 135° or 45° : s. $\angle$KDC = 25° $6\frac{1}{4}'$; hence 180° − 135° − 25° $6\frac{1}{4}'$ = 180° − 160° $6\frac{1}{4}'$ = 19° $53\frac{3}{4}'$ = $\angle$K; then as radius : s. $\angle$K :: KD 25 : DL = 8·507778, the double of which is DE = 17·015556 = the side of the square. And therefore 17·015556 × 17·015556 = 289·529 is the area of the square.

QUEST. 14. Given the base AC = 15, the area 45, and the ratio of AB to BC as 2 to 3; to find the sides AB, BC, of the triangle?



First, the area 45 ÷ $7\frac{1}{2}$ ($\frac{1}{2}$AC = 6 = the perpendicular BD. Then take AE : EC :: AB : BC, that is 5 = 2 + 3 : AC 15 :: $\begin{cases} 2 : 6 = \text{AE}, \\ 3 : 9 = \text{EC}; \end{cases}$ and suppose BF = BA, as also the lines BE, EF, and the perpendicular FG to be drawn; and we shall have the two triangles ABE, EBF, equal to each other in every respect; but

$$\tfrac{1}{2}\text{BD} \times \begin{Bmatrix} \text{AE} \\ \text{EC} \end{Bmatrix} = 3 \times \begin{cases} 6 = 18 = \triangle\text{ABE}, \\ 9 = 27 = \triangle\text{EBC}; \end{cases}$$
their difference is $9 = \triangle$EFC;

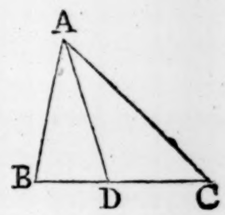
or $\triangle$EFC = $\triangle$ABC − ABFE = $\triangle$ABC − 2 $\triangle$ABE = 45 − 36 = 9; hence 9 ($\triangle$EFC) ÷ $4\frac{1}{2}$ ($\frac{1}{2}$EC) = 2 = the perpendicular FG; but EG = $\sqrt{\text{EF}^2 - \text{FG}^2}$ = $\overline{6^2 - 2^2}$ = $\sqrt{32}$, and GC = EC − EG = 9 − $\sqrt{32}$; whence, by similar triangles, FG : GC :: BD : DC = 27 − $3\sqrt{32}$, and, consequently, AD = AC − DC = $3\sqrt{32} - 12$; wherefore, lastly, AB = $\sqrt{\text{BD}^2 + \text{DA}^2}$ = $\sqrt{6^2 + (3\sqrt{32} - 12)^2}$ = $3\sqrt{2^2 + (\sqrt{32} - 4)^2}$ = $6\sqrt{13 - 8\sqrt{2}}$ = 7·79146; and, as 2 : 3 :: AB : BC = $9\sqrt{13 - 8\sqrt{2}}$ = 11·68719.

From

From the foregoing solution the following construction is evident.

Having drawn HI parallel to AC, and at such distance from it as is expressed by the perpendicular BD, found from the given area and base, and taken AE to EC in the given proportion of AB to BC; with the center E and radius EA describe an arc meeting in F, with a line KL drawn parallel to AC, and at such distance from it as is denoted by a third proportional to EC, EC — EA, and BD; then through F draw CB meeting HI in B, and join B, A.

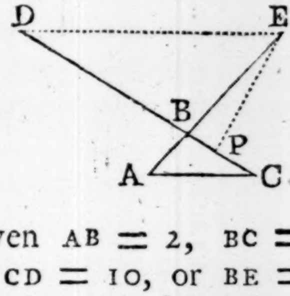
QUEST. 15. Given the segments BD = 10, and DC = 14, of the base BC, made by a line AD bisecting the vertical angle A, and the sum of the sides BA + AC = 48; to find the area.



If a line bisect any angle of a triangle and cut the opposite side, it is known that the segments of that side are to each other, as the other sides adjacent to them.

Therefore as BC 24 : BA + AC 48 :: $\begin{cases} \text{BD } 10 : 20 = \text{BA}, \\ \text{DC } 14 : 28 = \text{CA}; \end{cases}$
hence the area is $\sqrt{36 \times 8 \times 12 \times 16} = 6 \times 2 \times 2 \times 4\sqrt{2 \times 3} = 96\sqrt{6} = 235{\cdot}15100154$.

QUEST. 16. Having given the continuations of the two sides of the vertical angle of a given triangle, to find the area of the triangle formed by the said continuations, and a line connecting their extremities.

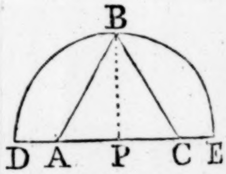


That is, Given AB = 2, BC = 3, AC = 4; and AE = 7, CD = 10, or BE = 5, BD = 7; required DE and the area of the triangle BDE.

Firſt, $\sqrt{\frac{9}{2}\times\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}} = \frac{3}{4}\sqrt{15}$ = the area of the triangle ABC. But ſince the vertical angles at B are equal, and triangles, having one angle in each equal, are as the rectangles of the ſides about the equal angles; we ſhall have $6 : 35 :: \frac{3}{4}\sqrt{15} : \frac{35}{8}\sqrt{15} =$ $16{\cdot}944302$ = the area of the triangle BDE.

Then, having drawn EP perpendicular to BC, $\frac{35}{8}\sqrt{15} \div \frac{7}{2}(\frac{1}{2}DB) = \frac{5}{4}\sqrt{15} = EP$; but $\sqrt{BE^2 - EP^2}$ $= \sqrt{25 - \frac{15}{16}\times 25} = \sqrt{\frac{25}{16}} = \frac{5}{4} = BP$, and $DB + BC$ $= 7 + \frac{5}{4} = \frac{33}{4} = DP$; hence $\sqrt{DP^2 + PE^2} =$ $\sqrt{(\frac{33}{4})^2 + 15\times(\frac{5}{4})^2} = \frac{1}{4}\sqrt{33^2 + 15\times 5^2} = \frac{1}{4}\sqrt{1464}$ $= \frac{1}{2}\sqrt{366} = 9{\cdot}565563235 = DE$.

QUEST. 17. Given the area 100 of the equilateral triangle ABC, whoſe baſe falls on the diameter, and vertex in the middle of the arc of a ſemicircle; required the diameter DE of the ſemicircle.



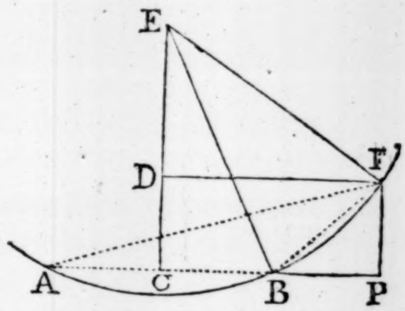
Since the area of an equilateral triangle is equal to the ſquare of the ſide drawn into $\frac{1}{4}\sqrt{3}$, we ſhall have the ſide $AB = \sqrt{\frac{4\times 100}{\sqrt{3}}}$, and the half of it $AP = \sqrt{\frac{100}{\sqrt{3}}} = \frac{10}{\sqrt[4]{3}}$; whence the radius of the circle or perpendicular of the triangle BP is = area $\div AP = 100 \div \frac{10}{\sqrt[4]{3}} = 10\sqrt[4]{3}$; the double of which is $20\sqrt[4]{3} = 26{\cdot}32148026$, the diameter required.

Otherwiſe. Since $AP = BP\sqrt{\frac{1}{3}}$, therefore $AP \times BP$ $= BP^2\sqrt{\frac{1}{3}} = 100$ the area; hence $BP\sqrt[4]{\frac{1}{3}} = 10$, and $2BP = 20\sqrt[4]{3}$, the diameter.

QUEST. 18. The area of a triangular meadow is 100 perches or ſquare poles, and the length of one of its ſides is 40 poles, which is alſo a mean propor-

tional between the other two sides; from which it is required to find the unknown fences.

Construction.



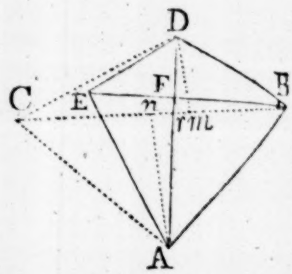
Make AB $= 40 =$ the base, or given side; on the middle C of which raise CD perpendicular and equal to $\frac{200}{40} = 5$, the perpendicular height of the triangle. Then, since the rectangle of two sides of a triangle drawn into the sine of their included angle, is equal to double the area, we shall have $\frac{200}{40 \times 40} = \frac{1}{8}$ for the sine of the vertical angle; therefore draw BE making the angle CBE equal to the complement of that angle, and meeting CD produced in E; so will E be the center of the circle circumscribing the triangle. Therefore, with the center E and radius EB or EA, describe an arc ABF meeting DF, drawn parallel to AB, in F; then draw AF, FB, and it is evident that AFB will be the triangle required.

Calculation.

Draw FP perpendicular to AB produced. Then, since $\frac{1}{8}$ is the s. of $\angle$E, it will be as $\frac{1}{8}$: 1 s. $\angle$C :: CB 20 : 160 $=$ BE; or thus, by geometry, $\frac{AF \times FB}{2FP} = \frac{40 \times 40}{10} = 160 =$ BE. Again CE $= \sqrt{EB^2 - BC^2} = \sqrt{160^2 - 20^2} = 20\sqrt{63} = 60\sqrt{7}$; hence DE $=$ CE $-$ CD $= 60\sqrt{7} - 5$, and DF $= \sqrt{EF^2 - ED^2} = 5\sqrt{15 + 24\sqrt{7}}$; but BP $=$ DF $-$ CB $= 5\sqrt{15 + 24\sqrt{7}} - 20$, and therefore BF $= \sqrt{FP^2 + BP^2} = 10\sqrt{8 + 6\sqrt{7} - 2\sqrt{15 + 24\sqrt{7}}} = 24{\cdot}80833$, and consequently FA $= \frac{AB^2}{BF} = \frac{40 \times 40}{24{\cdot}80833} = 64{\cdot}49447$.

QUEST. 19. The four ſides of a field, whoſe diagonals are equal to each other, are known to be 25, 35, 31, and 19 poles, in a ſucceſſive order; from whence the content of the field is required.

Conſtruction.



Make AB and AC perpendicular to one another, and each equal to one of the given ſides, as 35; with the centers B, C, and radii equal to the two remaining oppoſite ſides 25 and 31, deſcribe two arcs, interſecting in D; draw AD, and BE perpendicular and equal to it; draw AE and DE; and ABDE will be the trapezium.*

Calculation.

Draw BC meeting AD in r, upon which let fall the perpendiculars An, Dm. Then $BC = \sqrt{AB^2 + AC^2} = 35\sqrt{2}$, and $An = Bn = \frac{1}{2}BC = \frac{35\sqrt{2}}{2} = \frac{35}{\sqrt{2}}$; again, $Bn = \frac{BD^2 + BC^2 - CD^2}{2BC} = \frac{25^2 + 2 \times 35^2 - 31^2}{70\sqrt{2}} = \frac{2114}{70\sqrt{2}} = \frac{151\sqrt{2}}{10}$; and $nm = Bn - Bm = (\frac{35}{2} - \frac{151}{10}) \times \sqrt{2} = \frac{12}{5}\sqrt{2}$; but

* For, by the conſtruction, AB, BD are the two ſides 35, 25, and the diagonal BE = AD; ſo that we have only to ſhew that AE, ED are equal to the other two ſides, 31, 19.—Now, by the conſtruction the angles CAD, ABE, being each the complement of EAD, are equal to each other; and, by the conſtruction CA and AD, AB and BE, about thoſe equal angles, are alſo equal; and therefore the remaining ſides CD, AE are likewiſe equal, that is AE = 31.—But, by the property of all triangles, $AB^2 - AE^2 = BF^2 - FE^2 = BD^2 - DE^2$; and, therefore, $DE = \sqrt{BD^2 + AE^2 - AB^2} = \sqrt{25^2 + 31^2 - 35^2} = 19$. *Q.E.D.*

but $DM = \sqrt{BD^2 - Bm^2} = \sqrt{25^2 - 2(\frac{151}{10})^2} = \frac{1}{10}\sqrt{250^2 - 2 \times 151^2} = \frac{1}{10}\sqrt{16898}$; and, by similar triangles, $An + Dm : nm :: An : nr = \frac{420\sqrt{2}}{175+\sqrt{8449}}$; hence $Ar =$

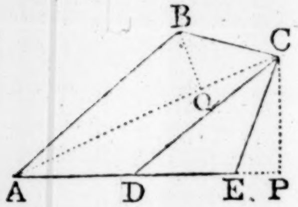
$\sqrt{An^2 + nr^2} = \frac{\sqrt{\frac{1}{2} \times 35^2 \times (175+\sqrt{8449})^2 + 2 \times 420^2}}{175+\sqrt{58449}}$;

and, by similar triangles, $nr : nm :: Ar : AD = \sqrt{793+7\sqrt{8449}} = BE$.—Then, because AD is equal and perpendicular to BE, by rule 2 prob. 3 we obtain the area $= AD \times BE \times \frac{1}{2}$ sine $\angle F = \frac{1}{2} BE^2 = \frac{1}{2} \times (793+7\sqrt{8449}) = 718{\cdot}214547$ square poles $= 4$ac. 1 ro. $38{\cdot}214547$ perches.

Otherwise.

Having found $Bm = \frac{151}{10}\sqrt{2}$; we shall have, as $BD : Bm :: 1 : \frac{151}{250}\sqrt{2} = {\cdot}604\sqrt{2} = {\cdot}85418499$ the cosine of the $\angle DBm = 31^\circ\ 20'$; but $\angle ABm = 45^\circ$, consequently $\angle DBA = 76^\circ\ 20'$; then as $AB + BD$ 60 : $AB - BD$ 10 :: tang. $\frac{1}{2}\angle BDA + \frac{1}{2}\angle BAD = 51^\circ\ 50'$: tang. $\frac{1}{2}\angle BDA - \frac{1}{2}\angle BAD = 11^\circ\ 58'$, hence $51^\circ\ 50' - 11^\circ\ 58' = 39^\circ\ 52' = \angle BAD$; lastly, s. $\angle BAD$: s. $\angle ABD$ :: BD : DA $= 37{\cdot}897 =$ BE, which is nearly equal to $\sqrt{793+7\sqrt{8449}} = 37{\cdot}90025$, the value above found.

QUEST. 20. The trapezium ABCD is to be measured; and, because of some obstructions from wood, water, &c, or for want of proper instruments, there can be taken only the following measures: viz. AB 58·5 chains, BC 27·3, CD 50, and DA 32 chains; also in AD, continued, there is measured any distance DE 27·5, and the distance from E to C is 32·5 chains; required the area.



Draw the diagonal AC, and on AD, produced, let fall the perpendicular CP.

Becaufe DC^2 is $= DE^2 + EC^2 + 2DE \times EP$, we ſhall have $EP = \frac{DC^2 - EC^2 - DE^2}{2DE} = \frac{50^2 - 32{\cdot}5^2 - 27{\cdot}5^2}{55} = 12{\cdot}5$; but $PC = \sqrt{CE^2 - EP^2} = \sqrt{32{\cdot}5^2 - 12{\cdot}5^2} = 30$, therefore $\frac{1}{2}AD \times PC = 16 \times 30 = 480$ the area of the triangle ACD.

Again, $AP = AD + DE + EP = 32 + 27{\cdot}5 + 12{\cdot}5 = 72$; therefore $AC = \sqrt{AP^2 + PC^2} = \sqrt{72^2 + 30^2} = 78$.

Alſo, letting fall the perpendicular BQ, ſince $AB^2 = AC^2 + BC^2 - 2AC \times CQ$, we ſhall have $CQ = \frac{AC^2 + BC^2 - AB^2}{2AC} = 21{\cdot}84$; and hence $BQ = \sqrt{BC^2 - CQ^2} = 16{\cdot}38$; therefore $\frac{1}{2}AC \times QB = 39 \times 16{\cdot}38 = 638{\cdot}82$ = the triangle ABC.

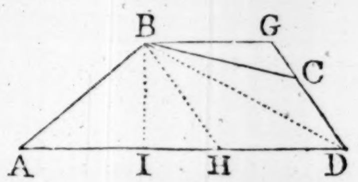
Whence $480 + 638{\cdot}82 = 1118{\cdot}82$ ſquare chains $= 111{\cdot}882$ acres $=$ 111 ac. 3r. 21·12 perches is the whole area required.

QUEST. 21. To find the length of a chord cutting off any part, as ſuppoſe $\frac{1}{3}$, from a circle whoſe diameter is 289.

The area of the whole tabular circle, or of the circle whoſe diameter is 1, is ·78539816; the $\frac{1}{3}$ of which is ·26179939 nearly, which is a ſegment ſimilar to the ſegment to be cut off. Now, the verſed ſine anſwering to this area, in the table of circular ſegments, is ·36753395, which taken from the whole diameter

diameter 1 leaves ·63246605 for the other part of the diameter; then, because the semi-chord is a mean proportional between the two segments of the diameter, we shall have $2\sqrt{\cdot 63246605 \times \cdot 36753395} = \cdot 96426162 =$ the chord of the tabular similar segment. Therefore as 1 : 289 :: ·96426162 : 278·6716 = the chord line required.

QUEST. 22. In the unpassable field ABCD there are measured the sides AB 29, AD 59, and DC 15 chains: also, by drawing BG parallel to AD till it meet with DC, produced, in G, BG measures 23; and GC, 10 chains; required the content.



Draw BH parallel to GD: then is BGDH a parallelogram, and consequently $BH = GD = 25$, and $HD = BG = 23$; hence $AH = AD - HD = 36$.

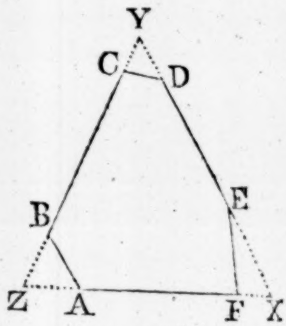
But, letting fall the perpendicular BI, $IH = \frac{AH^2 + BH^2 - AB^2}{2AH} = 15$, and $BI = \sqrt{BH^2 - IH^2} = 20$. Whence $(AD + BG) \times \frac{1}{2}BI = 82 \times 10 = 820$ is the area of the trapezoid ABGD.

Again, drawing the diagonal BD, $HD \times \frac{1}{2}BI = 23 \times 10 = 230 =$ the triangle BHD = the triangle BGD; but the triangles BGD, BGC, being of the same height, are to each other as their bases GD, GC; that is, DG : GC :: triangle BDG : triangle $BCG = \frac{2}{5} \times 230 = 2 \times 46 = 92$.

Whence, by taking the triangle BGC from the trapezoid ABGD, we have $820 - 92 = 728$ square chains = 72a. 3r. 8 perches = the area required.

Note. It is evident that this figure will be constructed, by first forming the triangle ABH, and joining the parallelogram BGDH to it.

QUEST. 23. To find the area of the hexagonal field ABCDEF, the sides BC, DE, AF, being produced to meet in the points X, Y, Z; there are given the following dimensions, ZB 39, BC 118, CY 25, YD 30, DE 100, EX 65, XF 23, FA 109, and AZ 37 chains.



First, by prob. 2 sect. 1, the area of the triangle ZYX is 14196 square chains.

But, it appears from rule 2 of the same problem, that the areas of triangles, having one angle common, are as the rectangles of the sides including the common angle; therefore

$ZX \times ZY = 169 \times 182 : 37 \times 39 :: 14196 : 666 = \triangle ZBA,$

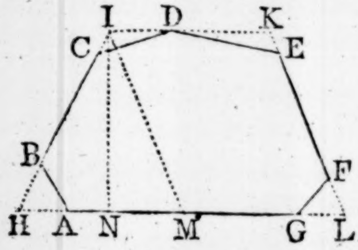
$YZ \times YX = 182 \times 195 : 25 \times 30 :: 14196 : 300 = \triangle CYD,$

$XY \times XZ = 195 \times 169 : 65 \times 23 :: 14196 : 644 = \triangle EXF,$

whose sum 1610
taken from 14196 the $\triangle$ ZYX
leaves 12586 the area
of the hexagon ABCDEF required.

QUEST. 24. It is required to find the area of the heptagon ABCDEFG inscribed in the trapezoid HIKL, the dimensions being as below:

Viz. HB 34, BC 85, CI 17, ID 42, DK 66, KE 13, EF 91, FL 26, LG 29, GA 160, and AH 33.



First,

First, drawing IM parallel to KL, and letting fall the perpendicular IN, IM will be $= KL = 130$, $HM = HL - IK = 114$; hence $NM = \dfrac{IM^2 + HM^2 - HI^2}{2\,HM}$ $= 50$, and $IN = \sqrt{IM^2 - NM^2} = 120$.

Hence $\frac{1}{2} IN \times (IK + HL) = 60 \times 330 = 19800 =$ the trapez. IKLH.

But a trapezoid is to a triangle, having each one angle equal, as the rectangle under the sum of the parallel sides and the other side including that angle, in the trapezoid, is to the rectangle under the sides including the angle, in the triangle.

Or, As that one of the two sides in the trapezoid, about the angle, which is not one of the parallel sides, is to half the distance of the parallel sides, so is the rectangle under the two sides of the triangle about the angle, to its area.* Therefore

$$136 : 60 :: 33 \times 34 : 495 = \triangle HBA,$$
$$136 : 60 :: 17 \times 42 : 315 = \triangle CID,$$
$$130 : 60 :: 66 \times 13 : 396 = \triangle DKE,$$
$$130 : 60 :: 26 \times 29 : 348 = \triangle GFL,$$

the sum is 1554
which taken from 19800 the trapezoid,
leaves 18246 square chains
= 1824a. 2 r. 16 perches.

Note. This is constructed as the 22d.

QUEST.

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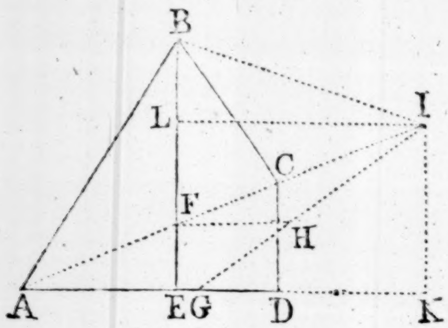
* For $AH \times HB : \triangle BAH :: (\text{radius} : \frac{1}{2}s.\angle H ::)\ HI : \frac{1}{2}IN ::$ (by equal mult.) $HI \times \overline{IK + HL} : \frac{1}{2}IN \times \overline{IK + HL} =$ the trapezium.

In the same manner, $GL \times LF : \triangle GLF :: LK : \frac{1}{2}IN :: LK \times \overline{IK + HL} : \frac{1}{2}IN \times \overline{IK + HL}$.

But the angles I and K are the supplements of the angles H and L, and have, therefore, the same sines; and, consequently, the rule will be the same for them.

QUEST. 25. In the quadrangular field ABCD, if AD be 46·8, DC 19·5, the perpendicular BE 44·8 chains, and D a right angle; also if AB be to BC as 8 to 5; it is required to find the content of the trapezium ABCD, and the length of the sides AB, BC.



Construction.

Draw AD = 46·8, and DC = 19·5 making a right angle D with it; join A, C; take AF : FC :: 8 : 5, or AF : AC :: 8 : 13; make AG = AF, and FH parallel to it and = to FC; draw GHI meeting AC produced in I; with the center I and radius IF, describe an arc meeting in B, a line drawn parallel to, and at the distance, of, the perpendicular 44·8 from AD; and B will be the other point of the figure required.*

Calculation.

First, $AC = \sqrt{AD^2 + DC^2} = 50·7$, and, by the construction, 13 : 8 :: AC : 31·2 = AF; hence FC

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* DEMONSTRATION.

Because, by similar triangles, AI : AG or AF :: FI : FH or FC, we have, by division, AI : FI :: FI : CI, or joining IB, AI : BI :: BI : CI; but the $\angle$I, included by these proportional lines, is common to both the triangles ABI, CBI, therefore those triangles are similar, and consequently AB : BC :: AI : BI or FI :: AG : FH :: 8 : 5.

FC = AC − AF = 19·5. Then, by similar triangles, 11·7 (= AG − FH or AF − FC) : 31·2 (= AF or AI − FI) :: 19·5 (= FH or FC) : 52 = FI or BI; whence AI or AF + FI = 83·2. Then, by similar triangles again, (drawing IK perpendicular to AD produced) as AC 50·7 : AI 83·2 ::

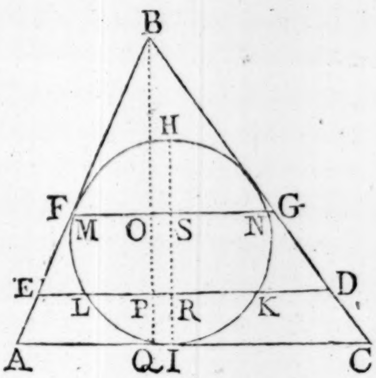
$$\begin{cases} \text{AD } 46·8 : 76·8 = \text{AK}, \\ \text{DC } 19·5 : 32 \quad = \text{KI}, \end{cases}$$

whence, (drawing IL parallel to KE), LB or BE − KI = 12·8, and EK or LI $= \sqrt{BI^2 - BL^2}$ = 50·4; and hence AE = AK − EK = 26·4, and AB $= \sqrt{AE^2 + EB^2}$ = 52; also 8 : 5 :: AB 52 : 32·5 = BC.

But $\frac{1}{2}$AE × EB	=	591·36 triang. ABE,
And $\frac{1}{2}$ED × (BE + CD)	=	655·86 trapez. EBCD,
Their sum is - - -		1247·22 sq. chains
		or 124 ac. 2 r. 35·52 p.

the area of the quadrangle ABCD required.

QUEST. 26. Given the three sides AB 13, AC 14, and BC 15, of a triangle ABC, divided into three equal parts by the two lines FG, ED, both parallel to AC; it is required to find the areas of the two segments MHN, KIL, and zone LMNK, into which the inscribed circle is cut by those lines.



First,

First, $\sqrt{21 \times 8 \times 7 \times 6} = \sqrt{7^2 \times 3^2 \times 4^2} = 7 \times 3 \times 4$ $= 84$ the area of the triangle ABC.

Then $84 \div 7$ ($\frac{1}{2}$ AC) $= 12$ is the perpendicular BQ; and $84 \div 21$ ($\frac{1}{2}$AB $+ \frac{1}{2}$AC $+ \frac{1}{2}$BC) $= 4$ is the radius, or 8 is the diameter HI of the inſcribed circle.

Now, the triangles BFG, BED, BAC, are ſimilar; but ſimilar triangles are as the ſquares of their like dimenſions; alſo thoſe triangles are to one another as the numbers 1, 2, 3, reſpectively; therefore

$$\sqrt{3} : BQ\ 12 :: \begin{cases} \sqrt{1} : 12\sqrt{\frac{1}{3}} = 4\sqrt{3} = BO, \\ \sqrt{2} : 12\sqrt{\frac{2}{3}} = 4\sqrt{6} = BP; \end{cases}$$

Hence $BQ - BP = 12 - 4\sqrt{6} = PQ =$ the verſed ſine IR,
and $BQ - BO = 12 - 4\sqrt{3} = OQ =$ the verſed ſine IS;
and $IH - IS = 4\sqrt{3} - 4 =$ the verſed ſine SH.

Then, dividing the verſed ſines HS, IR, by the diameter, we have $\frac{4\sqrt{3}-4}{8} = \frac{\sqrt{3}-1}{2} = \cdot366025404$,

and $\frac{12-4\sqrt{6}}{8} = \frac{3-\sqrt{6}}{2} = \cdot275255128$;

the correſponding tabular verſed ſines; to which, in the table of circular ſegments, belong the areas

	·26034449,
and	·17577983,
whoſe ſum	·43612432,
taken from	·78539816,
leaves	·34927384,

the area correſponding to the zone. Then each of theſe being multiplied by 64, the ſquare of the diameter, gives the ſeveral areas following, viz.

$$64 \times \begin{cases} \cdot26034449 = 16\cdot66204736 = \text{the ſegment LIK}, \\ \cdot17577983 = 11\cdot24990912 = \text{the ſegment MHN}, \\ \cdot34927384 = 22\cdot35352576 = \text{the zone MLKN}. \end{cases}$$

PART III.

MENSURATION OF SOLIDS.

GENERAL DEFINITIONS.

1. A Solid, or body, is a figure extended in every direction. It is commonly said to consist of length, breadth, and thickness, which are three of its extensions; of which the direction of each is perpendicular to those of the other two.

2. The measure of a solid is called its solidity, capacity, or content.

3. By the mensuration of solids then are determined the spaces included by contiguous surfaces; and the sum of the measures of these including surfaces, is the surface or superficies of the body.

4. Solids are measured by cubes, whose sides are inches, feet, yards, or any other assigned quantity; and hence the solidity of a body is said to be so many cubic inches, feet, yards, &c, as will fill its capacity or space, or another of an equal magnitude

5. The least solid measure is the cubic inch, other cubes being taken from it according to the proportion in the following table.

Table

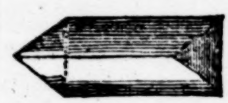
Table of Solid Measure.

Cubic inches	cubic feet				
1728	1	cubic yards			
46656	27	1	cub. poles		
7762392	4492$\frac{1}{8}$	166$\frac{3}{8}$	1	c. turl.	
946793088000	287496000	10648000	64000	1	c. mile
254358061056000	147197952000	5451776000	32768000	512	1

SECTION I.

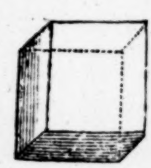
OF PRISMS, PYRAMIDS, AND THE SPHERE, WITH THE PARTS INTO WHICH SOME OF THEM MAY BE CUT BY PLANES.

1. A Prism is a solid, or body, whose ends are any plane figures, which are equal and similar; and its sides are parallelograms.

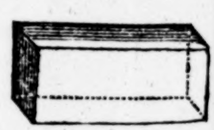


A prism is called a triangular prism, when its ends are triangles; a square prism, when its ends are squares; a pentagonal prism, when its ends are pentagons; and so on.

2. A cube is a square prism, having six sides, which are all squares. It is like a die, having its sides perpendicular to one another.

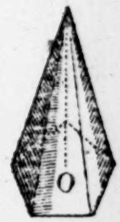


3. A parallelopipedon is a solid having six rectangular sides, every opposite pair of which are equal and parallel.



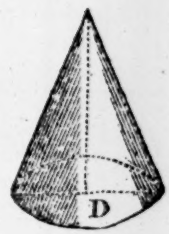
4. A cylinder is a round prism; having circles for its ends.

5. A pyramid is a solid having any plane figure for a base, and its sides are triangles whose vertices meet in a point at the top, called the vertex of the pyramid.



The pyramid takes names according to the figure of its base, like the prism; being triangular, or square, or hexagonal, &c.

6. A cone is a round pyramid; having a circular base.



7. A sphere is a solid bounded by one continued convex surface, every point of which is equally distant from a point within, called the center.—The sphere may be conceived to be formed by the revolution of a semicircle about its diameter, which remains fixed.

8. The axis of a solid, is a line drawn from the middle of one end, to the middle of the opposite end; as between the opposite ends of a prism. Hence the axis of a pyramid, is the line from the vertex to the middle of the base, or the end on which it is supposed to stand. And the axis of a sphere, is the same as a diameter, or a line passing through the center, and terminated by the surface on both sides.

9. When the axis is perpendicular to the base, it is a right prism or pyramid; otherwise it is oblique.

10. The height or altitude of a solid, is a line drawn from its vertex or top, perpendicular to its base.—This is equal to the axis in a right prism or pyramid;

pyramid; but in an oblique one, the height is the perpendicular ſide of a right-angled triangle, whoſe hypotenuſe is the axis.

11. Alſo a priſm or pyramid is regular or irregular, as its baſe is a regular or an irregular plane figure.

12. The ſegment of a pyramid, ſphere, or any other ſolid, is a part cut off the top by a plane parallel to the baſe of that figure.

The ſection made by the plane, is a plane ſimilar to the baſe of the figure; and every ſection of a ſphere is a circle. If the ſection be made through the center, it is a great circle of the ſphere, having the ſame diameter with the ſphere; if not, it is a little circle.

13. A fruſtum, truncus, or trunk, is the part remaining at the bottom after the ſegment is cut off.

If a fruſtum be cut by a plane diagonally paſſing through the extremity of one ſide at the leſs end, and through the extremity of the oppoſite ſide at the greater end, the two parts into which it is cut, are called ungulas or hoofs; the greater hoof being that including the greater end; and the leſs, that including the leſs.

14. A zone of a ſphere, is a part intercepted between two parallel planes; and it is the difference between two ſegments. When the ends, or planes, are equally diſtant from the center, on both ſides, the figure is called the middle zone.

15. The ſector of a ſphere, is compoſed of a ſegment leſs than a hemiſphere or half ſphere, and of a cone having the ſame baſe with the ſegment, and its vertex in the center of the ſphere.

16. A circular ſpindle, is a ſolid generated by the revolution of a ſegment of a circle about its chord, which remains fixed.

17. A

17. A wedge is a ſolid having a rectangular baſe, and two of the oppoſite ſides ending in an acies or edge.

18. A priſmoid is a ſolid having for its two ends any diſſimilar parallel plane figures of the ſame number of ſides, and all the upright ſides of the ſolid trapezoids.—If the ends of the priſmoid be bounded by diſſimilar curves, it is ſometimes called a cylindroid.

19. An ungula, or hoof, is a part cut off a ſolid by a plane oblique to the baſe.

PROBLEM I.

To find the Surface of a Priſm.

GENERAL RULE.

It is evident, that, if the area of each ſide and end be calculated ſeparately, the ſum of thoſe areas will be the whole ſurface of any priſm, whether right or oblique; or, indeed, of any other body whatever.* —But for a right priſm obſerve the following

PARTICULAR RULE.

Multiply the perimeter of the end by the height, and the product will be the ſum of the ſides, or upright ſurface.

If the ends of the priſm be regular plane figures, multiply the perimeter of the end by the ſum of the height of the priſm and the radius of the circle inſcribed in the end, and the product will be the whole ſurface.

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* The ſurfaces of ſimilar priſms, and indeed of any other ſimilar bodies, are as the ſquares of their like lineal dimenſions. This follows from their being compoſed of ſimilar plane figures, alike placed.

Perimeter, is the compass or sum of all sides which bound any figure of what kind soever whether rectilinear or mixed

EXAMPLE I.

What is the upright ſurface of a triangular priſm whoſe length is 20 feet, and the ends of its baſe each 18 inches?

Here $18 \times 3 = 54$ inches $= 4\frac{1}{2}$ feet is the perimeter of the baſe.

Therefore $4\frac{1}{2} \times 20 = 90$ ſquare feet is the upright ſurface.

Again, by rule 2 problem 4 ſection 1, we have $2 \times \frac{3}{2} \times \frac{3}{2} \times \cdot 433013 = 1\cdot 9485585 =$ the area of the two ends.

Therefore $91\cdot 9485585$ is the whole ſurface.

EXAMPLE II.

What is the ſurface of a cube, the length of each of whoſe ſides is 20 feet?

Firſt, $20 \times 20 = 400$ is the area of one ſide.
And $400 \times 6 = 2400$ is the whole ſurface required.

EXAMPLE III.

What muſt be paid for lining a rectangular ciſtern with lead at 2d a lb, the lead being 7 feet to the lb, ſuppoſing the length within ſide to be 3 feet 2 inches, the breadth 2 feet 8 inches, and height 2 feet 6 inches?

Firſt, $(38 + 32) \times 2 \times 30 = 70 \times 60 = 4200$ ſquare inches, the two ſides and two ends together.

Then, $38 \times 32 = 1216$ is the area of the bottom.

Therefore $4200 + 1216 = 5416$ ſquare inches $= 37\frac{11}{18}$ ſquare feet, is the whole area.

And $37\frac{11}{18} \times 7 = 263\frac{5}{18}$ lb, is the whole weight.

Therefore 1 lb : 2d :: $263\frac{5}{18}$: 2l 3s $10\frac{5}{9}$d, the coſt.

EXAMPLE IV.

What is the convex ſurface of a round priſm, or cylinder, whoſe length is 20 feet, and the diameter of whoſe baſe is 2 feet?

Firſt, $3 \cdot 1416 \times 2 = 6 \cdot 2832$ is the circumference.

Therefore $6 \cdot 2832 \times 20 = 125 \cdot 664$ is the convex ſurface.

EXAMPLE V.

What is the whole ſurface of a cylinder whoſe length is 10 feet, and the circumference 3 feet?

Here $\frac{3}{2 \times 3 \cdot 1416} = \cdot 477463$ is the radius of the end.

Theref. $10 \cdot 477463 \times 3 = 31 \cdot 43239$ is the whole ſurface.

PROBLEM II.

To find the Solidity of a Priſm.

Multiply the area of the baſe by the height, and the product will be the ſolidity.*

* For, if we conceive to be cut off from the priſm, by a plane parallel to the ends, a part whoſe height is equal to the lineal meaſuring unit; and then imagine both its ends to be divided, in a ſimilar manner, into as many ſquares as are expreſſed by the area of each, the ſide of each of thoſe ſquares being equal to the lineal meaſuring unit; and, laſtly, ſuppoſe planes to be drawn through the correſponding lines of diviſion; it is evident that the part cut off will be divided, by thoſe planes, into as many cubes as there are ſquares in each end, and alſo having the ſame dimenſion with thoſe ſquares, viz. the lineal meaſuring unit; and this number of cubes is the meaſure of the part.

But the magnitude of the whole priſm, or, indeed, of any other of an equal baſe, is to the magnitude of the part whoſe height is the lineal meaſuring unit, as the length of the whole is to that unit (1).

And

EXAMPLE I.

How many ſolid feet are in a ſquare priſm whoſe length is $5\frac{1}{2}$ feet, and each ſide of its baſe $1\frac{1}{3}$ feet?

$1\frac{1}{3} \times 1\frac{1}{3} = \frac{4}{3} \times \frac{4}{3} = \frac{16}{9}$ is the area of the end.
And $\frac{16}{9} \times 5\frac{1}{2} = \frac{16}{9} \times \frac{11}{2} = \frac{88}{9} = 9\frac{7}{9}$ ſolid feet, is the content.

EXAMPLE II.

How many ale gallons of water will the ciſtern hold, whoſe length, breadth, and height, are 3 feet 2 inches, 2 feet 8 inches, and 2 feet 6 inches?

$38 \times 32 \times 30 = 36480$ is the ſolidity in inches.
And $36480 \div 282 = 6080 \div 47 = 129\frac{17}{47}$, the ale gallons.

EXAMPLE III.

What is the capacity of a cylinder whoſe height, and the circumference of its baſe, are each 20 feet?

Firſt, $\frac{20}{3\cdot1416} =$ the diameter, and $10 \times \frac{10}{3\cdot1416} = \frac{100}{3\cdot1416} = \frac{25}{\cdot7854}$ is the area of the end.

Then $\frac{25}{\cdot7854} \times 20 = \frac{2500000}{3927} = 636\cdot61828$ is the content.

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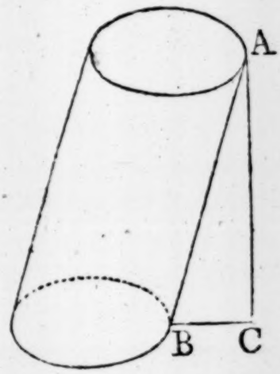
And therefore the meaſure of the whole is equal to that of the part as often repeated as there are lineal meaſuring units in the height; that is, equal to the baſe drawn into the height.

And that the rule is true for oblique priſms as well as right ones, will be evident by conceiving, according to the method of CAVALERIUS, a right and an oblique priſm, of equal baſes and heights, to be made up of an indefinite number of equal thin plates, all parallel to the baſe; for the priſms being both of a height, the one of them will require as many of ſuch plates to compoſe it as the other.

EXAMPLE IV.

What is the capacity of the oblique cylinder, whose axe, and the circumference of its base, are each 20 feet; the axe making an angle of 75° with the base?



As in the last example, the area of the base is $\frac{25}{\cdot 7854}$ or $\frac{125000}{3927}$.

But, as radius : sin. ∠B 75° :: AB 20 : 19·318516 = AC, the height of the cylinder.

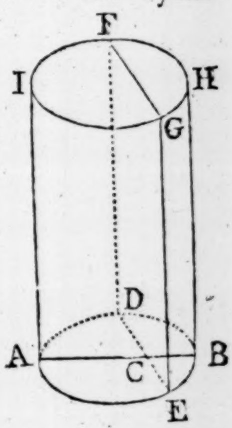
Therefore $\frac{125000}{3927} \times 19{\cdot}318516 = 125000 \times {\cdot}0049194082 =$ 614·92602 is the capacity required.

Otherwise.

As radius : sin. ∠B :: 636·61828 (the capacity in the last example) : 636·61828 × ·9659258 = 614·92602 the capacity, the same as before.

EXAMPLE V.

Suppose the right cylinder whose length is 20 feet, and diameter 3 feet, be cut by a plane parallel to, and at the distance of, 1 foot from its axe: required the solidities of the two prisms into which the cylinder is cut.



$\frac{CB}{BA} = \frac{\frac{1}{2}}{3} = \frac{1}{6} = {\cdot}1\dot{6} =$ the tab. versed sine; to which in the table of circular segments corresponds

the area - - - ·08604117
which taken from ·78539816
leaves the other seg. ·69935699

Then

Then multiplying each by 3^2 or 9 gives

$9 \times \cdot 08604117 = \cdot 77437053 =$ seg. DBE,
$9 \times \cdot 69935699 = 6 \cdot 29421291 =$ seg. DAE.

Hence

$20 \times \cdot 77437053 = 15 \cdot 48741 =$ slice FGHBED,
$20 \times 6 \cdot 29421699 = 125 \cdot 88434 =$ slice FGIAED.

PROBLEM III.

To find the Surface of a Right Pyramid.

Multiply the perimeter of the base by the slant height, or length of the side, and half the product will, evidently, be the surface, or the sum of the areas of all the triangles which form it.*

EX-

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* *Corol.* 1. Hence, because, in a right cone, the circumference of the base into half the side, is equal to the curve surface, and into half the radius, is equal to the base; therefore as the radius of the base is to the side, so is the base to the curve surface.—And the same is true of any other pyramid whose base is a regular figure, viz. as the radius of the circle inscribed in the base is to the slant height, so is the base to the surface.

Corol. 2. Let there be a cone and cylinder of the same base and altitude, the common altitude being equal to the radius of the base; then the base, the surface of the cone, and the surface of the cylinder, are to one another as the numbers 1, $\sqrt{2}$, and 2, and so are in continual proportion.

And the same is true of any other regular prism and pyramid, of the same base and altitude, the altitude being equal to the radius of the circle inscribed in the base.

Moreover, the curve surface of a cylinder, is equal to a circle whose radius is a mean proportional between the side of the cylinder and the diameter of its base. And the curve surface of a cone, is equal to the circle whose radius is a mean proportional between the side of the cone and the radius of its base.

4

EXAMPLE I.

What is the surface of a triangular pyramid, the slant height being 20, and each side of the base 3?

First, $3 \times 3 = 9$ is the perimeter of the base.
Then $9 \times 10 = 90$ is the surface required.

EXAMPLE II.

Required the surface of a square pyramid whose slant height is 20, and each side of the base 3.

Here $3 \times 4 = 12$ is the perimeter of the base.
Therefore $12 \times 10 = 120$ is the surface sought.

EXAMPLE III.

Required the convex surface of a circular pyramid, or cone, whose slant side is 20, and the circumference of the base 9.

Here $10 \times 9 = 90$ is the convex surface required.

PROBLEM IV.

To find the Surface of the Frustum of a Right Pyramid.

Multiply the sum of the perimeters of the ends by the slant height, and half the product will be the surface.*

EXAMPLE I.

How many square feet are in the surface of the frustum of a square pyramid, whose slant height is

* DEMONSTRATION.

For the surface is composed of a number of equal trapezoids, whose common height is equal to the slant height of the frustum, and the sums of whose parallel sides, make up the perimeters of the ends of the frustum.

10 feet, each side of the greater base being 3 feet 4 inches, and each side of the less 2 feet 2 inches?

Here $4 \times (40 + 26) = 4 \times 66 = 264$ is the sum of the perimeters.

Then $264 \times 10 \times 6$ square inches $= 11 \times 10 = 110$ feet, is the surface required.

EXAMPLE II.

If a segment of 6 feet slant height be cut off a cone whose slant height is 30 feet, and circumference of its base 10 feet; what will be the surface of the frustum?

As 30 = the height of the whole
: 6 = the height of the upper part
:: 10 = the bottom circumference
: 2 = the circumference at the top of the frustum.

Then $6 \times 24 = 144$ is the surface required.

PROBLEM V.

To find the Solidity of a Pyramid.

Multiply the base by the perpendicular height, and $\frac{1}{3}$ of the product will be the content.*

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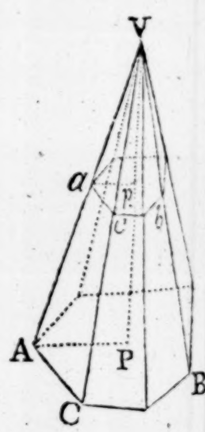


* Let VAB be a pyramid whose base is any plane figure, regular or irregular, VP a perp. to the base in P, and meeting the section ab, parallel to the base, in p; and put A for the area of the base AB, a for that of the section ab, h the height PV of the pyramid, and x the height Pp of the frustum AabB.

First, the parallel sections AB, ab, are to each other as the squares of their distances, VP, Vp, from the vertex. For, because they are similar plane figures, which are as the squares of their like sides, we shall have $A : a :: AC^2 : ac^2$: (by sim. triangles) $AV^2 : aV^2$ :: (by similar triangles again) PV^2 (hh) : $pV^2 = (\overline{h-x}|^2)$.

Hence

EXAMPLE I.

Required the solidity of a triangular pyramid whose height is 30, and each side of the base 3.

First, $3^2 \times {\cdot}433013 = 3{\cdot}897117 =$ area of base.
Then $3{\cdot}897117 \times 10 = 38{\cdot}97117$ is the solidity.

EX-

Hence we find $a = (h - x)^2 \times \frac{A}{hh}$, and $a\dot{x} = \frac{A\dot{x}}{hh} \times (h - x)^2$, the fluent of which is $Ax \times (1 - \frac{x}{h} + \frac{xx}{3hh}) =$ the content of the frustum $AabB$.

Corol. 1. Since $\sqrt{A} : \sqrt{a} :: h : h - x$, it will be $\sqrt{A} - \sqrt{a} : \sqrt{A} :: x ; h = \frac{x\sqrt{A}}{\sqrt{A} - \sqrt{a}}$, which being substituted for h in the quantity $Ax \times (1 - \frac{x}{h} + \frac{xx}{3hh})$, expressing the content of the frustum, that content will become $\frac{1}{3}x \times (A + a + \sqrt{Aa})$.

Corol. 2. Or, if S be a side, or any other line, in the greater end, and s a similar side or line in the less, since $S - s : S :: x : h$, hence $\frac{x}{h} = \frac{S - s}{S}$, which being substituted instead of it, gives $\frac{1}{3}Ax \times \frac{SS + Ss + ss}{SS}$ for the value of the said frustum.

Corol. 3. If $x = h$, then $a = 0$, and the general expression becomes $\frac{1}{3}Ah =$ the whole pyramid; which is our rule.

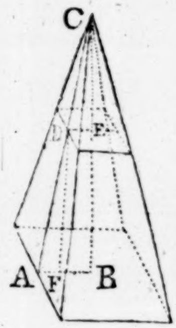
Corol. 4. If the base be a regular figure; let S be one of its sides, s a side of the less end of the frustum, and n the area of a similar figure whose side is 1, to be found in the table in page 114: then $A = nSS$, and $a = nss$; hence the solidity of the frustum will be $(SS + Ss + ss) \times \frac{1}{3}nx = \frac{S^3 - s^3}{S - s} \times \frac{1}{3}nx$, and that of the pyramid $= \frac{1}{3}nSSh$.

Corol. 5. If $f = \frac{1}{2}S + \frac{1}{2}s$ the half sum of the sides, and $d = \frac{1}{2}S - \frac{1}{2}d$ the half dif. then the frustum will be $= (3f^2 + d^2) \times \frac{1}{3}nx$.

Corol.

EXAMPLE II.

Required the solidity of the square pyramid, each side of whose base is 30, and the slant height 25.



First, $30 \times 30 = 900 =$ the base.

But $BC = \sqrt{AC^2 - AB^2} = \sqrt{25^2 - 15^2}$ $= \sqrt{5^2 \times 5^2 - 3^2 \times 5^2} = \sqrt{4^2 \times 5^2} =$ $4 \times 5 = 20$, the perpendicular height.

Then $300 \times 20 = 6000$ is the solidity.

EX-

Corol. 6. If the base be a square; then $n = 1$, and the last expressions become $\frac{1}{3}ssh$ for the whole pyramid, and $(SS + Ss + ss) \times \frac{1}{3}x$, or $\frac{S^3 - s^3}{S - s} \times \frac{1}{3}x$, or $(3f^2 + d^2)\frac{1}{3}x$, for the frustum.

Corol. 7. If the base be a circle; then $n = \cdot 7854$, $\cdot 2618 ssh =$ the cone, and $(SS + Ss + ss) \times \cdot 2618 x$, or $\frac{S^3 - s^3}{S - s} \times \cdot 2618 x$, or $(3f^2 + d^2) \times \cdot 2618 x =$ the frustum; where S is the diameter of the base, and s that of the top.

Corol. 8. If $a = A$; then $(A + a + \sqrt{Aa}) \times \frac{1}{3}x$ becomes $3A \times \frac{1}{3}x = Ax =$ the prism.

Corol. 9. Hence a prism is to a pyramid of the same base and height, as 3 to 1, and to the frustum of the same base and height, as 1 to $1 - \frac{x}{h} + \frac{xx}{3hh}$, or as 3 to $1 + \frac{a}{A} + \sqrt{\frac{a}{A}}$, or as 3 to $\frac{SS + Ss + ss}{SS}$ or $1 + \frac{s}{S} + \frac{ss}{SS}$.

Corol. 10. Similar pyramids are as the cubes of their like sides. For the pyramid is as Ah, or as h^3, or as VA^3 because A is as hh by similar triangles.

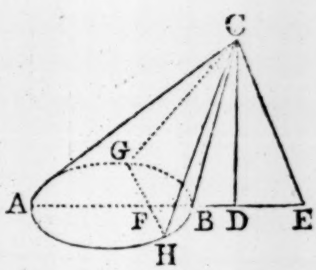
Corol. 11. All similar solids are as the cubes of their like sides. This follows from their being composed of similar pyramids.

EXAMPLE III.

Wanting to measure an oblique cone; and having taken the circumference of the base equal to 40 feet, and the outward angle which the side of the cone, where it was shortest, made with the horizon, 85°; going thence in a direct line towards the part to which the vertex inclined, to the distance of 100 feet, I found there the angle of elevation of the vertex of the cone to be 50°: required the solidity.

$$\text{The } \angle B + \angle E = 85^\circ + 50^\circ = 135^\circ$$
which taken from 180
leaves $\angle BCE$ 45

Then s. $\angle BCE$ 45°	9·8494850
s. $\angle E$ 50° -	9·8842540
BE 100 -	2·0000000
BC - -	2·0347690



And radius - -	10·0000000
s. $\angle B$ 85° -	9·9983442
BC - -	2·0347690
DC 107·9228	2·0331132

But $\frac{40}{3 \cdot 1416} = \frac{10}{\cdot 7854} = 12 \cdot 732396$ is the diameter.

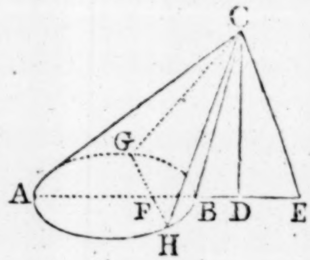
And $12 \cdot 732396 \times 10 = 127 \cdot 32396$ is the base.

Theref. $\frac{1}{3}DC \times 127 \cdot 32396 = 42 \cdot 44132 \times 107 \cdot 9228$ $= 4580 \cdot 386$ is the content of the cone.

EXAMPLE IV.

There is a cone whose perpendicular altitude is 40 feet, and the circumference of its base 30 feet, which is cut by a plane drawn from the vertex, and passing through the base at the distance of 2 feet from the center;

center; what is the ſolidity of the whole cone, and of the two pyramids, GHBC, GHAC, into which it is cut?



Here $\frac{30}{3\cdot1416} = 9\cdot549299$ is the diameter of the baſe.

And its half $4\cdot774648$ is the radius.

Therefore $4\cdot774648 - 2 = 2\cdot774648$ is the verſed ſine of the leſs ſegment of the circle.

Then, to find its area by the table of circular ſegments, $2\cdot774648 \div 9\cdot549299 = \cdot2905611$ is the tabular verſed ſine, to which anſwers the tabular

ſegment $\cdot18955709$;

and this taken from $\cdot78539816$

leaves the other tab. ſeg. $\cdot59584107$.

Then each of theſe ſegments multiplied by the ſq. of the diam. give $17\cdot2856$ for the leſs baſe GBHG,

and $54\cdot3341$ for the greater baſe GAHG.

Then $\frac{40}{3} \times 17\cdot2856 = 230\cdot474\frac{2}{3}$ = the leſs pyramid,

and $\frac{40}{3} \times 54\cdot3341 = 724\cdot454\frac{2}{3}$ = the greater pyr.

their ſum is $954\cdot929\frac{1}{3}$ = the whole cone.

PROBLEM VI.

To find the Solidity of the Fruſtum of a Pyramid.

Add into one ſum the areas of the two ends and the mean proportional between them, multiply the ſum by the perpendicular height, and $\frac{1}{3}$ of the product will be the ſolidity.

That is, If A be the area of the greater end,

a that of the leſs, and b the height,

Then $(A + a + \sqrt{Aa}) \times \frac{1}{3}b$ will be the ſolidity.

Note

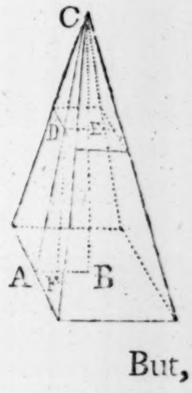
Note 1. If the ends be regular polygons, the particular rule for them will be easier, thus: Add together the square of a side of each end of the frustum, and the product of those sides, multiply the sum by the height, and the product by the tabular area answering to the particular figure of the ends, and $\frac{1}{3}$ of the last product will be the content.

Or, Divide the difference of the cubes of the said sides by their difference, and multiply the quotient by the height, and the tabular area, and take $\frac{1}{3}$ of the product.

Note 2. If the ends be circles, the frustum will be that of a cone, and then multiply ·2618, namely $\frac{1}{3}$ of ·7854, by the height, and the product either by the quotient arising from the division of the difference of the cubes of the diameters by the difference of the diameters, or by the sum arising from the addition of the square of each diameter and the product of the diameters, or by the sum arising from the square of the half difference of the diameters added to triple the square of the half sum.*

EXAMPLE I.

How many solid feet are there in a tree whose bases are squares, each side of the one being 15 inches, and each side of the other 6, and the length along the side measures 24 feet?



Here $15 \times 15 = 225$, the greater base,
and $6 \times 6 = 36$, the less,
and $15 \times 6 = 90$, their mean.

their sum is 351,
and $\frac{1}{3}$ of it is 117.

But,

* All these rules come from the demonstration of prob. 5, and its corollaries.

But, $\overline{AB - DE} = 7\frac{1}{2} - 3 = 4\frac{1}{2} = AF$, and $\sqrt{AD^2 - AF^2}$ $\sqrt{(24 \times 12)^2 - (4\frac{1}{2})^2} = 287{\cdot}9649$ inches $=$ DF, the perpendicular.

Therefore $117 \times 287{\cdot}9649 = 33691{\cdot}8933$ inches $= 19{\cdot}49762$ feet is the solidity.

EXAMPLE II.

If a cask, which is two equal conic frustums joined together at the bases, have its bung diameter 28 inches, its head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold?

Here $20^2 \times {\cdot}7854 = 314{\cdot}16$ is the area at the end,
and $28^2 \times {\cdot}7854 = 615{\cdot}7536 =$ area bung circle,
and $20 \times 28 \times {\cdot}7854 = 439{\cdot}824$ their mean propor.
their sum is - - - $1369{\cdot}7376$,
and its third part is - $456{\cdot}5792$,
which multiplied by 40, the length of both frustums together, produces $18263{\cdot}168$ solid inches; which divided by 231, the inches in a wine gallon, thus,

$$231 = 3 \times 7 \times 11 \left\{ \begin{array}{l} 3)18263{\cdot}168 \\ \hline 7)6087{\cdot}7226 \\ \hline 11)869{\cdot}6746 \\ \hline \end{array} \right.$$

gives $79{\cdot}0613$ wine gallons.

Or $(20^2 + 28^2 + 20 \times 28) \times 40 \times {\cdot}2618 = 1744 \times 10{\cdot}472 = 18263{\cdot}168$, the solidity the same as before.

PROBLEM VII.

To find the Solidity of a Wedge.

To twice the length of the base add the length of the edge, multiply the sum by the breadth of the base,

and

and the product by the height of the wedge, and $\frac{1}{6}$ of the last product will be the solidity.

That is, using the letters in the demonstration below, $(2\text{L} + l) \times \frac{1}{6}bh$ is the content.*

Note. If the length of the edge be equal to the length of the base, the wedge will be equal to half a prism of the same base and height, or equal to half the product of the base and height.

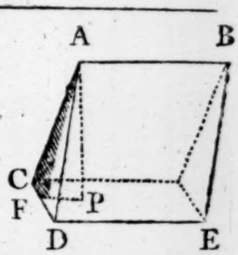
EXAMPLE I.

How many solid feet are in a wedge whose base is 3 feet 4 inches long, and 10 inches broad; and each end is inclined to the base in an angle of 70°; the edge being 2 feet 6 inches long?

First, $\text{FP} = \frac{1}{2}\text{L} - \frac{1}{2}l = 5$ inches.

And as rad. : tang. ∠F 70° :: FP 5 : AP = 13·737387 = h.

Then, $\frac{1}{6}bh \times (2\text{L} + l) = 10 \times 110 \times 2{\cdot}2895645 = 1100 \times 2{\cdot}2895645 = 2518{\cdot}52095$ solid inches $= 1{\cdot}45747$ solid feet.



* DEMONSTRATION.

Put $\text{L} = \text{BC}$ the length of the base,
$l = \text{EF}$ the length of the edge,
$b = \text{AB}$ the breadth of the base,
$h = \text{EP}$ the height of the wedge.

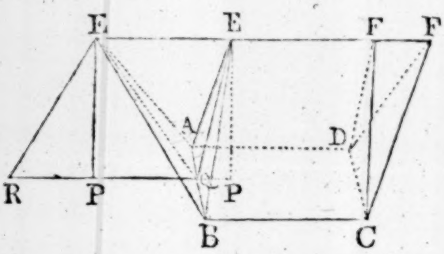
Then, since it is evident that, according as the edge is shorter or longer than the base, the wedge is greater or less than half a prism of the same height and breadth with the wedge, and length equal to that of the edge, by a pyramid of the same height and breadth at the base also, and the length of whose base is equal to the difference of the lengths of the edge and base of the wedge; we shall have $\frac{1}{2}blh \pm \frac{1}{3}bh \times (\pm \text{L} \mp l) = \frac{1}{2}blh + \frac{1}{3}bh \times (\text{L} - l) = \frac{1}{6}bh \times (3l + 2\text{L} - 2l) = \frac{1}{6}bh \times (2\text{L} + l)$. *Q.E.D.*

Corol. If $l = \text{L}$, the rule will become $\frac{1}{6}bh \times (3\text{L}) = \frac{1}{2}bh\text{L}$ $= \frac{1}{2}$ a prism of the same base and height, as it ought.

Scholium. It is evident that, whether the two ends, or the two sides of the wedge, be equally or unequally inclined to the base, it will make no difference in the rule.

EXAMPLE II.

If the length and breadth of the base of a wedge be 35 and 15 inches, and the length of the edge 55; what will be its solidity in feet, supposing the base to make an angle of $115°$ with one end of the wedge, and of $125°$ with the other.



Let EP be perpendicular to the base, ER parallel to the end DCF, and RPQ perpendicular to AB.

Then $RQ = EF - BC = 55 - 35 = 20$, and the angles at R and Q, or the supplements of the angles $115°$ and $125°$, are $65°$ and $55°$; and consequently the angle $REQ = 60°$. Hence,

As $s.\angle E\ 60° : s.\angle Q\ 55° :: RQ : RE = 18{\cdot}9175$; and $s.\angle P\ 90° : s.\angle R\ 65° :: RE : EP = 17{\cdot}14508$, the height of the wedge, or h.

Wheref. $\frac{1}{6}bh \times (2L + l) = \frac{1}{6} \times 15 \times 17{\cdot}14508 \times (70 + 55) = 8{\cdot}57254 \times 5 \times 125 = 5357{\cdot}8375$ solid inches $= 3{\cdot}1006$ solid feet, the content required.

PROBLEM VIII.

To find the Solidity of a Prismoid.

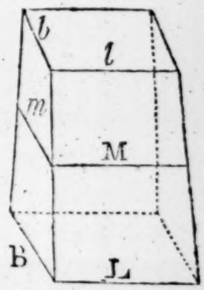
GENERAL RULE.

To the sum of the areas of the two ends, add four times the area of a section parallel to, and equally distant from, both ends; multiply the last sum by the height, and $\frac{1}{6}$ of the product will be the solidity.

PAR-

PARTICULAR RULE.*

Or, If the bases be dissimilar rectangles, take two corresponding dimensions, and multiply each by the sum of double the other dimension of the same end and the dimension of the other end corresponding to this last dimension; then multiply the sum of the products by the height, and $\frac{1}{6}$ of the last product will be the solidity.



That is, if L, l be one dimension of each corresponding to each other, and B, b the other corresponding dimensions, and h the height.

Then $[(2L + l)B + (2l + L)b] \times \frac{1}{6}h =$ the solidity.

Note. Corresponding dimensions are those which are connected by a side of the solid, as is evident in the figure.

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* DEMONSTRATION.

It is evident that the rectangular prismoid is composed of two wedges, whose bases are the two ends of the prismoid, and whose heights are each equal to that of the prismoid; therefore, by the last problem, its solidity will be $= [(2L + l)B + (2l + L)b] \times \frac{1}{6}h$. Which is the particular rule.

Corol. 1. Since $\frac{1}{2}L + \frac{1}{2}l = M$, and $\frac{1}{2}B + \frac{1}{2}b = m$, are the length and breadth of a section parallel to, and equally distant from, each end; the above rule $[(2L + l)B + (2l + L)b] \times \frac{1}{6}h$ or $(2BL + Bl + 2bl + bL) \times \frac{1}{6}h$, will become $(BL + bl + 4Mm) \times \frac{1}{6}h$. That is, the sum of the areas of the two ends and four times the section in the middle, multiplied by $\frac{1}{6}h$.

Corol. 2. This last rule will serve for any prismoid, or cylindroid, of whatever figures the ends may be; inasmuch as they may be conceived to be composed of an infinite number of rectangular prismoids. Which is the general rule.

EXAMPLE I.

How many solid feet of timber are in a tree whose ends are rectangles, the length and breadth of the one being 14 and 12 inches, and their corresponding sides of the other 6 and 4 inches; and the perpendicular length $30\frac{1}{2}$ feet?

First, by the general rule,

$$\left.\begin{array}{l} \frac{14+6}{2} = \frac{20}{2} = 10 \\ \frac{12+4}{2} = \frac{16}{2} = 8 \end{array}\right\} \text{the dimensions in the middle}$$

$$\begin{array}{lll} 10 \times 8 \times 4 = 320 & = & \text{four times the middle area} \\ 14 \times \phantom{8 \times{}} 12 = 168 & = & \text{area of the greater end} \\ 6 \times 4 = 24 & = & \text{area of the less end} \end{array}$$

their sum is 512 square inches $= \frac{32}{9}$ square feet.

Then $\frac{32}{9} \times \frac{30\frac{1}{2}}{6} = \frac{16 \times 30\frac{1}{2}}{9 \times 3} = \frac{162\frac{2}{3}}{9} = 18\frac{2}{27}$ feet, the solidity.

Secondly, by the particular rule,

Here $L = 14$, $B = 12$, $l = 6$, $b = 4$ inches, and $h = 30\frac{1}{2}$ feet; therefore $[(2L + l)B + (2l + L)b] \times \frac{1}{6}h = \frac{34 \times 12 + 26 \times 4}{6 \times 144} \times 30\frac{1}{2} = 18\frac{2}{27}$ feet, the same as before.

EXAMPLE II.

What is the capacity of a waggon whose inside dimensions are thus: at the top, the length and breadth $81\frac{1}{3}$, and 55 inches; at the bottom, the length 41 and breadth $29\frac{1}{2}$; and the perpendicular depth $47\frac{1}{4}$?

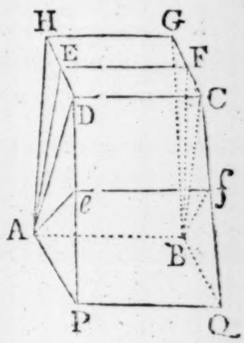
Here $[(2L + l)B + (2l + L)b] \times \frac{1}{6}h = (102 \times 55 + 81{\cdot}75 \times 29{\cdot}5) \times 47{\cdot}25 \times \frac{1}{3} = 8021{\cdot}625 \times 15{\cdot}75 = 126340{\cdot}59375$ cubic inches $= 448{\cdot}01629$ ale gallons, the content required.

PRO-

PROBLEM IX.

To find the Solidity of the two Parts, called Ungulas or Hoofs, into which the Frustum of a Rectangular or a Square Pyramid, or a Rectangular Prismoid, is cut by a Plane Inclined to its Base.

CASE I.



If the plane, passing through A and B, cut the end in EF, between GH and DC; it will cut off the wedge AEHGFB; whose base is EFGH, edge AB, and height the same with that of the frustum, or prismoid; and the remaining part AEDCFBQP will be a prismoid.

Then, by problem 7, find the content of the wedge ABFEHG, and that of the prismoid ABQPDCFE by problem 8.

EXAMPLE.

If the frustum of a square pyramid be cut by a plane, passing through one side of the less end, and through the middle of the greater end; what are the contents of the two parts, supposing each side of the greater end to be 15 inches, each side of the less 6, and the slant height 24 feet?

Since, by example 1 problem 6, the perpendicular height is 287·9649 inches, therefore,

First, $\frac{1}{6}(2 \times 15 + 6) \times 7\frac{1}{2} \times 287{\cdot}9649 = 45 \times 287{\cdot}9649 = 12958{\cdot}4205$ inches $= 7{\cdot}499086$ feet, the content of the wedge.

 And

And $\frac{1}{6}[(30+6)\times 7\frac{1}{2}+(12+15)\times 6]\times 287\cdot 9649$ $= \frac{1}{6}(18\times 15+27\times 6)\times 287\cdot 9649 = 72\times 287\cdot 9649$ $= 20733\cdot 4728$ inches $= 11\cdot 998536$ feet, the prismoid.

CASE II.

If the plane paſs through DC, as well as AB, the ſolid is thereby divided into two wedges or hoofs ABCGHD, ABCQPD, whoſe two baſes are the ends or baſes of the ſolid.

And then the contents of the two parts will be found by problem 7.

EXAMPLE.

Let there be taken here the ſame figure as in the laſt example, to find the content of the two wedges into which it is cut.

Firſt, $\frac{1}{6}(30+6)\times 15\times 287\cdot 9649 = 90\times 287\cdot 9649$ $= 25916\cdot 841$ inches $= 14\cdot 99817$ feet, the greater wedge.

And $\frac{1}{6}(12+15)\times 6\times 287\cdot 9649 = 27\times 287\cdot 9649$ $= 7775\cdot 0523$ inches $= 4\cdot 49945$ feet, the leſs wedge.

CASE III.

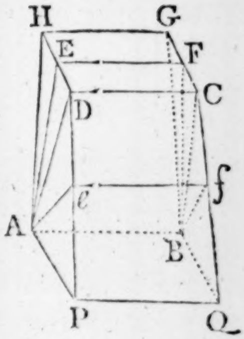
If the plane cut the ſide of the ſolid in *ef*; the part cut off ABQ*fe*P will be a wedge, whoſe baſe is AQ; and which being taken from the whole figure, will leave the content of the part AB*fe*DCGH.

EXAMPLE.

Let there be taken here the ſame figure, ſuppoſing the plane to cut the ſide PC in *ef* at the diſtance of 10 feet from PQ, which let be now ſuppoſed one ſide of the leſs end AQ; to find the ſolidity of the two parts.

Firſt,

First, As PD 24 : Pe 10 :: 287·9649 the height of the whole solid : 119·98536 the height of the wedge AQe.



And as PD 24 : Pe 10 :: DC − PQ 9 : ef − PQ $= \frac{15}{4} =$ 3·75.

Hence $ef = 3{\cdot}75 + 6 = 9{\cdot}75$ = the length of the edge.

Wherefore $\frac{1}{6}(12 + 9{\cdot}75) \times 6 \times 119{\cdot}98536 = 21\frac{3}{4} \times 119{\cdot}98536 = 87 \times 29{\cdot}99634 = 2609{\cdot}68158$ inches = 1·51023 feet, the solidity of the wedge AQe.
which taken from 19·49762 the whole frustum found by prob. 8,
leaves 17·98739 for the content of the part ABfeDCHG.

PROBLEM X.

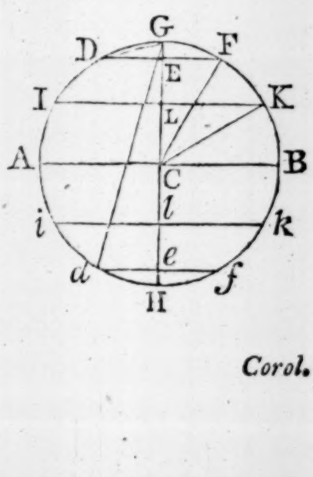
To find the Curve Surface of a Sphere, or of any Segment or Zone of it.

Multiply the circumference of the sphere by the height of the part required, and the product will be the curve surface, whether it be segment, zone, hemisphere, or the whole sphere.*

 Note.

* DEMONSTRATION.

Put d = the diameter AB or 2CG, x = CE the height of the zone ADFB, y = DE, z = AD, $p = 3{\cdot}14159$, and s = the surface required.



Then $\dot{s} = 2py\dot{z}$; but $y : \frac{1}{2}d :: \dot{x} : \dot{z}$, or $2y\dot{z} = d\dot{x}$; therefore $\dot{s} = pd\dot{x}$, and consequently $s = pdx$; viz. the product of the circumference of the sphere and height of the zone; for pd is = the circumference of the circle whose diameter is d.

Corol.

Note. The height of the whole ſphere is its diameter.

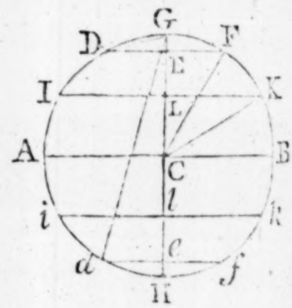
EXAMPLE I.

If the diameter or axe of the earth be $7957\frac{3}{4}$ miles, what is the whole ſurface, ſuppoſing it a perfect ſphere?

Firſt, $7957\frac{3}{4} \times 3.141592 = 25000$ miles, very near, = the circumference.

Then $7957\frac{3}{4} \times 25000 = 198943750$ ſquare miles = the whole ſurface required. And the half is 99471875 = the ſurface of the hemiſphere.

EX-



Corol. 1. When x becomes = CG = $\frac{1}{2}d$, s will be = $\frac{1}{2}pdd$ = the ſurface of the hemiſphere. And conſequently that of the whole ſphere is pdd, the product of the circumference and height or diameter.

Corol. 2. To or from $\frac{1}{2}pdd$, the ſurface of the hemiſphere, add or ſubtract pdx, that of the zone ADFB, and the remainder $pd \times (\frac{1}{2}d \pm x) = pd \times$ GE will be the ſurface of the ſegment DGF, viz. the product of the circumference and height. So that the rule is general. *Q. E. D.*

Corol. 3. The ſurfaces of ſpheres, and alſo of their ſimilar parts, are to each other as the ſquares of their diameters. For, by exterminating the common given quantity p, they are as d, the diameter, into the height of the part; but the heights of ſimilar parts are as the diameters; therefore &c.

Corol. 4. The ſurfaces of any ſegments or zones of a ſphere are to each other, or to that of the whole ſphere, as their heights. For pd is common to them all.

Corol. 5. Or the ſurface of any ſegment or zone of a ſphere, is as its height.

Corol.

EXAMPLE II.

To find the surface of the two frigid zones of the earth.

Note. The frigid zones are the two opposite segments DGF, dHf, in which each of the arcs DG, dH, or half the breadth of the zone, is $23\frac{1}{2}$ degrees.

Draw the radius CF; then the angle FCE $= 23\frac{1}{2}$ degrees, and its complement EFC $= 66\frac{1}{2}$ degrees; also the angle E $= 90$ degrees.

Then, as s. ∠E : s. ∠F :: FC $= 3978\frac{7}{8}$: CE $= 3648{\cdot}8675054$.
Which taken from CG — — $3978{\cdot}8750000$,
leaves the height GE — — $330{\cdot}0074946$.
And the circumference is 25000.

Hence $25000 \times 330{\cdot}0074946 = 8250187{\cdot}365$ = the surface of the segment DGF or frigid zone.

Corol. 6. The surface of any segment or zone, is equal to 4 times a circle whose diameter is a mean proportional between its height and the diameter of the sphere. For this circle is $= \frac{1}{4}p \times (\sqrt{dx})^2 = \frac{1}{4}pdx$.

Corol. 7. Hence the surface of a segment DGF is equal to 4 times the circle whose diameter is the chord DG drawn from the vertex to the extremity of the base; or equal to a circle whose radius is that chord. For DG $= \sqrt{\text{GH} \times \text{GE}} = \sqrt{dx}$.

Corol. 8. And hence the surface of a sphere is equal to 4 times the area of a great circle of it; that is, 4 times the area of a circle of the same diameter with the sphere. And consequently the surface of an hemisphere is double the area of its base.

Corol. 9. Or, the surface of a sphere is equal to a circle whose diameter is double to that of the sphere.

Corol. 10. The surface of any segment or zone of a sphere, is equal to the curve surface of a cylinder of the same height with it, and whose diameter is equal to that of the sphere.

Corol. 11. Hence the surface of a whole sphere, or of a hemisphere, is equal to the curve surface of the circumscribed cylinder.

EXAMPLE III.

To find the ſurface of the torrid zone IK*ki*, which extends to the diſtance of AI or A*i* $23\frac{1}{2}$ degrees on each ſide from AB the diameter.

Here the ∠LKC = $23\frac{1}{2}$ degrees.

Then, s. ∠L : s. ∠LKC :: CK : CL = 1586·572.

And the height L*l* = 2CL = 3173·144.

Hence 25000 × 3173·144 = 79328600 ſquare miles = the ſurface of the torrid zone.

EXAMPLE IV.

Required the convex ſurface of each of the temperate zones IDFK, *idfk*, which are included between the frigid and torrid zones.

Here, by the 1ſt ex. CE = 3648·8675
and by the 2d ex. CL = 1586·572
their difference is LE = 2062·2955 = the height.

Therefore 25000 × 2062·2955 = $51557387\frac{1}{2}$ ſquare miles = each temperate zone.

Hence the two frigid zones	=	16500375
the two temperate zones	=	103114775
the torrid zone	=	79328600
their ſum is		198943750

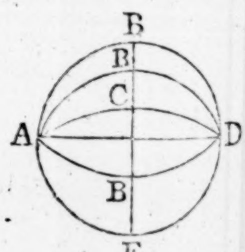
or the ſurface of the whole ſphere.

PROBLEM XI.

To find the Lunar Surface ABDCA, *included between two Great Circles* ABD, ACD *of a Sphere.*

RULE I.

Multiply the diameter AD by the breadth BC of the ſurface in the middle, that is, by the arc which meaſures the angle BAC of inclination of the two circles; and the product will be the ſurface ABDCA.



That

That is, if d = AD the diameter,
and a = the length of the arc BC.
Then ad is the surface.

* RULE II.

As four right angles, to the surface of the sphere;
Or, As one right angle, to a great circle of the sphere;
So is the angle made by the two great circles,
To the surface included by them.

EXAMPLE.

Required the surface included by two great circles forming an angle of 25 degrees; the diameter of the sphere being 10 feet.

1. By rule 1.

First, $3 \cdot 1415926 \times 10 = 31 \cdot 415926$ is the circumf.

And 360 : 25 (:: 72 : 5 :: 144 : 10) :: 31·415926 : 2·181661 the greatest breadth of the surface.

Then $2 \cdot 181661 \times 10 = 21 \cdot 81661$ is the surface req.

2. By

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* DEMONSTRATION.

If the great circle BCE, whose poles are A and D, be conceived to be divided into an indefinite number of equal parts, and great circles be conceived to be drawn through the points of division, and through A and D: it is evident that the whole surface will be divided, by the circles, into the same number of parts, similar and equal to one another; and that, therefore, the surface included between any two of those circles, will be as the number of parts, or as the arc of the circle BE included by them; wherefore as the whole circumference BE is to the arc BC, or as four right angles to the angle BAC, so is the surface of the sphere to the lunar surface ABDCA; or also as one right angle is to the angle BAC, so is the area of a great circle of the sphere ($\frac{1}{4}$ of the surface) to the lunar surface. Which is rule 2.

Corollary. If d be the diameter, and c the circumference of the sphere, and a the arc BC; then, from the process above, $c : a :: cd$ (the surface of the sphere) $: ad =$ the lunar surface. Which is rule 1.

2. By rule 2.

Here $31{\cdot}415926 \times 10 = 314{\cdot}15926$ is the surface of the sphere.

And $360 : 25 :: (144 : 10 ::)\ 314{\cdot}15926 : 21{\cdot}81661$ the surface required.

PROBLEM XII.

To find the Area of a Spherical Triangle; that is, the Spherical Surface included by the Arcs of Three Great Circles of the Sphere Intersecting one another.

As 8 right angles or 720°,
To the surface of the sphere;
Or, As 2 right angles or 180°,
To a great circle of the sphere;
So is the excess of the 3 angles above 2 right angles,
To the area of the triangle.*

That

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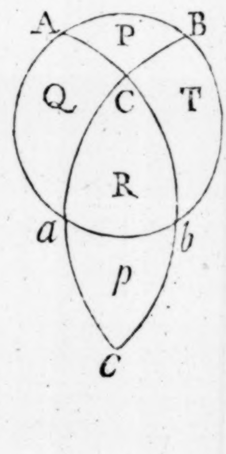
* DEMONSTRATION.



For having produced all the sides of the triangle ABC (P), till they intersect again, and form the triangle p, which by the principles of the sphere will be equal to the former triangle P; put $\frac{1}{2}s = \frac{1}{2}$ the surface of the sphere $= P + Q + R + T$. Then, by the last problem,

$$\begin{cases} 180^\circ : A :: \frac{1}{2}s : P + T, \\ 180 : B :: \frac{1}{2}s : P + Q, \\ 180 : C :: \frac{1}{2}s : P + R = p + R; \text{ hence} \end{cases}$$

$$180 : A + B + C :: \tfrac{1}{2}s : 3P + T + Q + R = 2P + \tfrac{1}{2}s,$$

and as $180 : A + B + C - 180 :: \frac{1}{2}s : 2P :: \frac{1}{4}s : P = s \times \dfrac{A + B + C - 180}{720}$. *Q. E. D.*

Corollary. When $P = 0$, then $(A + B + C \text{ or})\ s = 180$; but when $P = \frac{1}{2}pdd$, half the surface of the sphere, then $s - 180 = 360$, or $s = 540$: consequently $A + B + C$ is always between 180 and 540, that is, greater than 2, and less than 6, right angles.

That is, $pdd \times \frac{s-180}{720} =$ the area of the triangle; putting $p = 3{\cdot}14159$,
$d =$ the diameter of the sphere,
$s =$ the sum of the 3 angles of the triangle.

EXAMPLE.

If the angles be 55, 60, and 85 degrees; what is the trilineal surface; supposing the diameter to be 10?

Here $\cdot7853982 \times 102 = 78{\cdot}53982 = \frac{1}{4}$ of the surface of the sphere. And $55 + 60 + 85 - 180 = 20$.

Then $180 : 20$, or $9 : 1 :: 78{\cdot}53982 : 8{\cdot}72664$, the area of the triangle required.

PROBLEM XIII.

To find the Area of a Spheric Polygon, or to find the Spherical Surface included by Any Number of Intersecting Great Circles.

As 8 right angles or 720°,
To the surface of the sphere;
Or, As 2 right angles or 180°,
To a great circle of the sphere;
So is the excess of all the angles, above the product of 180 and 2 less than the number of angles,
To the area of the spherical polygon.

That is, putting $n =$ the number of angles,
$s =$ the sum of all the angles,
$d =$ the diameter of the sphere,
$p = 3{\cdot}14159$ &c.

Then

Then $pd^2 \times \frac{s - 180 \times (n-2)}{720}$ = the area of the ſpherical polygon.*

EXAMPLE.

What ſurface is included by the intercepted arcs of five interſecting great circles of a ſphere, of 10 feet diameter; ſuppoſing the ſum of the angles formed by thoſe arcs to be 640 degrees.

Here the ſurface of the ſphere is $3{\cdot}1415926 \times 102 = 314{\cdot}15926$.

And $180 \times (n - 2) = 180 \times 3 = 540$; which taken from 640, leaves 100.

Then as 720 : 100 :: 314·15926 : 43·63323, the area of the polygon required.

PRO-

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* DEMONSTRATION.

For if the polygon be ſuppoſed to be divided into as many triangles as it has ſides, by great circles drawn to all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by the laſt problem, it will be, in any one triangle,

As 720 : s :: ſum of its angles — 180 : its area. Theref. by compoſ. as 720 : s :: s + all the vertical angles — $180n$

: ſum of all the triangles or area of the polygon.

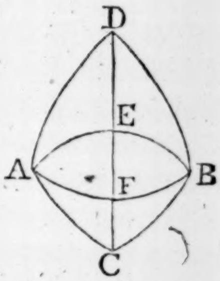
But all the vertical angles = 360 or 180×2. Therefore

As 720 : s :: $s - 180 \times (n - 2)$: s $\times \frac{s - 180 \times (n-2)}{720}$, the area of the polygon. *Q. E. D.*

Corollary. When the polygon is = 0, then s is $= 180 \times (n-2)$; and when the polygon is = the ſemi ſpheric ſurface, then $s = 180 \times (n - 2) + 360 = 180 \times n$; conſequently s the ſum of all the angles of any polygon, is always between $180 \times (n - 2)$ and $180 \times n$, that is, leſs than n times 2 right angles, but greater than $n - 2$ times 2 right angles, n being the whole number of angles.

PROBLEM XIV.

To find the Surface AEBF *included between Two Interfecting Little Circles of a Sphere.*



Through the poles D, C, and interfections A, B, of the two little circles AFB, AEB, draw the great circles AD, BD, AC, BC; and alfo the great circle DEFC.

By prob. 12 find the area of the triangle BCD, having firft found the angles at B, C, and D, from the given fides, by the principles of fpherical trigonometry.

By prob. 10 find the furface of the fegment cut off by the circle of which AFB is a part, thus, viz. As the diameter is to the verfed fine of the arc DF, or DB, fo is the furface of the fphere to that of the fegment.

Then as 4 right angles, to the ∠BDC,
So is that furface, to the part of it DBF.

In the fame manner find the furface CBE.

Then from the fum of DBF and CBE take the triangle DBC, and the remainder will be the part BEF; the double of which will be the whole AEBFA.

PROBLEM XV.

To find the Solidity of a Sphere or Globe.

* RULE I.

Multiply the furface by $\frac{1}{3}$ of the radius, or by $\frac{1}{6}$ of the diameter; and the product will be the folidity.

RULE

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* DEMONSTRATION.

The fphere may be confidered as conftituted of an infinite number of pyramids, whofe bafes compofe the fpheric furface, and

RULE II.

Multiply the cube of the diameter by ·5236, and the product will be the solidity.

That is ($\frac{1}{6}$ of 3·1416 or) ·5236d^3 = the solidity.

EX-

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and all the vertices meeting in the center, their common height being equal to the radius of the sphere. And consequently the sphere, or any spherical pyramid, being a part contained within right lines drawn from the surface to the center, is equal to a pyramid whose base is equal to the spherical surface, and height equal to the radius. And therefore the surface of the whole, or of any such part, being drawn into $\frac{1}{3}$ of the radius, will give the solidity, as in rule 1.

Corol. 1. Since the surface of the sphere is $= pdd$, we shall have $pdd \times$ ($\frac{1}{3}$ of the radius or) $\frac{1}{6}d = \frac{1}{6}pd^3 =$ ·523598775 &c. $\times d^3$ (which is rule 2) $= \frac{2}{3}$ of a cylinder of the same diameter and height.

Corol. 2. If h be the height, or versed sine, of any segment; then, since $pdh =$ its surface, $\frac{1}{6}pd^2h =$ ·523598775 &c. $\times d^2h$ will be the solidity of the spheric pyramid, or cone, whose base is the surface of the segment.

Corol. 3. Hence spheres and their similar pyramids, and also any other similar parts of them, are as the cubes of the diameters.

Corol. 4. If to or from $\frac{1}{6}pd^2h$, the spheric cone, be added or subtracted $\frac{1}{4}p \times (\mp\frac{1}{2}p \pm h) \times \frac{4}{3}h \times (d-h) = \mp\frac{1}{6}pd^2h \pm \frac{1}{2}pdh^2 \mp \frac{1}{3}ph^3$, the cone whose base is the same with the base of the segment, and whose vertex is in the center, the sum or difference $\frac{1}{2}pdh^2 - \frac{1}{3}ph^3 = \frac{1}{3}ph^2 \times (\frac{3}{2}d - h)$, will be the spheric segment whose height is h, either greater or less than the hemisphere, or of whatever magnitude h is, not exceeding d. Or if r be = the radius of the segment's base, since $dh = r^2 + h^2$, the segment will be $\frac{1}{2}pr^2h + \frac{1}{6}ph^3 = \frac{1}{6}ph \times (3r^2 + h^2)$.

Corol. 5. Hence the difference between two segments whose heights are H, h, and the radii of their bases R, r, will give for the frustum or zone $\frac{1}{6}p \times (3\text{R}^2\text{H} + \text{H}^3 - 3r^2h - h^3)$; which, putting a for the altitude of the frustum, and exterminating H and h by means of the two equations $(\text{R}^2 + \text{H}^2)h = (r^2 + h^2)\text{H}$, and $a = \text{H} - h$, will become $\frac{1}{2}ap \times (\text{R}^2 + r^2 + \frac{1}{3}a^2)$.

I *Corol.*

EXAMPLE.

Supposing the earth to be spherical, and its diameter $7957\frac{3}{4}$ miles, what is its solidity?

1. By rule 1.

By exam. 1 of prob. 10 the surface is 198943750. Then $198943750 \times 7957\frac{3}{4} \times \frac{1}{6} = 263857437760$ miles is the solidity.

2. By the 2d rule.

Here $\cdot5236d^3 = \cdot5236 \times (7957\frac{3}{4})^3 = 263858149120$ miles = the solidity by this rule. The difference arising by taking the number ·5236 rather too great.

PROBLEM XVI.

To find the Solidity of the Segment of a Sphere.

RULE I.

To three times the square of the radius of its base, add the square of its height; multiply the sum by the height, and the product by ·5236, for the solidity.

That is, if $r =$ DE the radius of its base,

$h =$ GE the height;

Then $\cdot5236h \times (3rr + hh) =$ the solidity of the segment DGF. By cor. 4 to the last problem. [See the figure in page 198.]

RULE

Corol. 6. If one end of the frustum pass through the center, then $R^2 = \frac{1}{4}d^2 = r^2 + a^2$, and the last theorem will become $ap \times (r^2 + \frac{2}{3}a^2) = ap \times (\frac{1}{4}d^2 - \frac{1}{3}a^2)$.

Corol. 7. Hence the middle zone, or the double of the last expression, will be $2ap \times (r^2 + \frac{2}{3}a^2) = 2ap \times (\frac{1}{4}d^2 - \frac{1}{3}a^2)$; where a is half its altitude, and r half the diameter of each end. But if A be its whole altitude, and D the diameter of each end, those theorems will become $\frac{1}{4}Ap \times (D^2 + \frac{2}{3}A^2) = \frac{1}{4}Ap \times (d^2 - \frac{1}{3}A^2) = \frac{1}{8}Ap \times (d^2 + 2r^2)$.

RULE II.

From three times the diameter of the ſphere, ſubtract twice the height of the fruſtum; multiply the difference by the ſquare of the height, and the product by $\cdot 5236$, for the ſolidity.

That is,

If $d =$ GH the diam. of the ſph.
$h =$ GE the height of the fruſt. [*See fig. page* 198.]

Then $\cdot 5236 h^2 \times (3d - 2h)$ = the ſolidity of DGF. By cor. 4 to the laſt problem.

EXAMPLE.

What is the ſolidity of each of the frigid zones of the earth; the axe being $7957\frac{3}{4}$ miles, and half the breadth, or arc DG, of the zone being $23\frac{1}{2}$ degrees?

By rule 2.

As 1 = tabular radius : $3978\frac{7}{8}$ = radius of the earth :: $\cdot 0829399$ = tab. verſed ſine of $23\frac{1}{2}$ degrees : $330\cdot 0074946$, the verſed ſine or height of the ſegment.

Then $\cdot 5236 h^2 \times (3d - 2h) = \cdot 5236 \times 330\cdot 0074946^2 \times 23213\cdot 2350108 = 1323679710$, the content.

By rule 1.

As 1 : $3978\frac{7}{8}$ :: $\cdot 3987491$ = tabular ſine of $23\frac{1}{2}$ degrees : $1586\cdot 57282526$, the radius of the baſe.

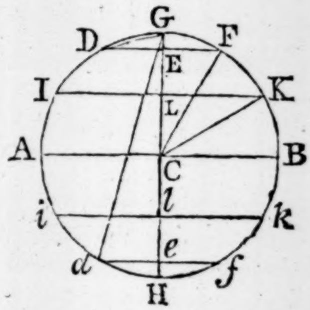
Then $\cdot 5236 h \times (3r^2 + h^2) = \cdot 5236 \times 330\cdot 0074946 \times 7660544\cdot 936 = 1323680299\cdot 69$, the ſolidity.

PRO-

PROBLEM XVII.

To find the Solidity of a Frustum or Zone of a Sphere.

Add together the squares of the radii of the ends, and $\frac{1}{3}$ of the square of their distance, or of the height; multiply the sum by the said height, and the product again by 1·5708 for the content.



That is, $(R^2 + r^2 + \frac{1}{3}h^2) \times \frac{1}{2}ph$ = the solidity of the frustum whose height is h, and the radii of its ends R and r, p being 3·1416. By cor. 5 to prob. 15.

EXAMPLE I.

What is the solidity of the frustum of a sphere, the diameter of whose great end is 4 feet, the diameter of the less 3 feet, and the height $2\frac{1}{2}$ feet?

Here $(R^2+r^2+\frac{1}{3}h^2)\times 1{\cdot}5708h = (2^2+1{\cdot}5^2+\frac{1}{3}\times 2{\cdot}5^2) \times 1{\cdot}5708 \times 2\frac{1}{2} = 8\frac{1}{3} \times 3{\cdot}927 = 32{\cdot}725$, the solidity of the frustum required.

EXAMPLE II.

What is the solidity of each temperate zone of the earth, extending from $23\frac{1}{2}$ degrees to $66\frac{1}{2}$ degrees of latitude, and the diameter of the earth being $7957\frac{3}{4}$ miles?

By the example to the last problem, the radius of the top is 1586·57282526.

And as $1 : 3978\frac{7}{8} :: {\cdot}9170601$ = tabular sine of $66\frac{1}{2}$ degrees : 3648·86750538, the radius of the base.

Alſo by example 4 prob. 10, the height is 2062·2955.

Then $(R^2 + r^2 + \frac{1}{3}h^2) \times 1·5708h = 1724913{6} \times 2062·2955 \times 1·5708 = 55877778668$, the ſolidity of each temperate zone.

Otherwiſe.

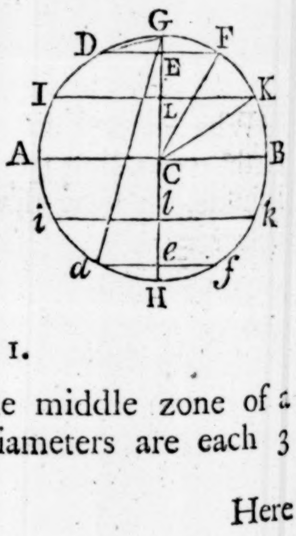
Since the radii of the ends of this zone, are the ſines of $23\frac{1}{2}$ and $66\frac{1}{2}$ degrees, which are complements the one of the other, the ſum of the ſquares of thoſe radii will be equal to the ſquare of the radius of the ſphere; and therefore $(\frac{1}{4}dd + \frac{1}{3}hh) \times 1·5708h = 17249136 \times 2062·2955 \times 1·5708 = 55877778668$, the content, as before.

PROBLEM XVIII.

To find the Solidity of the Middle Zone of a Sphere.

Multiply, either the ſum of the ſquare of the diameter of the end and $\frac{2}{3}$ of the ſquare of the height, or the difference between the ſquare of the diameter of the ſphere and $\frac{1}{3}$ of the ſquare of the height, by the height, and the product by ·7854 for the content.

That is, $(DD + \frac{2}{3}hh) \times ·7854h$ or $(dd - \frac{1}{3}hh) \times ·7854h$ is the content of the middle zone whoſe height is h, the diameter of each end D, and the diameter of the ſphere d. By cor. 6. to prob. 15.



EXAMPLE I.

Required the ſolidity of the middle zone of a ſphere whoſe top and bottom diameters are each 3 feet, and height 4 feet.

Here $(DD + \frac{2}{3}bb) \times \cdot 7854h = (3^2 + \frac{2}{3} \times 4^2) \times 4 \times \cdot 7854 = 59 \times 4 \times \cdot 2618 = 61 \cdot 7848$, the content required.

EXAMPLE II.

What is the solidity of the torrid zone of the earth, which extends to $23\frac{1}{2}$ degrees on each side of the equator; the diameter of the earth being $7957\frac{3}{4}$ miles?

By the example to prob. 16, the sine of $23\frac{1}{2}$ degrees, or $\frac{1}{2}$ the height of the zone, is $1586 \cdot 57282526$, and the whole height is $3173 \cdot 14565052 = h$.

Then $(dd - \frac{1}{3}hh) \times \cdot 7854h =$
$(7957 \cdot 75^2 - \frac{1}{3} \times 3173 \cdot 14565052^2) \times 3173 \cdot 14565052 \times \cdot 7854 = 149455081137$, the content.

SCHOLIUM.

From the last three problems we find that

The two frigid zones	=	2647359420
The two temperate zones	=	111755557336
The torrid zone	=	149455081137
	whose sum	263857997893

is the whole sphere, nearly the same as found in prob. 15.

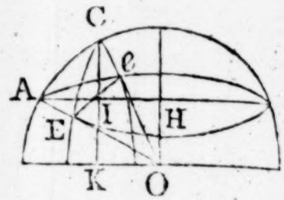
PROBLEM XIX.

To find the Solidity of the Second Segment of a Sphere.

DEFINITION.

A second segment is a part cut off a segment by a plane perpendicular to the base.

RULE.



By prob. 14 find the curve surface of the second segment AECe, which being drawn into $\frac{1}{3}$ of the radius of the sphere, will produce the content of the spheric sector OEACe; from which if there be taken the pyramid OEAe, whose base is the segment AEe, and height OH; and from the remainder be taken the pyramid OECe, whose base is the segment CEe, and height KO, or HI; it is evident that the last remainder will be the second segment AECe: for these two pyramids and the second segment compose the spheric pyramid.

PROBLEM XX.

To find the Surface of a Circular Spindle, or of any Segment or Frustum of it.

From the product of the height of the solid and radius of the revolving arc, subtract the product of the said arc and central distance; multiply the remainder by 3·1416; and double the product will be the surface described by that arc; whether it be the whole or any part of the spindle.*

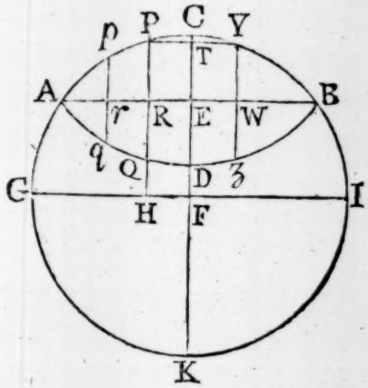
That

* DEMONSTRATION.

Put $z =$ the arc CP, $x =$ its sine RE, r and c for the radius and central distance, and $p = 3{\cdot}14159$. Then the fluxion of the surface $\dot{s}$ is $= 2p\dot{z} \times \text{RP} = 2p \times (\dot{z}\sqrt{rr - xx} - c\dot{z})$; but, by the property of the circle, $\dot{z}\sqrt{rr - xx}$ is $= r\dot{x}$; therefore $\dot{s} = 2p \times (r\dot{x} - c\dot{z})$, and $s = 2p \times (rx - cz) = 2p \times (r \times \text{RE} - c \times \text{PC})$.

Corol.

That is,
$r =$ FC the rad. of the arc,
$c =$ FE the central dist.
$b = r$R the height,
$a =$ Pp its revolving arc,
$p = 3{\cdot}1416$.



Then $(br - ac) \times 2p$ = the surface of that part pPQq.

EXAMPLE.

Required the surface of a circular spindle whose greatest diameter is 30, and its length 40 inches.

Here, by the property of the circle, EK $=$ AE$^2 \div$ EC $= 400 \div 15 = 26\frac{2}{3}$; and $26\frac{2}{3} + 15 = 41\frac{2}{3}$ is the diameter CK, or $20\frac{5}{6} =$ the radius of the circle.

And

Corol. 1. When RE $=$ AE, the rule becomes $2p \times$ ($r \times$ AE $- c \times$ AC) for the surface of half the spindle, or $2p \times$ ($r \times$ AB $- c \times$ ACB) for that of the whole.

Corol. 2. If from the surface of the semi-spindle be taken that of the frustum, there will remain $2p \times$ ($r \times$ AR $- c \times$ AP) for that of the segment APQ: so that the rule is general.

Corol. 3. When E coincides with F, c vanishes, and the spindle becomes a sphere; and then the theorem becomes barely $2prx$, the same with that before found for the sphere.

Corol. 4. From cor. 1, it appears that the radius is to the cosine of an arc, always in a greater proportion than that of the arc to its sine, or of the double arc to its chord. But those ratios approximate to an equality as the arc diminishes, till, when the arc vanishes, they become accurately equal, and the arc and chord vanish also in a ratio of equality.—Consequently, in small arcs, the arc is a fourth proportional to the cosine, sine, and radius, nearly. Thus, the sine and cosine of an arc of 1 degree are ·0174524 and ·9998477 to the radius 1; then ·9998477 : ·0174524 :: 1 : ·0174550 = the arc nearly, the true figures being ·0174533.

And EF $=$ FC $-$ CE $= 20\frac{5}{6} - 15 = 5\frac{5}{6}$ the central distance.

Also, as $20\frac{5}{6} : 20 :: 1 : 24 \div 25 = 96 \div 100 =$ ·96 the tabular sine of the arc AC, to which belong 73·7398 degrees, the double of which 147·4796 are the number of degrees in the whole arc ACB. And therefore by rule 1 prob. 6 sect. 1 part 2, we have $·01745329 \times 147·4796 \times 20\frac{5}{6} = 53·62508$ for the length of the revolving arc.

Therefore, by the rule, $(hr - ac) \times 2p$
$= (20\frac{5}{6} \times 40 - 53·62508 \times 5\frac{5}{6}) \times 6·2832$
$= (25 \times 40 - 53·62508 \times 7) \times \frac{5}{6} \times 6·2832$
$= 624·62444 \times 5·236 = 3270·5335$, the surface required.

PROBLEM XXI.

To find the Solidity of a Circular Spindle.

From $\frac{1}{3}$ of the cube of half the length of the spindle, subtract the product of the central distance and half the generating circular segment; multiply the remainder by 4 times 3·14159, and the product will be the content of the spindle.*

That

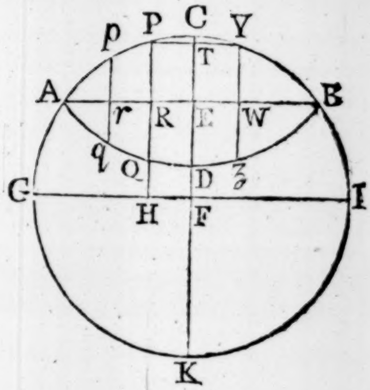
---

* GENERAL INVESTIGATION.

Putting RE $= x$, EF $= c$, $3·14159 = p$; the fluxion of the solid $\dot{s}$ will be $= p\dot{x} \times \text{PR}^2 = p\dot{x} \times (\text{HP} - c)^2 = p\dot{x} \times (\text{HP}^2 - c \times (2\text{HP} - c) = p\dot{x} \times \text{HP}^2 - c \times (2\text{RP} + c) = p\dot{x} \times (r^2 - c^2 - x^2 - 2c \times \text{RP})$;

and $s = px \times (r^2 - c^2 - \frac{1}{3}x^2 - \frac{2c}{x} \times \text{RPCE})$
$= p \times [(\text{AE}^2 - \frac{1}{3}x^2)\, x - 2c \times \text{RPCE}] =$ the frustum generated by RPCE.

Corol.



That is,

If $l =$ AE the half length,
$c =$ EF the central dist.
$a =$ the area ACE,
$p = 3{\cdot}14159$.

Then $(\frac{1}{3}l^3 - ac) \times 4p =$ the whole spindle ACBDA.

EXAMPLE.

If the length of a circular spindle be 40, and its greatest diameter 30; what is its solidity?

By the last problem the radius is $20\frac{5}{6}$, the central distance $5\frac{5}{6}$, and the length of the whole arc $53{\cdot}62508$; hence, $53{\cdot}62508 \times 20\frac{5}{6} =$ double the sector FAB, (lines being supposed to be drawn from the center to A and B); and $40 \times 5\frac{5}{6} =$ double the triangle FAB; consequently $53{\cdot}62508 \times 20\frac{5}{6} - 40 \times 5\frac{5}{6} = (53{\cdot}62508 \times 25 - 40 \times 7) \times \frac{5}{6} = 883{\cdot}8558 =$ double the segment ACB; $\frac{1}{4}$ of which, or $220{\cdot}9639$, is half that segment.

Then $(\frac{1}{3}l^3 - ac) \times 4p = (\frac{1}{3} \times 20^3 - 220{\cdot}9639 \times 5\frac{5}{6}) \times 4 \times 3{\cdot}1416 = (\frac{4}{3} \times 20^2 - 220{\cdot}9639 \times 1\frac{1}{6}) \times 20 \times 3{\cdot}1416 = 275{\cdot}542 \times 62{\cdot}832 = 17312{\cdot}858$, the solidity required.

Corol. 1. When x becomes $=$ AE, the theorem above becomes $(\frac{1}{3}\text{AE}^3 - c \times \text{ACE}) \times 2p = \frac{1}{2}$ the spindle, and $(\frac{1}{3}\text{AE}^3 - c \times \text{ACE}) \times 4p =$ the whole spindle.

Corol. 2. If from half the spindle be taken the frustum, there will remain $p \times (\text{AE}^2 \times \text{AR} - \frac{1}{3}(\text{AE}^3 - x^3) - 2c \times \text{APR}) = p \times (\frac{1}{3}\text{AR}^2 \times (3\,\text{AE} - \text{AR}) - 2c \times \text{APR})$ for the segment PAQ of the spindle.

Corol. 3. If E coincide with F, the spindle will become a sphere, c will vanish, and the theorems above become $\frac{4}{3}p \times \text{GF}^3 = \frac{1}{6}p \times \text{GI}^3$ for the whole sphere; $p \times \text{HF} \times (\text{GF}^2 - \frac{1}{3}\text{HF}^2)$ for the spheric frustum; and $\frac{1}{3}p \times \text{GH}^2 \times (3\,\text{GF} - \text{GH})$ for the spheric segment; which are the same with th heorems i problems 15, 16, and 18.

PROBLEM XXII.

To find the Content of the Middle Frustum or Zone of a Circular Spindle.

From the ſquare of half the length of the ſpindle, take $\frac{1}{3}$ of the ſquare of half the length of the middle zone; and multiply the remainder by the ſaid half length of the zone; from the product ſubtract the product of the generating circular area, and central diſtance; then the remainder drawn into 2 times 3·14159 will be the content of the middle zone.

That is, putting
l = EF half the length of the zone,
L = EA half the length of the ſpindle,
c = FE the central diſtance,
a = the generating area RPYW.
Then $[(LL - \frac{1}{3}ll)l - ac] \times 2p$ = the zone PYZQ.

EXAMPLE.

If a caſk, in the form of the middle fruſtum of a circular ſpindle, have its head diameter 24, bung diameter 32, and length 40 inches; how many ale gallons will it hold?

Here CE − PR = 16 − 12 = 4 = TC, and $PT^2 \div TC = 400 \div 4 = 100 = TK$; hence 100 + 4 = 104 = the diameter, and 52 = the radius of the generating circle.

FC − CE = 52 − 16 = 36 = EF the central diſtance; and $\sqrt{AF^2 - FE^2} = \sqrt{52^2 - 36^2} = 4\sqrt{13^2 - 9^2} = 4\sqrt{88} = 8\sqrt{22}$ = AE half the length of the ſpindle.

PR × RW = 12 × 40 = 480 = the area RPTYW.
And the ſegment PCY, is 107·5185.

The

The sum of these two is $587{\cdot}5185$ = the generating area RPCYW.

Then $[(LL - \frac{1}{3}ll)l - ac] \times 2p =$
$[(64 \times 22 - \frac{1}{3} \times (400 \times 20 - 36 \times 587{\cdot}5185] \times 2 \times 3{\cdot}14159 = 4342{\cdot}7879 \times 6{\cdot}28318 =$
$27286{\cdot}5411256$ the solidity in inches.

Then $27286{\cdot}5411256 \div 282 = 96{\cdot}7608$ ale gallons.

PROBLEM XXIII.

To find the Content of the Segment of a Circular Spindle.

From 3 times half the length of the spindle subtract the height of the segment, and multiply the remainder by the square of the said height; from $\frac{1}{3}$ of the product subtract double the product of the generating area and central distance; then the remainder drawn into $3{\cdot}14159$, will produce the content.

That is, $[(3EA - AR) \times \frac{1}{3}AR^2 - 2FE \times APR] \times p =$ the solidity of PAQ.

EXAMPLE.

If the length of the whole spindle be 12, and its greatest diameter 9; what will be the content of a segment of it whose height is 1?

$\frac{AE^2}{EC} = \frac{36}{4\frac{1}{2}} = \frac{72}{9} = 8 = EK$; $KE + EC = 8 + 4\frac{1}{2} = 12\frac{1}{2} =$ diam. and $6\frac{1}{4} =$ the radius of the generating circle; $FC - CE = 6\frac{1}{4} - 4\frac{1}{2} = 1\frac{3}{4} = EF$ the central distance; $\sqrt{FP^2 - PT^2} = \sqrt{(6\frac{1}{4})^2 - 5^2} = \sqrt{(\frac{5}{4})^2 \times (5^2 - 4^2)} = 3\frac{3}{4} = TF$, and $CF - FT = 6\frac{1}{4} - 3\frac{3}{4} = 2\frac{1}{2} = TC$.

By the table of circular segments, the half segment PCT is $8{\cdot}736234375$, to which adding $RT = 5 \times 2 = 10$, makes $18{\cdot}736234375$ = the area RC.

Also

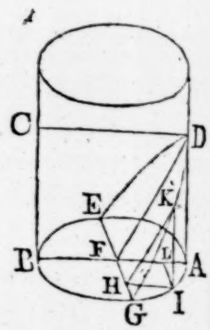
Alſo the half ſegment ACE is 19·886763281.

Their difference = 1·150528906 is the generating area APR.

Therefore $[(3\,AE - AR) \times \frac{1}{3} AR^2 - 2\,FE \times APR] \times p = (17 \times \frac{1}{3} - 3\frac{1}{2} \times 1{\cdot}150528906) \times 3{\cdot}14159 = 1{\cdot}639815495 \times 3{\cdot}14159 = 5{\cdot}1516323$, the content required.

PROBLEM XXIV.

To find the Curve Surface of the Ungula DEAGD *of a Cylinder.*



Put h = the height AD,
v = AF the verſ. ſine of AE,
d = the diameter AB,
a = the arc EAG of the baſe,
s = the right ſine FG,
c = the coſine of the $\frac{1}{2}$ arc.

Then $\frac{ds - ac}{v} \times h$ is the convex ſurface.*

That

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* DEMONSTRATION.

For, having drawn HI, IK parallel to FA and AD reſpectively, and joined the points H, K; ſince it is evident that the ſurface is generated by the motion of IK along the arc AIG, KI × the fluxion of IA will be the fluxion of the ſurface. Therefore put z = AI, x = its ſine IL, and y = its coſine; then HI $= y - c$; and, by ſimilar triangles, FA : AD :: HI : IK $= \frac{h}{v} \times (y - c)$; and hence the fluxion of the ſurface, or $\dot{z} \times$ IK is $\frac{h}{v} \times (y\dot{z} - c\dot{z}) = \frac{h}{v} \times (\frac{1}{2}d\dot{x} - c\dot{z})$: the fluent of which is $= \frac{h}{v} \times (\frac{1}{2}dx - cz) =$ (when AI = AG) $\frac{h}{v} \times (\frac{1}{2}ds - \frac{1}{2}ac)$; the double of which is $\frac{h}{v} \times (ds - ac)$ = the whole convex ſurface DEAGD.

Corol.

That is, from the product of the diameter and sine, subtract the product of the arc and cosine, and multiply the difference by the height, and divide by the versed sine.

Note 1. When F is the center of the base; then $v = s = \frac{1}{2}d$, and $c = 0$; and then the theorem becomes dh, viz. the product of the diameter and height equal to the curve surface.

Note 2. When AF exceeds $\frac{1}{2}$AB, then ac must be added.

EXAMPLE I.

Given the diameter AB 100, the height AD 140, and the versed sine AF 10: required the curve surface.

Here $d = 100$, $h = 140$, and $v = 10$: therefore $\frac{1}{2}d - v = 50 - 10 = 40 = c$.
And $\sqrt{\frac{1}{4}dd - cc} = \sqrt{2500 - 1600} = \sqrt{900} = 30 = s$.

But $\frac{s}{\frac{1}{2}d} = \frac{30}{50} = \frac{3}{5} = \cdot 6$ is the sine reduced to the radius 1; to which, in a table of sines, belong $36^\circ\ 52\cdot 268' = 36\cdot 87113$ degrees.

Then, by rule 1 for the length of a circular arc, $\cdot 01745329 \times 36\cdot 87113 \times 100 = 64\cdot 352252$ is the arc a.

Whence $\frac{ds - ac}{v} \times h = (3000 - 2574\cdot 09008) \times 14 = 425\cdot 90992 \times 14 = 5962\cdot 73888 =$ the convex surface required.

EX-

Corol. 1. If F be the center; then $v = s = \frac{1}{2}d$, and $c = 0$; and then the theorem becomes barely $dh = 4$ times the triangle FDA.

Corol. 2. When AF exceeds $\frac{1}{2}d$, c is negative, and then $- ac$ becomes $+ ac$.

Corol. 3. If F coincide with B; then $s = 0$, and $c = -\frac{1}{2}v$; and the theorem becomes $\frac{1}{2}ah =$ the surface of the half cylinder.

EXAMPLE II.

If the diameter and height be 100 and 140, as before, and the section be made through the center of the base, or $v = \frac{1}{2}d = 50$; what is the convex surface?

Here, by note 1, $db = 100 \times 140 = 14000 =$ the convex surface required.

EXAMPLE III.

Supposing d and h still the same, and $v = 90$; to find the convex surface.

Here $\frac{1}{2}d - v = 50 - 90 = -40 = c$, $s = 30$ the same as before, but it is here the sine of the supplemental arc, which therefore is $180 - 36{\cdot}87113 = 143{\cdot}12887$ degrees. Hence $\cdot01745329 \times 143{\cdot}12887 \times 100 = 249{\cdot}807013 =$ the arc a. Or the arc may be sooner found by only subtracting the arc in the first example, viz. $64{\cdot}352252$, from $314{\cdot}159265$, the whole circumference.

Then, by note 2, $\frac{ds + ac}{v}h = \frac{14}{9}(3000 + 9992{\cdot}28052) = \frac{14}{9} \times 12992{\cdot}28052 = 20210{\cdot}21414$ the convex surface required.

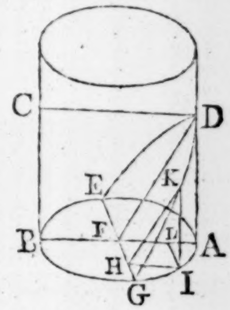
PROBLEM XXV.

To find the Solidity of the Hoof of a Cylinder.

From $\frac{2}{3}$ of the cube of the right sine, subtract the product of the base and cosine of half the arc of the base; then multiply the difference by the height, and divide by the versed sine, the quotient will be the

the solidity required. That is, putting, as before

h = the height AD
v = the versed sine AF
s = the right sine FG
c = the cosine $= \frac{1}{2}$AB — AF
b = the base, or area of the seg. GAEG;



Then $\frac{\frac{2}{3}s^3 - bc}{v} \times h$ = the solidity.*

Note 1. If F be the center, that is, if the base be equal to the semicircle, then $v = s$, and $c = 0$; and therefore $\frac{2}{3}hss = \frac{1}{6}ddh$ is the solidity in that case.

Note 2. If v exceed $\frac{1}{2}d$, that is, if the base exceed the semicircle, then c is negative, and bc must be added.

EX-

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* DEMONSTRATION.

For, the fluxion of the solid is = the △ HIK drawn into the fluxion of LI, which fluxion will, therefore, be $\dot{x} \times \frac{h}{2v} \times$ HI^2 (using the same characters as in the demonstration of the last problem) $= \frac{h\dot{x}}{2v} \times (y-c)^2 = \frac{h\dot{x}}{2v} \times (yy - 2cy + cc) = \frac{h\dot{x}}{2v} \times (\frac{1}{4}dd - xx - 2cy + cc) = \frac{h\dot{x}}{2v} \times (\frac{1}{4}dd - xx - dc + 2c \times \text{AL} + cc) = \frac{h\dot{x}}{2v} \times (ss - xx - 2c \times \text{FL})$; whose fluent, $\frac{hx}{2v} \times (ss - \frac{1}{3}xx) - \frac{hc}{v} \times$ area FAIH, when I coincides with G, is $\frac{h}{2v} \times (\frac{2}{3}s^3 - bc)$, the double of which is $\frac{h}{v} \times (\frac{2}{3}s^3 - bc)$ = the content of the solid DEAGD required.

Corol. 1. If F fall in the center of the base; then $c = 0$, and $s = v = \frac{1}{2}d$; and the rule will be $\frac{1}{6}ddh$.

Corol. 2. If AF exceed FB, c will be negative, and then $-bc$ will become $+bc$.

Corol. 3. If F fall in B; $s = 0$, and $c = -\frac{1}{2}v$; and then the theorem becomes $\frac{1}{2}bh$ = half the cylinder.

EXAMPLE I.

If the diameter AB be 50, the height AD 120, and the verſed ſine AF 10; what is the ſolidity of the hoof?

Or ſuppoſing a cylindric veſſel ABCD, containing a fluid, to be placed in ſuch a poſition that the ſurface of the fluid, diſpoſing itſelf parallel to the horizon, may cut the baſe in GE, leaving 40 inches of the diameter dry, and the ſide of the cylinder in D, 120 inches diſtant from the baſe; to find how many ale gallons are in it; the diameter of the baſe being 50 inches.

Here $h = 120$, $d = 50$, and $v = 10$. Then $\frac{1}{2}d - v = 25 - 10 = 15 = c$, and $\sqrt{\frac{1}{4}dd - cc} = \sqrt{25^2 - 15^2} = \sqrt{40 \times 10} = 20 = s$.

And, to find the baſe by the table of ſegments, $\frac{v}{d} = \frac{10}{50} = \cdot 2$; this being found in the column of verſed ſines, oppoſite to it is the area $\cdot 1118238$: hence $50 \times 50 \times \cdot 1118238 = 279 \cdot 5595 = b$ is the ſegment or baſe.

Then $\frac{\frac{2}{3}s^3 - bc}{v}h = 12 \times (\frac{2}{3} \times 8000 - 15 \times 279 \cdot 5595) = 12 \times (5333\frac{1}{3} - 4193 \cdot 3925) = 12 \times 1139 \cdot 9408 = 13679 \cdot 2896 =$ the ſolidity in inches; which, divided by 282, the inches in a gallon, give $48 \cdot 50939$ ale gallons for the content.

EXAMPLE II.

Suppoſe the cylinder ſo placed, that the ſurface of the liquor may biſect the baſe, and riſe up the ſide to the ſame diſtance of 120 inches from the baſe: to find the content.

Here, by note 1, we have $\frac{1}{6}ddh = 50 \times 50 \times 20 = 50000$ ſolid inches $= 177 \cdot 3049645$ gallons, for the content in this caſe.

EX-

EXAMPLE III.

Suppose, now, the same vessel so placed, as that the surface of the liquor may leave only 10 inches of the diameter dry, still rising to the same distance of 120 inches along the side; to find the content.

Here the part of the cylinder's base left dry, is equal to the base in the first example, viz. 279·5595 which, therefore, taken from

$50 \times 50 \times \cdot78539816 = 1963\cdot4954$, the whole circle, leaves $1683\cdot9359 = b$, the base of the ungula in this example.

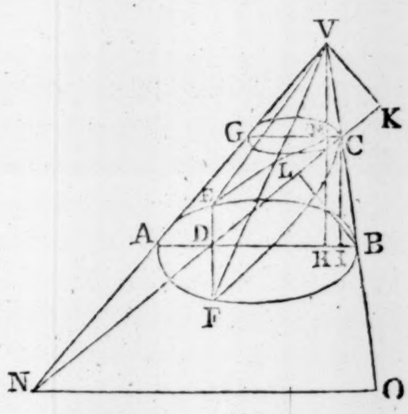
Now $v = 40$, $c = -15$, and $s = 20$.

Whence $\frac{\frac{2}{3}s^3 - bc}{v}h = (\frac{2}{3} \times 8000 + 25259\cdot0385)\frac{120}{40}$ $= 30592\cdot3718 \times 3 = 91777\cdot1154$ solid inches $=$ $325\cdot45076$ gallons, the content in this case.

PROBLEM XXVI.

To find the Solidity of the Elliptic Hoofs of the Frustum of a Cone, made by a Plane cutting Diagonally the Opposite Extremities of the Ends.

* From the square of the diameter of the base of the hoof, subtract the product of the diameter of the other



* Let AEBF be the base of a cone, or of any other pyramid, right, or oblique; AVB a section through the vertex by a plane perpendicular to the base; and EVF, ECF two other sections perpendicular to AVB, the former through the vertex, and the latter through the side, at C, between V and B. On AB let fall the perpendiculars VH, CI; and on DC the perpendiculars VK, BL; and draw GC parallel to AB, and meeting AV and VH in G and M. Then

other end of the frustum, and a mean proportional between the diameters; divide the difference by the difference of the diameters; multiply the quotient by the height of the hoof, and the product by the diameter of its base; so shall the last product multiplied by ·2618 give the content of the hoof.—That is, putting

$\text{D} = \text{AB}$ the diameter of the greater end,
$d = \text{GC}$ the less diameter, and
$h = \text{CI}$ the perpendicular height.*

Then $\dfrac{\text{D}^2 - d\sqrt{\text{D}d}}{\text{D} - d} \times 0{\cdot}2618\,\text{D}h =$ the greater hoof ABC,

And $\dfrac{\text{D}\sqrt{\text{D}d} - d^2}{\text{D} - d} \times 0{\cdot}2618\,dh =$ the less hoof AGC.

A plane being conceived to be drawn through C and A.—The proof is in corollary 2.

EX-

Then it is evident that EFBV is a pyramid whose base is EFB (A), altitude VH (a); and therefore its content is equal to $\frac{1}{3}$ Aa; and that EFCV is a pyramid whose base is EFC (B), height VK (b), and therefore its content equal to $\frac{1}{3}$ Bb; and moreover that the difference of these pyramids, or $\frac{1}{3}$ Aa − $\frac{1}{3}$ Bb is the content of the hoof EFBC.

But, by the similar triangles ABV, GVC, it is

$\text{AB} - \text{CG} : \text{CI}$ or $\text{HV} - \text{VM} :: \text{AB} : \text{HV}$, or $a = \dfrac{\text{AB} \times \text{CI}}{\text{AB} - \text{GC}}$;

also $\text{AB} - \text{GC} : \text{CI} :: \text{AB} : \text{HV} :: \text{GC} : \text{VM} = \dfrac{\text{GC} \times \text{CI}}{\text{AB} - \text{GC}}$;

and DC : BD :: (by the similar triangles ICD, DBL) CI : BL :: (because of the similar triangles BCI and CVM, VKC and CLB)

$\text{VM}\ (= \dfrac{\text{GC} \times \text{CI}}{\text{AB} - \text{GC}}) : \text{VK}\ (b) = \dfrac{\text{GC} \times \text{CI} \times \text{DB}}{\text{DC} \times (\text{AB} - \text{GC})}$.

Wherefore the hoof EFBC will be

$= \dfrac{\frac{1}{3}\text{CI}}{\text{AB} - \text{GC}} \times (\text{A} \times \text{AB} - \text{B} \times \dfrac{\text{GC} \times \text{DB}}{\text{DC}}$; which is a general theorem for the hoof of any pyramid.

Corol. 1. If the base be circular, or the pyramid a cone, and the angle CDB be less than the angle VAD; or, which is the same, if CD and VA, produced, intersect in N; the section ECF will

EXAMPLE I.

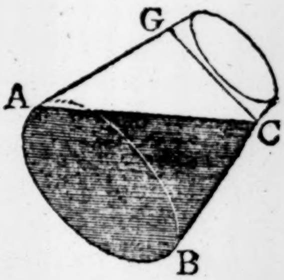
If a conical vessel, whose bottom diameter is 30 inches, be inclined to the horizon till the liquor in it just cover the bottom, and its surface, disposing itself into the position AC, cut the side BC of the vessel in C, at the distance of 18 inches from the bottom AB; how many wine gallons of liquor are in it, supposing the diameter GC of the vessel at C to be 19·2 inches?

Here $D = 30$, $d = 19·2$, and $h = 18$.

Whence $\frac{D^2 - d\sqrt{Dd}}{D - d} \times ·2618\,Dh = \frac{30^2 - 19·2\sqrt{30 \times 19·2}}{30 - 19·2} \times 30 \times 18 \times ·2618 = \frac{439\frac{1}{5} \times 30 \times 18 \times ·2618}{10\frac{4}{5}} = \frac{2196 \times 30 \times 18 \times ·2618}{54} = 2196 \times ·2618 \times 10 =$ 5749·128 inches. Which being divided by 231, thus,

$$231 = 3 \times 7 \times 11 \left\{ \begin{array}{r} 3)5749·128 \\ \hline 7)1916·376 \\ \hline 11)\ 273·768 \\ \hline \end{array} \right.$$

gives 24·888 wine gallons.



 EX-

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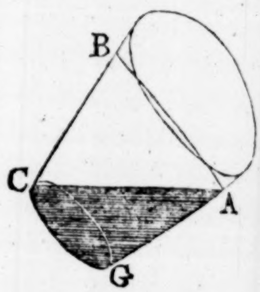
will be a segment of an ellipse whose transverse axe is CN, and conjugate $\sqrt{NO \times GC}$, NO being drawn parallel to AB, and meeting VB produced in O. And then the above general theorem will become $\frac{\frac{1}{3}CI}{AB - GC} \times$ (AB $\times$ circular segment EBF $- \frac{GC \times DB}{DC} \times$ elliptic segment ECF) =, by prob. 6 sect. 3 part 3, $\frac{\frac{1}{3}CI}{AB - GC} \times$ (AB $\times$ cir. seg. EBF $- \frac{GC \times DB \times \sqrt{NO \times GC}}{DC \times CN}$ $\times$ cir. seg. whose diameter is CN and height CD) =, since similar segments

EXAMPLE II.

Let there be taken here the ſame dimenſions as in the laſt example, ſuppoſing the veſſel to be narroweſt at the bottom, to find how many ale gallons are in it.



Here $\frac{D\sqrt{Dd-dd}}{D-d} \times \cdot 2618\, db =$

$\frac{30\sqrt{30 \times 19\frac{1}{5} - 19{\cdot}2^2}}{30 - 19\frac{1}{5}} \times \cdot 2618 \times 19\frac{1}{5} \times 18 =$

$\frac{351\frac{9}{25} \times 19\frac{1}{5} \times 18 \times \cdot 2618}{10\frac{4}{5}} =$

$\frac{351\frac{9}{25} \times 96 \times 18 \times \cdot 2618}{54} =$

$117\frac{3}{25} \times 96 \times \cdot 2618 = 117{\cdot}12 \times 96 \times \cdot 2618 =$ 2943·55 inches. Which being divided by 282, give 10·43, for the number of ale gallons required.

PRO-

ſegments are as the ſquares of their diameters, $\frac{\frac{1}{3}\text{CI}}{\text{AB} - \text{GC}} \times ($AB $\times$ circular ſegment EBF $- \frac{\text{GC} \times \text{DB} \times \text{CN} \times \sqrt{\text{NO} \times \text{GC}}}{\text{DC} \times \text{AB}^2} \times$ ſeg. of the cir. AEBF whoſe height is $\frac{\text{AB} \times \text{DC}}{\text{CN}}) =$ the content of the elliptic hoof EFCB.

But, by ſim. triangles, GC $-$ AD : DC :: GC : CN $= \frac{\text{GC} \times \text{CD}}{\text{GC} - \text{AD}}$, and GC $-$ AD : DB :: GC : NO $= \frac{\text{GC} \times \text{DB}}{\text{GC} - \text{AD}}$; which values of NO and NC being ſubſtituted inſtead of them, in the above expreſſion of the elliptic hoof, will give

$\frac{\frac{1}{3}\text{CI}}{\text{AB} - \text{GC}} \times [\text{AB} \times \text{cir. ſeg. EBF} - \frac{\text{GC}^3}{\text{AB}^2} \times (\frac{\text{DB}}{\text{GC} - \text{AD}})^{\frac{3}{2}} \times$ ſeg. of the cir. AEBF whoſe height is $\frac{\text{AB} \times (\text{GC} - \text{AD})}{\text{GC}}] = \frac{\frac{1}{3}h}{\text{D} - d} \times [\text{D} \times \text{circ. ſeg. EBF} - \frac{d^3}{\text{D}^2} \times (\frac{\text{DB}}{\text{BD} - \text{D} + d})^{\frac{3}{2}}$ $\times$ ſegment of the circle AB whoſe height is $\frac{\text{D} \times (\text{DB} - \text{D} + d)}{d}]$

$=$

PROBLEM XXVII.

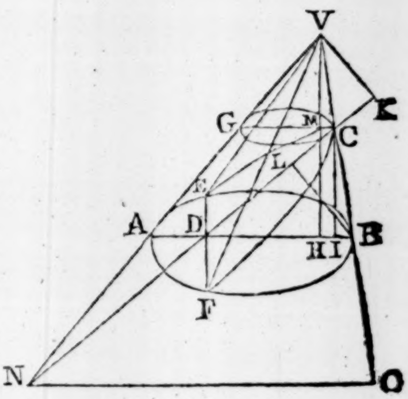
To find the Solidity of the Elliptic Hoofs of the Frustum of a Cone made by a Plane cutting off a Part of the Base.

1. From the versed sine, or height of the base of the hoof, subtract the difference of the diameters of the base and end of the frustum, and divide the remainder by the diameter of the said end; find the tabular segment whose versed sine is equal to the quotient; find also the tabular segment whose versed sine is expressed by the quotient of the versed sine of the base of the hoof divided by the diameter of the base of the frustum; multiply the former segment by the cube of the diameter of the end, and by the quotient of the versed sine of the base of the hoof divided by the difference between the said versed sine and the difference of the diameters, and by the root of the said quotient; and multiply the latter

 segment

= the elliptic hoof EFCB; putting h for the height of the hoof, and D and d for the diameters of the base and end, or top, of the frustum, respectively.

Or if we would reduce the circular segments in the last expression to those of a circle whose diameter is 1, to be found in the table of circular segments at the end of the book, that expression will become $\frac{\frac{1}{3}h}{D - d} \times [D^3 \times$ tab. seg. whose height is $\frac{BD}{D} - d^3 \times (\frac{BD}{BD - D + d})^{\frac{3}{2}} \times$ tab. seg. whose height is $\frac{BD - D + d}{d}] =$ the elliptic hoof EFCB. And this is the rule in prob. 27.



And

ſegment by the cube of the greater diameter; multiply the difference of theſe two products by $\frac{1}{3}$ of the height of the hoof, and the product divided by the difference of the diameters will give the content of the hoof required.

2. Subtract

And if this value of the hoof be taken from $\frac{1}{3}hn \times \frac{D^3 - d^3}{D - d}$ = that of the whole conic fruſtum, the remainder $\frac{\frac{1}{3}h}{D - d} \times : [D^3 - d^3) \times n - D^3 \times$ tab. ſeg. whoſe height is $\frac{BD}{D} + d^3 \times (\frac{BD}{BD - D + d})^{\frac{3}{2}} \times$ tabular ſeg. whoſe height is $\frac{BD - D + d}{d}]$ or $\frac{\frac{1}{3}h}{D - d} \times [-nd^3 + D^3 \times$ tab. ſeg. whoſe height is $\frac{AD}{D} + d^3 \times (\frac{BD}{BD - D + d})^{\frac{3}{2}} \times$ tabular ſeg. whoſe height is $\frac{BD - D + d}{d}]$ will expreſs the meaſure of the complemental elliptic hoof EFCGA.

Corol. 2. If the points D and A coincide, the ſection EFC will be a whole ellipſe, and the rules in corollary 1 will become $\frac{1}{3}Dhn \times \frac{DD - d\sqrt{Dd}}{D - d}$ = the elliptic hoof ACB, and $\frac{1}{3}Dhn \times \frac{D\sqrt{Dd} - dd}{D - d}$ = the complemental elliptic hoof ACG. And theſe are the rules in prob. 26.

Corol. 3. Since $\frac{h}{D - d} = \frac{H}{D}$, denoting the height of the whole cone by H, the laſt theor. will become $\frac{1}{3}nH \times (D^2 - d\sqrt{Dd})$ = ACB, and $\frac{1}{3}nH \times (D\sqrt{Dd} - dd)$ = ACG.

Corol. 4. But $\frac{1}{3}nHD^2$ = the whole cone VAB, and therefore $\frac{1}{3}nHD^2 - \frac{1}{3}nH \times (D^2 - d\sqrt{Dd}) = \frac{1}{3}nHd\sqrt{Dd}$ = the top part VAC. Which top part, conſequently, is to the whole cone VAB, as $\frac{1}{3}nHd\sqrt{Dd}$ to $\frac{1}{3}nHD^2$, or as $d^{\frac{3}{2}}$ to $D^{\frac{3}{2}}$; or the ſquare of the whole cone VAB is to the ſquare of the top part VAC, as D^3 to d^3.

Corol.

2. Subtract the measure of the hoof, above found, from that of the whole frustum, and the remainder will be the measure of the complemental hoof.

That is, if D and d be the extreme diameters of the frustum, and h its height; also P = the tabular segment whose versed sine is $\frac{BD}{D}$, Q = that whose versed sine is $\frac{BD - D + d}{d}$, and $n = \cdot 785398$ &c.

Then $(P \times D^3 - Q \times d^3 \times \frac{BD}{BD - D + d}\sqrt{\frac{BD}{BD - D + d}}) \times \frac{\frac{1}{3}h}{D - d}$ = the content of the elliptic hoof EFCB.

And

$[n(D^3 - d^3) - P \times D^3 + Q \times d^3 \times \frac{BD}{BD - D + d}\sqrt{\frac{BD}{BD - D + d}}] \times \frac{\frac{1}{3}h}{D - d}$ = to that of the complemental hoof EFCGA.

These are proved in corollary 1 to prob. 26.

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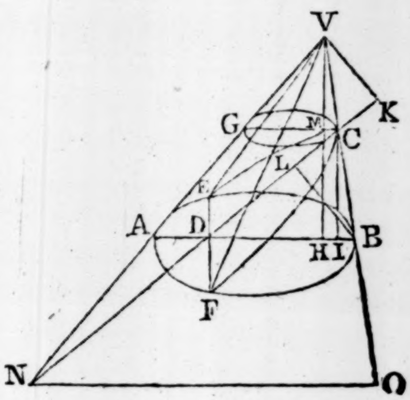


Corol. 5. If the angle CDB be equal to the angle VAB, or if the section ECF be parallel to the side AG of the cone, it will be a parabola whose axe is CD base EF = $2\sqrt{AD \times DB} = 2\sqrt{(D-d) \times d}$, and its area, by prob. 5 sect 4 part 3, = $\frac{2}{3}DC \times EF = \frac{4}{3}DC\sqrt{(D - d) \times d}$; and therefore the general theorem will become

$\frac{\frac{1}{3}h}{D - d} \times (D \times \text{cir. seg. } EBF - \frac{d(D-d)}{DC} \times \frac{4}{3}DC\sqrt{(D - d) \times d})$

$= \frac{1}{3}h \times (\frac{D^3}{D - d} \times \text{tab. seg. whose height is } \frac{D - d}{D} - \frac{4}{3}d\sqrt{(D - d) \times d})$

= the parabolic hoof EFBC.

Which

EXAMPLE I.

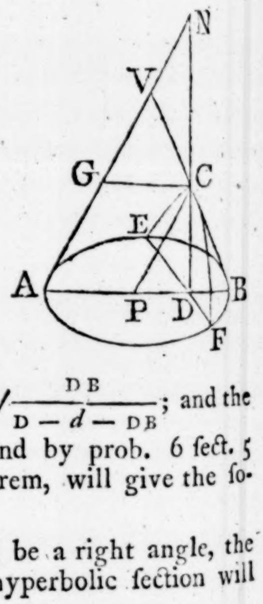
If a veſſel in the form of the fruſtum of a cone, whoſe bottom diameter is 30 inches, be inclined to the horizon till the ſurface of the liquor in it cut the bottom, leaving 10 inches of its diameter dry, and meeting the ſide in C at the diſtance CI of 18 inches from the baſe: how many Wincheſter gallons are in it, ſuppoſing the diameter GC of the veſſel at the top of the liquor to be $19\frac{1}{5}$ inches?

Here

Which being taken from $\frac{1}{3}hn \times \frac{D^3 - d^3}{D - d}$ = the meaſure of the whole fruſtum ABCG, the remainder

$\frac{1}{3}h \times [\frac{4}{3}d\sqrt{(D-d)d} - \frac{nd^3}{D-d} + \frac{D^3}{D-d} \times$ tab. ſeg. whoſe height is $\frac{d}{D}$],

or $\frac{\frac{1}{3}h}{D-d} \times [(D-d)\frac{4}{3}d\sqrt{d(D-d)} - nd^3 + D^3 \times$ tab. ſegment whoſe height is $\frac{D}{d}$,] will expreſs the meaſure of the complemental parabolic hoof EFCGA.



Corol. 6. If the ∠CDB exceed the ∠VAB; or, which is the ſame thing, if DC and AV, produced, interſect above V, as in N; the ſection ECF will be an hyperbola, whoſe tranſverſe axe is CN. But in the hyperbola, as well as in the other conic ſections, the tranſverſe axe $CN = \frac{GC \times CD}{PD}$, CP being drawn parallel to VA, or $= \frac{d \times CD}{D - d - BD}$, and the conjugate of each alſo $= GC\sqrt{\frac{DB}{PD}} = d\sqrt{\frac{DB}{D - d - DB}}$; and the area of the hyperbolic ſection being found by prob. 6 ſect. 5 part 3, and ſubſtituted in the general theorem, will give the ſolidity of the hyperbolic ungula.

Corol. 7. If D coincide with I, or CDB be a right angle, the tranſverſe and conjugate axes of the hyperbolic ſection will become $\frac{2dh}{D-d}$ and d reſpectively.

Here $D = 30$, $d = 19\frac{1}{5}$, $h = 18$, and $BD = 30 - 10 = 20$.

Whence $\frac{BD}{D} = \frac{20}{30} = \frac{2}{3} = \cdot6666\frac{2}{3}$, the tab. verſ. the tabular area for which is $\cdot55622573$; and $\frac{BD-D+d}{d} = \frac{20-30+19\frac{1}{5}}{19\frac{1}{5}} = \frac{46}{96} = \frac{23}{48} = \frac{2\cdot875}{6} = \cdot479\frac{2}{3}$, the tabular area anſwering to which is $\cdot37187178$.

Again, $D^3 = 27000$, which multiplied by $\cdot55622573$, the former area, produces $15018\cdot09471$.

And $d^3 \times \frac{BD}{BD-D+d} \times \sqrt{\frac{BD}{BD-D+d}} = (19\frac{1}{5})^3 \times \frac{20}{9\frac{1}{5}} \times \sqrt{\frac{20}{9\frac{1}{5}}} = 19\frac{1}{5} \times (\frac{19\frac{1}{5}}{9\frac{1}{5}})^2 \times 20\sqrt{184} = \frac{19\frac{1}{5} \times 48^2 \times 40}{23 \times 23} \times \sqrt{46} = 22686\cdot470698$, which multiplied by $\cdot37187178$, the latter area, gives $8436\cdot45824$.

Then $6581\cdot63647$, the difference of theſe two products, being multiplied by $\frac{6}{10\frac{4}{5}} = \frac{30}{54} = \frac{5}{9} = \frac{\frac{1}{3}h}{D-d}$, the quotient of $\frac{1}{3}$ of the height divided by the difference of the diameters, will produce $3656\cdot4647$ for the ſolidity in inches: which being divided by $268\frac{4}{5}$, the inches in a corn gallon, give $13\cdot6029$ corn or Wincheſter gallons, for the quantity of liquor in the veſſel.

EXAMPLE II.

If a veſſel, in form of the fruſtum of a cone, cloſe at both ends, be placed in ſuch a poſition, that the liquor may juſt cover the leſs end, and 10 inches of the diameter of the greater; what number of wine gallons are in it, ſuppoſing the diameters of the two ends to be 30 and $19\frac{1}{5}$ inches, and their diſtance 18 inches?

Here the part filled is the complemental hoof to that in the laſt example. Therefore if from $(D^2 + (D + d)d) \times \frac{1}{3}nh = (30^2 + 19\frac{1}{5} \times 49\frac{1}{5}) \times 6 \times$ $\cdot785398 = 8692\cdot661$, the content of the whole fruſtum, be taken $3656\cdot4647$, that of the hoof in the laſt example, the remainder $5036\cdot1963$ will be the content, in inches, of the complemental hoof; which being divided by 231, give $21\cdot8018$ wine gallons for the content required.

PROBLEM XXVIII.

To find the Parabolic Hoofs of the Fruſtum of a Cone; that is, of the Hoofs made by a Plane paſſing Parallel to the Side of the Cone.

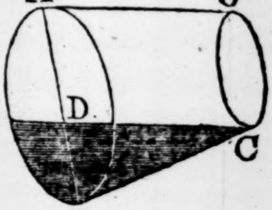
Multiply the baſe of the hoof by the greater diameter of the fruſtum, and divide the product by the difference of the diameters; from the quotient ſubtract $\frac{4}{3}$ of the product ariſing from the multiplication of the leſs diameter by the ſquare root of the product of the leſs diameter and difference of the diameters; then the remainder multiplied by $\frac{1}{3}$ of the height will be the content of the hoof required.

That is, $(\frac{A \times D}{D - d} - \frac{4}{3}d\sqrt{(D - d)d}) \times \frac{1}{3}h =$ the parabolic hoof EFCB; where D and d are the diameters AB and GC, of the fruſtum, h the height CI, and A the baſe EBF of the hoof. [*See the fig. page* 229.]

EXAMPLE.

If a veſſel, in the form of the fruſtum of a cone, be laid with its upper ſide parallel to the horizon, and if the greateſt diſtance of the ſurface of the liquor from the bottom, whoſe diameter is 30 inches, be 18 inches; how many ale gallons of liquor are in it, ſuppoſing the diameter of the veſſel, at the greateſt

greatest distance of the surface of the liquor from the bottom, to be $19\frac{1}{5}$ inches.



Here the dimensions are the same as in the last problems; and $BD = D - d = 30 - 19\frac{1}{5} = 10\frac{4}{5} = 10\cdot8$ the versed sine, or height, of the base; which divided by the diameter 30, gives $\cdot36 =$ the tabular versed sine; the area answering to which is $\cdot25455056$; which multiplied by 900, the square of the diameter, gives $229\cdot095504$ for the circular segment, or base of the hoof. Then $229\cdot0955 \times \frac{30}{10\cdot8} = \frac{2290\cdot955}{3\cdot6} = \frac{3818\cdot25834}{6} = 636\cdot37639$.

Again $\frac{4}{3}d\sqrt{(D - d) \times d} = \frac{4}{3} \times 19\frac{1}{5}\sqrt{10\frac{4}{5} \times 19\frac{1}{5}} = \frac{4}{3} \times 19\cdot2 \times \sqrt{\frac{54}{5} \times \frac{96}{5}} = \frac{4}{3} \times 19\cdot2 \times 14\cdot4 = 4 \times 6\cdot4 \times 14\cdot4 = 368\cdot64$.

Then $(636\cdot37639 - 368\cdot64) \times \frac{18}{3} = 267\cdot73639 \times 6 = 1606\cdot41834$ inches, the solidity; which being divided by 282, give $5\cdot696526$ ale gallons, for the quantity of liquor required.

Note 1. If from $8692\cdot661$, the content of the whole frustum, found in the second example of the last problem, be taken $1606\cdot418$, the content of the hoof, above found, the remainder $7086\cdot243$ will be that of the complemental parabolic hoof DAGC.

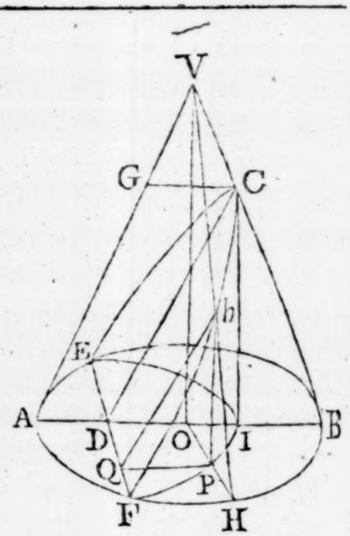
Note 2. If the section ECF be an hyperbola, the hoofs may then be found after the manner described in the sixth corollary to prob. 27.

PROBLEM XXIX.

*To find the Convex Surface of the Elliptic Hoofs of the Frustum of a Cone, made by a Plane cutting the Opposite Extremities of the Ends.**

To four times the ſquare of the height of the fruſtum, add the ſquare of the difference of the diameters of the baſe and end; divide the ſquare root of the ſum, by the difference of the ſaid diameters; and call the quotient Q.

Multiply



* Let the right-angled triangle VOH be conceived to revolve about the axe VO, the part Hh of the hypotenuſe deſcribing the ſurface of the ungula, while the part HP of the radius of the baſe of the cone, cut off by the perpendicular hP, deſcribes the ſpace FPIEBF.

Then ſince, by the ſimilar triangles, VOH, hPH, it is everywhere as HP : Hh :: HO or OB : HV or BV, the ſpaces generated by theſe lines will be, likewiſe, in the ſame ratio, viz. OB : BV :: the area FIEBF : the ſurface FCEBS $= \frac{BV}{BO} \times$ FIEBF.

And in the ſame manner, by reaſon of the ſimilar triangles DIC, QPh, QP being drawn parallel to DB, we have DC : DI :: FCEF = B = the ſpace deſcribed by Qh, : FIEF the ſpace deſcribed in the ſame time by QP $= \frac{DI}{DC} \times$ FCEF $= \frac{DI}{DC} \times$ B.

But the ſpace FIEBF = the circular ſeg. FBEF or A − FIEF $= A - \frac{DI}{DC} \times B$. And conſequently

$S = \frac{VB}{OB} \times FIEBF = \frac{VB}{OB} \times (A - \frac{DI}{DC} \times B) = \frac{CB}{IB} \times (A - \frac{DI}{DC} \times B)$ = the convex ſurface of the ungula EFBC.

And

Multiply half the ſum of the ſaid diameters by the ſquare root of their product; and call the product P. Then

1. Q multiplied by P and by ·785398, will give the convex ſurface of the oblique cone ACV.

That is, $\frac{n\sqrt{4b^2+(D-d)^2}}{D-d} \times \frac{D+d}{2}\sqrt{Dd} =$ the curve ſurface of ACV.

2. Q multiplied by the difference between P and the ſquare of the diameter of the baſe, and by ·785398, will give that of the hoof, ABC, including the baſe.

That is, $\frac{n\sqrt{4b^2+(D-d)^2}}{D-d} \times (DD - \frac{D+d}{2}\sqrt{Dd}) =$ the curve ſurface of ABC.

3. Q

And the method of proceeding will be the ſame for any other kind of pyramid.

Corol. 1. If from $\frac{VB}{OB} \times$ the circle AE, the convex ſurface of the whole cone, be taken that of the ungula, above found, the remainder $\frac{VB}{OB} \times (FAEF + \frac{DI}{DC} \times FCEF)$, will expreſs the convex ſurface of the remaining part EFCVA of the cone.

Corol. 2. And if from the value of the part laſt found be taken $\frac{VB}{OB} \times$ circle GC $= \frac{VC}{OI} \times$ circle GC, the convex ſurface of the top cone GVC, the remainder $\frac{VB}{OB} \times (FAEF + \frac{DI}{DC} \times FCEF -$ circle GC$)$, will expreſs that of the complemental ungula EFAGC.

Corol. 3. If the $\angle$CDB be leſs than the $\angle$GAB, or the ſect. FCE be an ellipſe, the ſurface of the ungula will be $\frac{VB}{OB} \times$ (cir. ſeg. FBE $- \frac{DI}{DC} \times$ ellipſe ſeg. FCE) = by proceeding as

3. Q multiplied by the difference between P and the ſquare of the top diameter, and by ·785398, will give the ſurface of the hoof ACG including the top.

That is, $\frac{n\sqrt{4h^2 + (D-d)^2}}{D-d} \times (\frac{D+d}{2}\sqrt{Dd - dd}) =$ the curve ſurface of ACG.

Where the letters denote the ſame Quantities as in the foregoing problems.—All theſe are proved in corollary 4.

EX-

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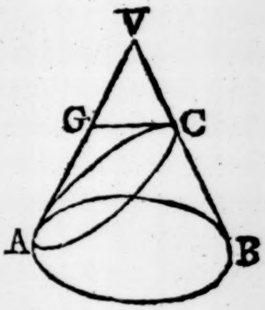
as in cor. 1 prob. 26, $\frac{VB}{OB} \times (EBF - \frac{GC^2 \times DI}{AB^2 \times (GC - AD)}\sqrt{\frac{DB}{GC - AD}}$ $\times$ ſeg. of the circle AB whoſe height is AB $\times \frac{GC - AD}{GC}) =$ $\frac{\sqrt{4h^2 + (D-d)^2}}{D-d} \times (EBF - \frac{dd}{DD} \times \frac{DB - \frac{1}{2}(D-d)}{DB - (D-d)}\sqrt{\frac{DB}{DB-(D-d)}}$ $\times$ ſeg. of the circle AB whoſe height is D $\times \frac{DB - (D-d)}{d}) =$ $\frac{\sqrt{4h^2 + (D-d)^2}}{D-d} \times$ ($D^2 \times$ tabular ſegment, whoſe height is $\frac{DB}{D} - d^2 \times \frac{DB - \frac{1}{2}(D-d)}{BD - (D-d)}\sqrt{\frac{DB}{DB - (D-d)}} \times$ tab. ſeg. whoſe height is $\frac{DB - (D-d)}{d}$), where h is the height, and D and d the diameters of the baſe and top of the fruſtum.

And the value of the ſurface of the remaining part EFCVA, in corollary 1, will become $\frac{VB}{OB} \times$ (cir. ſeg. FAE + $\frac{DI}{DC} \times$ ellip. ſeg. FCE) $= \frac{VB}{OB} \times (FAE + \frac{GC^2 \times DI}{AB^2 (GC - AD)}\sqrt{\frac{DB}{GC - AD}} \times$ ſeg. of the circle AB whoſe height is AB $\times \frac{GC - AD}{GC}) = \frac{\sqrt{4h^2 + (D-d)^2}}{D-d} \times$ $(FAE + \frac{dd}{DD} \times \frac{DB - \frac{1}{2}(D-d)}{DB - (D-d)}\sqrt{\frac{DB}{DB - (D-d)}} \times$ ſeg. of the

cir.

EXAMPLE.

Required the convex ſurfaces of the two hoofs of a conic fruſtum, whoſe baſe and top diameters are 30 and $19\frac{1}{5}$, and height 18 inches; the ſection being made through the contrary extremities of the diameters: together with that of the oblique cone ACV.



Here $D = 30$, $d = 19\frac{1}{5}$, and $h = 18$.

Then $Q = \frac{\sqrt{4h^2 + (D-d)^2}}{D-d} = \frac{\sqrt{4 \times 18^2 + 10{\cdot}8^2}}{10{\cdot}8} = \frac{\sqrt{4^2 + 1{\cdot}2^2}}{1{\cdot}2} = \frac{1}{3}\sqrt{10^2 + 3^2} = \frac{1}{3}\sqrt{109} = 3{\cdot}48010217$.

$P = \frac{D+d}{2}\sqrt{Dd} = 24\frac{3}{5}\sqrt{30 \times 19\frac{1}{5}} = 24\frac{3}{5} \times 24 = 590\frac{2}{5}$.——And then

I. Q

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cir. AB whoſe height is $D \times \frac{DB - (D-d)}{d}) = \frac{\sqrt{4h^2+(D-d)^2}}{D-d} \times$ $(D^2 \times$ tab. ſeg. whoſe height is $\frac{AD}{D} + d^2 \times \frac{DB - \frac{1}{2}(D-d)}{DB-(D-d)}$ $\sqrt{\frac{DB}{DB-(D-d)}} \times$ tab. ſeg. whoſe height is $\frac{DB-(D-d)}{d})$. Or it is $= \frac{\sqrt{4h^2+(D-d)^2}}{D-d} \times (FAE + \frac{dd}{DD} \times \frac{\frac{1}{2}(D+d) - AD}{d - AD}\sqrt{\frac{D-AD}{d-AD}}$ $\times$ ſegment of the circle AB whoſe height is $D \times \frac{d - AD}{d}) =$ $\frac{\sqrt{4h^2 + (D-d)^2}}{D-d} \times (D^2 \times$ tabular ſegment whoſe height is $\frac{AD}{D} + d^2 \times \frac{\frac{1}{2}(D+d) - AD}{d - AD}\sqrt{\frac{D-AD}{d-AD}} \times$ tab. ſeg. whoſe height is $\frac{d-AD}{d})$.

Alſo

1. $Q \times P \times \cdot785398 = 2\cdot73326528 \times 590\cdot4 = 1613\cdot7198213$ inches $= 11\cdot2063876$ feet, the convex surface of the oblique cone ACV.

2. $Q \times \cdot785398 \times (D^2 - P) = \cdot785398 \times 3\cdot48010217 \times (900 - 590\cdot4) = 2\cdot73326528 \times 309\cdot6 = 846\cdot2189307$ inches $= 5\cdot87652$ feet, that of the hoof ABC.

3. $Q \times \cdot785398 \times (P - d^2) = \cdot785398 \times 3\cdot48010217 \times (509\cdot4 - 368\cdot64) = 2\cdot73326528 \times 221\cdot76 = 606\cdot1289085$ inches $= 4\cdot2092285$ feet, that of the complemental hoof ACG.

PRO-

Also that of the complemental hoof in cor. 2, will become $\frac{VB}{OB}$ (cir. seg. FAE $+ \frac{DI}{DC} \times$ ellip. seg. FCE $-$ the cir. CG) $=$ $\frac{VB}{OB} \times (-n \times GC^2 + FAE + \frac{GC^2 \times DI}{AB^2 \times (GC - AD)}) \sqrt{\frac{DB}{GC - AD}} \times$ segment of the circle AB whose height is $AB \times \frac{GC - AD}{GC}) =$ $\frac{\sqrt{4b^2 + (D - d)^2}}{D - d} \times (-ndd + FAE + \frac{dd}{DD} \times \frac{\frac{1}{2}(D + d) - AD}{d - AD}$ $\sqrt{\frac{D - AD}{d - AD}} \times$ seg. of the circle AB whose height is $D \times \frac{d - AD}{d})$ $= \frac{\sqrt{4b^2 + (D - d)^2}}{D - d} \times (-nd^2 + D^2 \times$ tab. seg. whose height is $\frac{AD}{D} + d^2 \times \frac{\frac{1}{2}(D + d) - AD}{d - AD} \sqrt{\frac{D - AD}{d - AD}} \times$ tab. seg. whose height is $\frac{d - AD}{d})$.

Corol. 4. If D coincide with A, the rules in the last corollary will become $\frac{VB}{BO} \times (AB^2 \times n - n \times AB \times \sqrt{AI \times GC} =$ $\frac{n\sqrt{4b^2 + (D - d)^2}}{D - d} \times (D^2 - \frac{D + d}{2}\sqrt{Dd})$ for the convex surface of the ungula ABC.

And

PROBLEM XXX.

To find the Convex Surfaces of the Elliptic Ungulas of a Cone, made by a Plane cutting off a Part of the Base.

Multiply continually together the four following quantities, viz. the quotient arising from the division of

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And $\frac{VB}{OB} \times AI\sqrt{AB \times GC} \times n = 2n \times VB \times AI\sqrt{\frac{GC}{AB}} = \frac{n\sqrt{4b^2 + (D-d)^2}}{D-d} \times \frac{D+d}{2}\sqrt{Dd}$ for that of the oblique cone ACV.

Also $\frac{VB \times n}{OB} \times (AI\sqrt{AB \times GC} - GC^2) = \frac{n\sqrt{4b^2 + (D-d)^2}}{D-d} \times (\frac{D+d}{2}\sqrt{Dd-d^2})$ for that of the complemental elliptic hoof ACG.

Corol. 5. If DC be parallel to AV, or the section a parabola; since its area B is $\frac{4}{3}DC \times DF = \frac{4}{3}DC\sqrt{AD \times DB}$, the general theor. for the ungula will become $\frac{VB}{OB} \times (\text{seg. } FBE - \frac{4}{3}DI\sqrt{AD \times DB}) = \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ [seg. FBE to height $DB - \frac{2}{3}(D-d)\sqrt{d(D-d)}$] for the convex surface of the parabolic ungula FEBC.

And the rule in cor. 1 will be $\frac{VB}{OB} \times (\text{seg. } FAE + \frac{4}{3}DI\sqrt{AD \times DB}) = \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ [(seg. FAE to height $AE + \frac{2}{3}(D-d)\sqrt{d(D-d)}$] for that of the part AEFCV.

Also that in cor. 2 will be $\frac{VB}{OB} \times (\text{seg. } FAE + \frac{4}{3}DI\sqrt{AD \times DB} - AD^2 \times n) = \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ (segment FAE to height $AD + \frac{2}{3}(D-d)\sqrt{d(D-d)} - ndd$), for that of the complemental parabolic ungula FAECG.

Corol.

of the ſquare of the leſs diameter by that of the greater; the quotient ariſing from the diviſion of the difference between the part of the diameter of the baſe not cut off and half the ſum of the diameters, by the difference between the ſaid part of the diameter of the baſe and the leſs diameter; the ſquare root of the quotient ariſing from the diviſion of the part cut off the diameter of the baſe, by the difference between the part not cut off and the leſs diameter; and ſuch a ſegment of the baſe of the fruſtum, whoſe height is equal to the product ariſing from the multiplication

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Corol. 6. If the angle CDB be greater than the angle VAB, or the ſection be an hyperbola, its area being found, and ſubſtituted for B in the general rules, will give the ſurfaces of the hyperbolic ungulas.

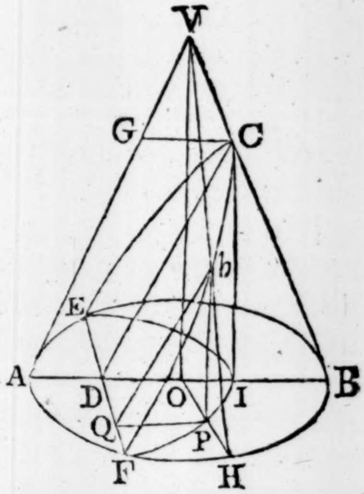
Corol. 7. If the hyperbolic ſection be perpendicular to the baſe, DI will vaniſh, and the general theorems will become $\frac{VB}{OB} \times$ ſeg. FBE $= \frac{CB}{IB} \times$ ſeg. FBE its height being IB $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ ſeg. of the circle AB, whoſe height is $\frac{D-d}{2}$, for the curve ſurface of the perpendicular ungula CIB.

Alſo $\frac{VB}{OB} \times$ ſegment FAE $= \frac{CB}{IB} \times$ ſegment whoſe height is AI $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ ſeg. of the cir. AB whoſe height is $\frac{D+d}{2}$, for that of the remaining part AICV.

And $\frac{VB}{OB} \times$ ſeg. FAE $-$ cir. CG $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ (ſeg. of the circle AB whoſe height is $\frac{D+d}{2} - nd^2$), for that of the complemental perpendicular ungula AICG.

tion of the quotient of the greater diameter divided by the leſs, into the difference between the leſs diameter and the part of the baſe diameter not cut off: and call the laſt product R.—Then

1. The difference between R and the baſe of the hoof, multiplied by the quantity Q, in the laſt problem, will give the convex ſurface of the ungula required.



That is, $\frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times$ (ſeg. FBE $- \frac{dd}{DD} \times \frac{\frac{1}{2}(D+d)-AD}{d-AD}$ $\sqrt{\frac{DB}{d-AD}} \times$ ſeg. of the circle AB whoſe height is $D \times \frac{d-AD}{d}$) = the convex ſurface of the hoof FEBC. Uſing ſtill the ſame letters as in the former problems.

2. The ſum of R and the remaining ſegment of the baſe, not included by the hoof above, multiplied by Q, will give the ſurface of the remaining part of the whole cone.

That is, $\frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times$ (ſeg. FAE $+ \frac{dd}{DD} \times \frac{\frac{1}{2}(D+d)-AD}{d-AD} \sqrt{\frac{DB}{d-AD}} \times$ ſegment of the circle AB whoſe height is $D \times \frac{d-AD}{d}$) = the convex ſurface of the part FEAVC.

3. From the ſurface of the part in the laſt article, ſubtract that of the little cone at the top, viz. the product of Q by the area of the top of the fruſtum; and the remainder will be the ſurface of the complemental ungula.

That is, $\frac{\sqrt{4b^2 + (\text{D} - d)^2}}{\text{D} - d} \times (-ndd + \text{seg. FAE} +$ $\frac{dd}{\text{DD}} \times \frac{\frac{1}{2}(\text{D} + d) - \text{AD}}{d - \text{AD}} \sqrt{\frac{\text{DB}}{d - \text{AD}}} \times$ seg. of the circle AB whose height is $\text{D} \times \frac{d - \text{AD}}{d}) =$ the convex surface of the complemental hoof FEAGC.—All these are proved in corollary 3 problem 29.

EXAMPLE.

Required the convex surfaces of the parts into which a cone is cut by a plane which enters the side at the distance of 18 inches from the base, and cuts off 20 inches of the diameter of the base; that diameter being 30, and the diameter at the top of the section $19\frac{1}{5}$ inches.

Here $\text{D} = 30$, $d = 19\frac{1}{5}$, $b = 18$, $\text{BD} = 20$, and $\text{AD} = 10$.

Whence $Q = \frac{\sqrt{4b^2 + (\text{D} - d)^2}}{\text{D} - d} = \frac{1}{3}\sqrt{109} =$ $3{\cdot}48010217$.

The seg. $\text{FBE} = 30^2 \times {\cdot}55622573 = 500{\cdot}603157$.

The seg. $\text{FAE} = 30^2 \times {\cdot}22917243 = 206{\cdot}255187$.

$\text{R} = (\frac{19\frac{1}{5}}{30})^2 \times \frac{14\frac{3}{5}}{9\frac{1}{5}} \sqrt{\frac{20}{9\frac{1}{5}}} \times$ seg. whose diameter is 30 and height $\frac{30 \times 9\frac{1}{5}}{19\frac{1}{5}}$

$= \frac{8^2 \times 146\sqrt{46}}{5^3 \times 23^2} \times 30^2 \times$ tabular seg. whose height is $\frac{23}{48}$

$= \frac{6^2 \times 8^2 \times 146\sqrt{46}}{5 \times 23^2} \times {\cdot}37187178 = 320{\cdot}761173$.

And $ndd = {\cdot}785398 \times (19\frac{1}{5})^2 = 289{\cdot}529179$.

Therefore

1. $Q \times (\text{FBE} - \text{R}) = 3{\cdot}48010217 \times 179{\cdot}841984$ $= 625{\cdot}868479 =$ the convex surface of the hoof FEBC.

2. Q

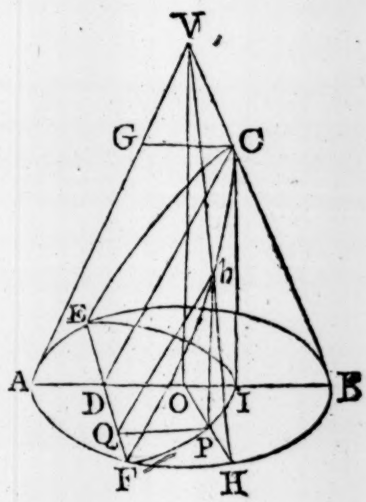
2. $Q \times (FAE + R) = 3{\cdot}48010217 \times 527{\cdot}01636 = 1834{\cdot}070779$ = that of the part FAECV.

3. $Q \times (FAE + R - ndd) = 3{\cdot}48010217 \times 237{\cdot}487181 = 826{\cdot}479654$ = that of the complemental hoof FAECG.

PROBLEM XXXI.

To find the Convex Surfaces of the Parabolic Hoofs made by a Plane cutting a Cone Parallel to one Side.

Multiply $\frac{2}{3}$ of the difference of the diameters of the base and top of the frustum, by the square root of the product of the said difference and the less diameter; and call the product S.—Then



1. The difference between S and the base of the hoof, multiplied by Q, in the 29th problem, will give the surface of the hoof.

That is, $\frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ [segment $FBE - \frac{2}{3}(D-d)\sqrt{d(D-d)}$] = the convex surface of the ungula FEBC. Using still the same letters.

2. The sum of S and the complemental base, multiplied by Q, will give the convex surface of the remaining part of the cone.

That is, $\frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ [segment $FAE + \frac{2}{3}(D-d)\sqrt{d(D-d)}$] = the surface of the part FAECV.

 3. The

3. The difference between the circle CG, and the ſum of S and the complemental baſe, multiplied by Q, will give the ſurface of the complemental hoof.

That is, $\frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times [\text{ſegment FAE} + \frac{2}{3}(D-d)\sqrt{d(D-d)} - ndd] =$ the convex ſurface of the complemental hoof FAECG.

EXAMPLE.

Required the convex ſurfaces of the parabolic hoofs of the fruſtum of a cone, whoſe height is 18, and its diameters 30 and $19\frac{1}{5}$ inches.

Here again $Q = 3{\cdot}48010217$.

$S = \frac{2}{3}(D-d)\sqrt{d(D-d)} = \frac{2}{3} \times 10\frac{4}{5}\sqrt{19\frac{1}{5} \times 10\frac{4}{5}} = 103{\cdot}68$.

The ſeg. FBE $= 30^2 \times$ tab. ſeg. whoſe verſed ſine is $\frac{10\frac{4}{5}}{30}$ or $\frac{3{\cdot}6}{10}$ or $\cdot 36 = 900 \times \cdot 25455055 = 229{\cdot}095495$.

The ſeg. FAE $= 30^2 \times$ tab. ſeg. whoſe height is $\frac{19\frac{1}{5}}{30}$ or $\frac{6{\cdot}4}{10}$ or $\cdot 64 = 900 \times \cdot 53084761 = 477{\cdot}762849$.

And ndd, as before, $= 289{\cdot}529179$.—Whence

1. $Q \times (\text{FBE} - S) = 3{\cdot}48010217 \times 125{\cdot}415495 = 436{\cdot}458737 =$ the convex ſurface of the hoof FEBC.

2. $Q \times (\text{FAE} + S) = 3{\cdot}48010217 \times 581{\cdot}442849 = 2023{\cdot}480521 =$ that of the part FAECV.

3. $Q \times (\text{FAE} + S - ndd) = 3{\cdot}48010217 \times 291{\cdot}91367 = 1015{\cdot}889396 =$ that of the complemental hoof FAECG.

PRO-

PROBLEM XXXII.

To find the Curve Surfaces of the Hyperbolic Hoofs of the Frustum of a Cone, made by a Plane cutting it Perpendicular to the Base.

1. Multiply the quantity Q, found as in the preceding problems, by the base of the hoof; and the product will be its surface.

That is, $\frac{VB}{OB} \times$ seg. FBE $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ seg. of the circle AB whose height is $\frac{D-d}{2} =$ the curve surface of the perpendicular hoof FEBC. [See the fig. at page 243.]

2. Multiply Q by the complemental base, and the product will be the surface of the complemental part of the whole cone.

That is, $\frac{VB}{OB} \times$ seg. FAE $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ seg. of the circle AB whose height is $\frac{D+d}{2}$ or AD or AI $=$ the curve surface of the part FAECV.

3. Multiply Q by the difference between the complemental segment and the circle GC, and the product will be the surface of the complemental hoof.

That is, $\frac{VB}{OB} \times$ (seg. FAE$-$cir. CG) $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d}$ $\times$ ($-ndd +$ seg. of the circle AB whose height is $\frac{D+d}{2}$) $=$ the curve surface of the complemental hoof FAECG. Using still the same letters as in the former problems.

Note. The two first articles are the same rule, and serve for the whole cone as well as for any perpendicular part.

EXAMPLE.

Required the curve ſurfaces of the perpendicular hoofs of a cone, whoſe baſe diameter is 30 inches; the height of the ſection being 18, and the diameter of the cone at the top of the ſection, $19\frac{1}{5}$ inches.

Here again $\begin{cases} Q = 3{\cdot}48010217, \\ ndd = 289{\cdot}529179. \end{cases}$

The ſegment FBE $= 30^2 \times$ tab. ſeg. whoſe height is $\frac{30 - 19\frac{1}{5}}{2 \times 30}$ or $\cdot 18 = 900 \times \cdot 09613453 = 86{\cdot}521077$.

The ſegment FAE $= 30^2 \times$ tab. ſeg. whoſe height is $\frac{30 + 19\frac{1}{5}}{2 \times 30}$ or $\cdot 82 = 900 \times \cdot 68926363 = 620{\cdot}337267$.

And FAE $- ndd = 330{\cdot}808088$.

Whence, multiplying theſe three each by Q or $3{\cdot}48010217$, we have

1. $Q \times$ FBE $= 301{\cdot}102188$ the ſurf. of FEBC.

2. $Q \times$ FAE $= 2158{\cdot}837068$ the ſurf. of FEACV.

3. $Q \times$ (FAE $- ndd$) $= 1151{\cdot}245945$ the ſurf. of FEACG.

SECT.

SECTION II.

OF THE REGULAR BODIES.

DEFINITIONS.

1. A Regular ſolid or body, is a ſolid contained under ſome number of like, equal, and regular plane figures.

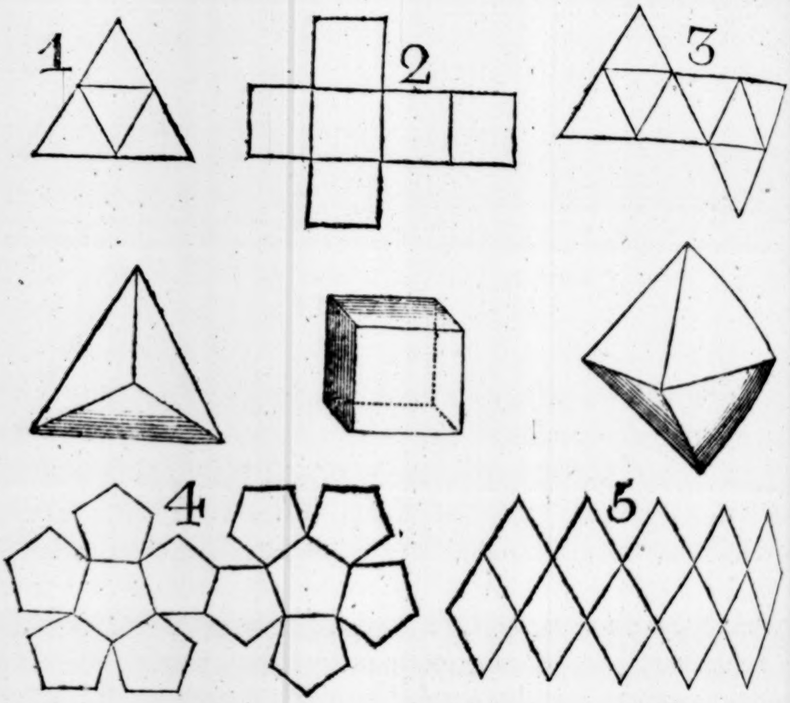
2. The plane figures, under which the ſolid is contained, are the faces of the ſolid. And the ſides of the plane figures, are the edges or linear ſides of the ſolid.

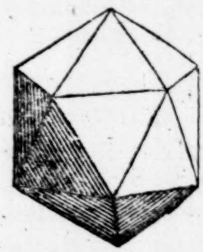
3. There are only five ſorts of regular ſolids, viz.
The tetraedron, or regular triangular pyramid, having four triangular faces;
The hexaedron, or cube, having ſix ſquare faces;
The octaedron, having eight triangular faces;
The dodecaedron, having twelve pentagonal faces;
And the icoſaedron, having twenty triangular faces.

PROBLEM I.

To Conſtruct or Form the Regular Solids.

Having deſcribed figures, as below, on paſteboard, or ſome other pliable matter, cut them out by the extreme ſides; and cut the other lines half through; then fold them at theſe lines, ſo cut, till the ſides meet; which being faſtened together with glue, or otherwiſe, you will have the form of the bodies. Namely, figure 1 will form the tetraedron; figure 2, the hexaedron; figure 3, the octaedron; figure 4, the dodecaedron; and figure 5, the icoſaedron.





PROBLEM II.

To find the Surface of a Tetraedron.

Multiply the ſquare of the linear edge by the root of 3, and the product will be the whole ſurface, or the ſum of the four faces.

That is, $A^2\sqrt{3}$ is the whole ſurface; A being the linear edge, or a ſide of one of the faces.*

EX-

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* DEMONSTRATION.

For, by rule 2 prob. 4 ſect. 1 part 2, one of the faces will be $= \frac{1}{4}\sqrt{3} \times A^2$; which being multiplied by 4, gives $A^2\sqrt{3}$ for all the four faces. *Q. E. D.*

EXAMPLE.

If each side of one face of a tetraedron be 1, required the whole surface.

Here $A = 1$; and therefore $A^2\sqrt{3} = \sqrt{3} = 1.7320508 =$ the surface required.

PROBLEM III.

To find the Solidity of a Tetraedron.

Multiply $\frac{1}{12}$ of the cube of the linear side by the root of 2, and the product will be the solidity of the tetraedron.

That is, $\frac{1}{12}A^3\sqrt{2}$ is the solidity.*

EX-

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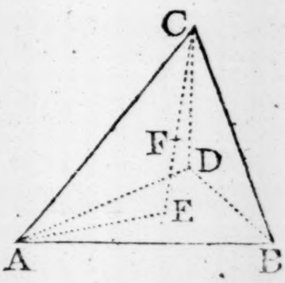
* DEMONSTRATION.



From one angle C of the tetraedron let fall a perpendicular CE upon the opposite side ADB, and draw EA.

Then $AC^2 = AE^2 + EC^2$; but, from the tables in page 81 and 84, it appears that $\frac{1}{3}AC^2 = \frac{1}{3}AB^2$ is $= AE^2$; therefore, by subtracting this equation from the former, $\frac{2}{3}AC^2$ will be $= CE^2$, and hence $CE = AC\sqrt{\frac{2}{3}}$. But ABD is $= \frac{1}{4}AB^2\sqrt{3} = \frac{1}{4}AC^2\sqrt{3}$. Then $\frac{1}{3}CE \times ABD = \frac{1}{3}AC\sqrt{\frac{2}{3}} \times \frac{1}{4}AC^2\sqrt{3} = \frac{1}{12}AC^3\sqrt{2}$ is the solidity. *Q. E. D.*

Corol. 1. A cube is to a tetraedron of the same linear side, as 1 is to $\frac{1}{12}\sqrt{2}$, or as 12 is to $\sqrt{2}$.

Corol. 2. Put r for the radius EF of the inscribed sphere, A for the linear side AC, B for the whole surface, and C for the solidity; and we shall have

$$r = \frac{3C}{B} = \frac{\frac{1}{4}A^3\sqrt{2}}{A^2\sqrt{3}} = \frac{1}{4}A\sqrt{\frac{2}{3}} = \frac{1}{12}A\sqrt{6}.$$

Corol. 3. Hence also r or FE is $= \frac{1}{4}EC$; for, by the demonstration, CE is $= A\sqrt{\frac{2}{3}}$.

Corol. 4. And hence the radius of the circumscribed sphere, is equal to 3 times that of the inscribed; that is, $R = FC = CE - EF = 4FE - EF = 3FE = 3r = \frac{1}{4}A\sqrt{6}$.

SCHO-

EXAMPLE.

If the linear ſide of a tetraedron be 1; required the ſolidity.

Here $A = 1$, and $\frac{1}{12} A \sqrt[3]{2} = \frac{1}{12}\sqrt{2} = \cdot 11785113$ = the ſolidity required.

PROBLEM IV.

To find the Surface or Solidity of a Hexaedron or Cube.

It is evident that 6 times the ſquare of the linear ſide, will be equal to the whole ſurface; and the cube of the linear ſide, equal to the ſolidity of the hexaedron, or cube.*

PRO-

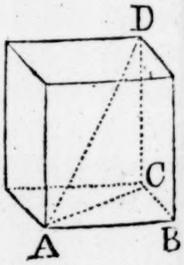
SCHOLIUM.

From this problem and its corollaries, together with the former problem, are deduced theſe equations.

1. $A = \sqrt{\frac{1}{3} B \sqrt{3}} = \sqrt[3]{6 C \sqrt{2}} = \frac{2}{3} R \sqrt{6} = 2r\sqrt{6}$.
2. $B = A^2 \sqrt{3} = 6 \sqrt[3]{C^2 \sqrt{3}} = \frac{8}{3} R^2 \sqrt{3} = 24 r^2 \sqrt{3}$.
3. $C = \frac{1}{12} A^3 \sqrt{2} = \frac{1}{36} B \sqrt{2 B \sqrt{3}} = \frac{8}{27} R^3 \sqrt{3} = 8 r^3 \sqrt{3}$.
4. $R = \frac{1}{4} A \sqrt{6} = \frac{1}{4} \sqrt{2 B \sqrt{3}} = \frac{3}{2} \sqrt[3]{\frac{1}{3} C \sqrt{3}} = 3r$.
5. $r = \frac{1}{12} A \sqrt{6} = \frac{1}{12} \sqrt{2 B \sqrt{3}} = \frac{1}{2} \sqrt[3]{\frac{1}{3} C \sqrt{3}} = \frac{1}{3} R$.

* SCHOLIUM.

Since, as is evident, the diameter of the inſcribed ſphere is equal to the linear ſide of the hexaedron, and the diameter of the circumſcribed ſphere is the diagonal $AD = \sqrt{AC^2 + CD^2} = \sqrt{AB^2 + BC^2 + CD^2} = \sqrt{3 AB^2} = AB\sqrt{3}$; if A, B, and C be put to denote the linear ſide, the ſurface, and the ſolidity of a hexaedron, and r, R for the radius of the inſcribed and circumſcribed ſphere; then in the hexaedron, or cube, we ſhall have theſe equations.



I

1. A

PROBLEM V.

To find the Surface of an Octaedron.

Multiply the square of the linear side by the root of 3, and double the product will be the surface. That is, $2A^2\sqrt{3}$ is the surface.*

EXAMPLE.

The linear side of an octaedron being 1, it is required to find the superficies.

Here $A = 1$, and $2A^2\sqrt{3} = 2\sqrt{3} = 3{\cdot}4641016$ = the surface required.

PROBLEM VI.

To find the Solidity of an Octaedron.

Multiply the cube of the linear side by the root of 2, and $\frac{1}{3}$ of the product will be the content of the octaedron.

That

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1. $A = \sqrt{\tfrac{1}{6}B} = \sqrt[3]{C} = \tfrac{2}{3}R\sqrt{3} = 2r.$
2. $B = 6A^2 = 6\sqrt[3]{C^2} = 8R^2 = 24r^2.$
3. $C = A^3 = \tfrac{1}{6}B\sqrt{\tfrac{1}{6}B} = \tfrac{8}{9}R^3\sqrt{3} = 8r^3.$
4. $R = \tfrac{1}{2}A\sqrt{3} = \tfrac{1}{2}\sqrt{\tfrac{1}{2}B} = \tfrac{1}{2}\sqrt{3} \times \sqrt[3]{C} = r\sqrt{3}.$
5. $r = \tfrac{1}{2}A = \tfrac{1}{2}\sqrt{\tfrac{1}{6}B} = \tfrac{1}{2}\sqrt[3]{C} = \tfrac{1}{3}R\sqrt{3}.$

* DEMONSTRATION.

By rule 2 prob. 4 sect. 1 part 2, one face is $= \tfrac{1}{4}A^2\sqrt{3}$; consequently 8 faces, or the whole surface, will be $= 2A^2\sqrt{3}$. *Q. E. D.*

That is, $\frac{1}{3}A^3\sqrt{2}$ is the solidity.*

EX-

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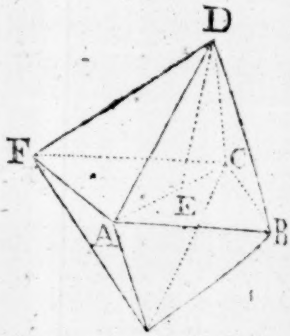
* DEMONSTRATION.



Let E be the middle of the diagonal AC of the solid, or the center of the solid; and draw DE, which will be equal to AE.

Then, since the solid is evidently composed of two equal square pyramids, each base of which, ABCF, is equal to the square of the linear side of the solid, and the altitude of each equal to DE or AE, half the diagonal of that square; we shall have $ABCF \times \frac{2}{3}AE = AB^2 \times \frac{1}{3}AC = \frac{1}{3}AB^2\sqrt{AB^2 + BC^2} = \frac{1}{3}AB^2\sqrt{2AB^2} = \frac{1}{3}AB^3\sqrt{2}$ for the solidity. *Q. E. D.*

Corol. 1. The radius AE of the circumscribed sphere is $= \frac{1}{2}AC = \frac{1}{2}\sqrt{2AB^2} = \frac{1}{2}AB\sqrt{2} = AB\sqrt{\frac{1}{2}}$.

Corol. 2. The radius of the inscribed sphere, is equal to the quotient arising from the division of 3 times the solidity by the whole surface, equal (from this problem and the last) $AB^3\sqrt{2} \div 2AB^2\sqrt{3} = \frac{1}{2}AB\sqrt{\frac{2}{3}} = \frac{1}{6}AB\sqrt{6} = AB\sqrt{\frac{1}{6}}$.

SCHOLIUM.

From the two last problems, and the corollaries, are deduced the following equations; in which A denotes the linear side of an octaedron, B the whole surface, C the solidity, r the radius of the inscribed sphere, and R the radius of the circumscribed sphere.

1. $A = \sqrt{\frac{B\sqrt{3}}{6}} = \sqrt[3]{\frac{3C\sqrt{2}}{2}} = R\sqrt{2} = r\sqrt{6}$.

2. $B = 2A^2\sqrt{3} = 6\sqrt[3]{\frac{C^2\sqrt{3}}{2}} = 4R^2\sqrt{3} = 12r^2\sqrt{3}$.

3. $C = \frac{1}{3}A^3\sqrt{2} = \frac{B\sqrt{B\sqrt{3}}}{18} = \frac{4}{3}R^3 = 4r^3\sqrt{3}$.

4. $R = \frac{1}{2}A\sqrt{2} = \sqrt{\frac{B\sqrt{3}}{12}} = \sqrt[3]{\frac{3}{4}C} = r\sqrt{3}$.

5. $r = \frac{1}{6}A\sqrt{6} = \frac{1}{6}\sqrt{B\sqrt{3}}, = \sqrt[3]{\frac{C\sqrt{3}}{12}} = \frac{1}{3}R\sqrt{3}$.

EXAMPLE.

If the linear side of an octaedron be 1, what is its solidity?

Here $A = 1$, and therefore $\frac{1}{3}A^3\sqrt{2} = \frac{1}{3}\sqrt{2} =$ ·47140452079 is the content required.

PROBLEM VII.

To find the Surface of a Dodecaedron.

To 1 add $\frac{2}{5}$ or $\frac{4}{10}$ of the root of 5; multiply the root of the sum by 15 times the square of the linear side; and the product will be the surface of the dodecaedron.

That is, $15A^2\sqrt{1 + \frac{2}{5}\sqrt{5}}$ is the surface.*

EXAMPLE.

If the linear side be 1, required the surface.

Here $A = 1$, and therefore $15A^2\sqrt{1 + \frac{2}{5}\sqrt{5}} = 15\sqrt{1 + \frac{2}{5}\sqrt{5}} = 20{\cdot}6457788075$ is the surface required.

PRO-

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* DEMONSTRATION.

By rule 2 prob. 4 sect. 1 part 2, the area of one face is $\frac{5}{4}A^2\sqrt{1 + \frac{2}{5}\sqrt{5}}$, and therefore 12 faces or the whole surface will be $12 \times \frac{5}{4}A^2\sqrt{1 + \frac{2}{5}\sqrt{5}} = 15A^2\sqrt{1 + \frac{2}{5}\sqrt{5}}$. *Q.E.D.*

PROBLEM VIII.

To find the Solidity of a Dodecaedron.

* To 21 times the root of 5 add 47, and divide the sum by 40; multiply the root of the quotient by 5 times

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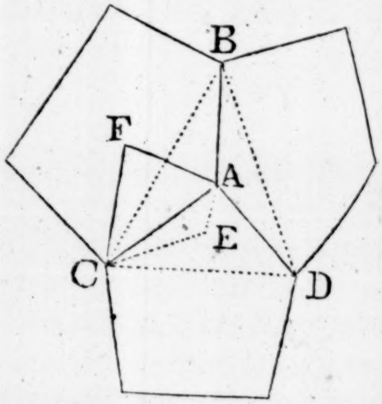
* DEMONSTRATION.



Let A be a solid angle of the dodecaedron; and let the extremities of the sides AB, AC, AD, of the faces which form the angle, be connected by the right lines BC, CD, DB, forming the equilateral triangle BCD within the solid; upon the center E of which triangle let fall the perpendicular AE, and to the center F of one of the faces draw the lines AF, CF.

The angle CAD contains 108 degrees, whose sine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$, to the radius 1; and the angle ADC contains 36 degrees, whose sine is $\frac{1}{4}\sqrt{10-2\sqrt{5}}$; therefore as $\sqrt{10-2\sqrt{5}} : \sqrt{10+2\sqrt{5}} ::$ $\mathrm{CA} : \mathrm{CD} = \mathrm{AC}\sqrt{\frac{10+2\sqrt{5}}{10-2\sqrt{5}}} = \mathrm{AC}\sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = \mathrm{AC}\sqrt{\frac{(5+\sqrt{5})^2}{25-5}}$ $= \frac{5+\sqrt{5}}{2\sqrt{5}}\mathrm{AC} = \frac{1+\sqrt{5}}{2}\mathrm{AC}$. And much after the same manner, we find $\mathrm{CE} = \frac{1}{3}\mathrm{CD}\sqrt{3} = \mathrm{CD}\sqrt{\frac{1}{3}} = \frac{1+\sqrt{5}}{2\sqrt{3}}\mathrm{AC}$.

Hence $\mathrm{EA} = \sqrt{\mathrm{AC}^2 - \mathrm{CE}^2} = \sqrt{\mathrm{AC}^2 - \left(\frac{1+\sqrt{5}}{2\sqrt{3}}\right)^2 \mathrm{AC}^2} =$ $\mathrm{AC}\sqrt{1 - \frac{3+\sqrt{5}}{6}} = \mathrm{AC}\sqrt{\frac{3-\sqrt{5}}{6}}$. Then because the chord of an arc is a mean proportional between its versed sine and the diameter, and AE is the versed sine of the arc whose chord is AC, and its diameter equal to that of the circumscribed sphere;

5 times the cube of the linear ſide; and the product will be the ſolidity.

That is, $5A^3\sqrt{\frac{47+21\sqrt5}{40}}$ is the ſolidity.

EX-

ſphere; we ſhall have $AC^2 \div 2AE = AC^2 \div 2AC\sqrt{\frac{3-\sqrt5}{6}}$ $= \frac{1}{2}AC\sqrt{\frac{6}{3-\sqrt5}} = \frac{1}{2}AC\sqrt{6\times\frac{3+\sqrt5}{9-5}} = \frac{1}{2}AC\sqrt{3\times\frac{6+2\sqrt5}{4}}$ $= AC\times\frac{1+\sqrt5}{4}\sqrt3 = \frac{\sqrt3+\sqrt{15}}{4}AC = R$ the radius of the circumſcribed ſphere.

Again, the angle AFC contains 72 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt5}$; and the angle ACF 54 degrees, whoſe ſine is $\frac{1+\sqrt5}{4}$; therefore as $\sqrt{10+2\sqrt5} : 1+\sqrt5 :: AC : AF =$ $\frac{1+\sqrt5}{\sqrt{(10+2\sqrt5)}}AC = AC\sqrt{\frac{5+\sqrt5}{10}}$.

But, ſince the radius of the circumſcribed ſphere is the hypotenuſe of a right-angled triangle, whoſe two legs are AF and the radius of the inſcribed ſphere, we ſhall have $\sqrt{R^2-AF^2} =$ $\sqrt{\left(\frac{\sqrt3+\sqrt{15}}{4}AC\right)^2 - \frac{5+\sqrt5}{10}AC^2} = AC\sqrt{\frac{18+6\sqrt5}{16}-\frac{5+\sqrt5}{10}}$ $= AC\sqrt{\frac{25+11\sqrt5}{40}} = r$ the radius of the inſcribed ſphere.

Then, becauſe the ſolidity of any regular ſolid is equal to the ſurface drawn into $\frac{1}{3}$ of the radius of the inſcribed ſphere, and the ſurface B equal to $15A^2\sqrt{\frac{5+2\sqrt5}{5}}$ by the laſt problem, we ſhall have $B\times\frac{1}{3}r = 15A^2\sqrt{\frac{5+2\sqrt5}{5}} \times \frac{1}{3}A\sqrt{\frac{25+11\sqrt5}{40}}$ $= 5A^3\sqrt{\frac{47+21\sqrt5}{40}} = C$ the ſolidity. *Q.E.D.*

Corollary.

EXAMPLE.

If the linear side be 1, required the solidity of the dodecaedron.

Here $A = 1$, and therefore $5A^3\sqrt{\frac{47+21\sqrt{5}}{40}} = 5\sqrt{\frac{47+21\sqrt{5}}{40}} = 7\cdot66311189605$ is the content.

PRO-

Corollary. From the demonstration it appears that the radius of the circumscribed sphere R is $= \frac{\sqrt{3}+\sqrt{15}}{4}A$, and that of the inscribed sphere r is $= A\sqrt{\frac{25+11\sqrt{5}}{40}}$.

SCHOLIUM.

Putting, as before, A, B, C for the linear side, surface, and solidity, and R, r for the radius of the circumscribed and inscribed sphere respectively; we shall have, in the dodecaedron,

1. $A = \sqrt{\frac{B\sqrt{5-2\sqrt{5}}}{15}} = \sqrt[3]{\frac{C\sqrt{470-210\sqrt{5}}}{5}} = \frac{\sqrt{15}-\sqrt{3}}{3}R = r\sqrt{50-22\sqrt{5}}$.

2. $B = 15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} = 3\sqrt[3]{10C^2\sqrt{130-58\sqrt{5}}} = 10R^2\sqrt{\frac{10-2\sqrt{5}}{5}} = 30r^2\sqrt{130-58\sqrt{5}}$.

3. $C = 5A^3\sqrt{\frac{47+21\sqrt{5}}{40}} = \frac{B}{30}\sqrt{\frac{B\sqrt{650+290\sqrt{5}}}{6}} = \frac{20R^3}{3}\sqrt{\frac{3+\sqrt{5}}{30}} = 10r^3\sqrt{130-58\sqrt{5}}$.

4. $R = \frac{\sqrt{15}+\sqrt{3}}{4}A = \sqrt{\frac{B\sqrt{10+2\sqrt{5}}}{40}} = \sqrt[3]{\frac{3C\sqrt{90-30\sqrt{5}}}{40}} = r\sqrt{15-6\sqrt{5}}$.

5. $r = \frac{A\sqrt{250+110\sqrt{5}}}{20} = \frac{1}{20}\sqrt{\frac{2B\sqrt{650+290\sqrt{5}}}{3}} = \sqrt[3]{\frac{C\sqrt{650+290\sqrt{5}}}{200}} = R\sqrt{\frac{5+2\sqrt{5}}{15}}$.

PROBLEM IX.

To find the Surface of an Icosaedron.

Multiply 5 times the square of the linear side by the root of 3, and the product will be the surface.

That is, $5A^2\sqrt{3}$ is the surface.*

EXAMPLE.

The linear side being 1; required the surface.

Here $5A^2\sqrt{3} = 5\sqrt{3} = 8{\cdot}66025403$.

PROBLEM X.

To find the Solidity of an Icosaedron.

To 7 add 3 times the root of 5, divide the sum by 2, multiply the root of the quotient by five-sixths of the cube of the linear side, and the product will be the solidity of the icosaedron.

That is, $\frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}}$ is the content.†

S EX-

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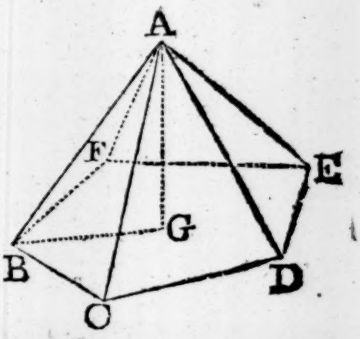
* DEMONSTRATION.

For one face is $= \frac{1}{4}A^2\sqrt{3}$, and therefore twenty faces, or the whole surface, will be $20 \times \frac{1}{4}A^2\sqrt{3} = 5A^2\sqrt{3}$. *Q. E. D.*

† DEMONSTRATION.



Let A be a solid angle, of the icosaedron, formed by five faces or triangles whose bases form the pentagon BCDEF, upon whose center G let fall the perpendicular AG, and draw BG.

In the demonstration of problem 8 it was found that BG is $= AB\sqrt{\frac{5+\sqrt{5}}{10}}$, and that the radius ρ of the circle circumscr. one face ABC is $= AB\sqrt{\frac{1}{3}}$. But the radius of the circumscr. sphere is

EXAMPLE.

The linear ſide of an icoſaedron being 1, required the ſolidity.

Here $A = 1$, and therefore $\frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{5}{6}\sqrt{\frac{7+3\sqrt{5}}{2}} = 2{\cdot}1816949905$ is the content.

PROBLEM XI.

To find the Superficies or Solidity of any Regular Body.

RULES.

1. Multiply the proper tabular area (taken from the following table) by the ſquare of the linear edge of the ſolid, for the ſuperficies.

2. Multiply

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is $R = \frac{AB^2}{2\,AG} = \frac{AB^2}{2\sqrt{AB^2 - BG^2}} = \frac{AB^2}{2\sqrt{AB^2 - \frac{5+\sqrt{5}}{10}AB^2}}$

$= \frac{AB}{2\sqrt{1 - \frac{5+\sqrt{5}}{10}}} = \frac{AB}{2\sqrt{\frac{5-\sqrt{5}}{10}}} = \frac{1}{2}AB\sqrt{\frac{10}{5-\sqrt{5}}} =$

$\frac{1}{2}AB\sqrt{\frac{10}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}}} = \frac{1}{2}AB\sqrt{\frac{10\times(5+\sqrt{5})}{25-5}} = AB\sqrt{\frac{5+\sqrt{5}}{8}}.$

And R is the hypotenuſe of a right-angled triangle, of which the one leg is the radius ρ of the circle circumſcribing the face, and the other the radius r of the inſcribed ſphere; therefore r is $= \sqrt{R^2 - \rho^2} = \sqrt{\frac{5+\sqrt{5}}{8}AB^2 - \frac{1}{3}AB^2} = AB\sqrt{\frac{15+3\sqrt{5}-8}{8\times 3}} = AB\sqrt{\frac{7+3\sqrt{5}}{24}}.$

Conſequently the ſolidity $C = \frac{1}{3}rB$, B being the whole ſurface, will be $\frac{1}{3}AB\sqrt{\frac{7+3\sqrt{5}}{24}} \times 5AB^2\sqrt{3} = \frac{5}{3}AB^3\sqrt{\frac{7+3\sqrt{5}}{8}} = \frac{5}{6}AB^3\sqrt{\frac{7+3\sqrt{5}}{2}}$. *Q.E.D.*

SCHO-

2. Multiply the tabular solidity by the cube of the linear edge, for the solid content.*

Surfaces and Solidities of Regular Bodies.

N° of Sides	Name	Surface	Solidity
4	Tetraedron	1·7320508	0·1178513
6	Hexaedron	6·0000000	1·0000000
8	Octaedron	3·4641016	0·4714045
12	Dodecaedron	20·6457788	7·6631189
20	Icosaedron	8·6602540	2·1816950

SCHOLIUM.

From this problem and the last may be deduced the following equations in the icosaedron, the letters being as before.

1. $A = \sqrt{\frac{B\sqrt{3}}{15}} = \sqrt[3]{\frac{6}{5}C\sqrt{\frac{7-3\sqrt{5}}{2}}} = R\sqrt{\frac{10-2\sqrt{5}}{5}}$
$= r\sqrt{42-18\sqrt{5}}$.

2. $B = 5A^2\sqrt{3} = 3\sqrt[3]{70\sqrt{3}-30\sqrt{15}} = 2R^2 \times (5\sqrt{3}-\sqrt{15})$
$= 3r^2 \times (7\sqrt{3}-3\sqrt{15})$.

3. $C = \frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{B}{18}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{10}B}$
$= \frac{2}{3}R^3\sqrt{10+2\sqrt{5}} = 10r^3 \times (7\sqrt{3}-3\sqrt{15})$.

4. $R = \frac{1}{2}A\sqrt{\frac{5+\sqrt{5}}{2}} = \frac{1}{2}\sqrt{\frac{5\sqrt{3}+\sqrt{15}}{30}B} = \sqrt[3]{\frac{3}{4}C\sqrt{\frac{5-\sqrt{5}}{10}}}$
$= r\sqrt{15-6\sqrt{5}}$.

5. $r = \frac{1}{2}A\sqrt{\frac{7+3\sqrt{5}}{6}} = \frac{1}{6}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{10}B}$
$= \frac{1}{2}\sqrt[3]{\frac{7\sqrt{3}+3\sqrt{15}}{30}C} = R\sqrt{\frac{5+2\sqrt{5}}{15}}$.

* DEMONSTRATION.

The numbers in the above table denote the surface and solidity of each body, when its edge is 1; and because, in similar bodies, the surfaces are as the squares of the linear edges, and the solidities as the cubes of the same; therefore the truth of the rules is manifest.

SECTION III.

OF SOLID RINGS.

DEFINITION.

BY a ring, in general, is meant a ſolid returning into itſelf; of which every ſection perpendicular to the axe, or line paſſing through the middle of the ſolid, is every-where the ſame figure, and of the ſame magnitude.

PROBLEM I.

To find the Surface of a Solid Ring.

Multiply the axe by the perimeter of a ſection perpendicular to it, and the product will be the ſurface.

EXAMPLE.

A workman having made for a jeweller a circular ring, or a ring whoſe axe forms the circumference of a circle; it is required to find the expence of the gilding at a penny the ſquare inch, the thickneſs of the ring, or the diameter of a ſection of it, being 2 inches, and the inner diameter, acroſs from ſide to ſide, 18 inches.

Here $18 + 2 = 20 =$ the diameter of the circle formed by the axe; conſequently $20 \times 3{\cdot}14159$ $=$ the length of the axe. But $2 \times 3{\cdot}14159 =$ the circumference of a ſection of it. Therefore $20 \times 3{\cdot}14159 \times 2 \times 3{\cdot}14159 = 40 \times 3{\cdot}14159^2 =$ $394{\cdot}785$ ſquare inches, nearly, $= 394{\cdot}785$ pence $=$ £ 1 12 $10\frac{3}{4}$ nearly, the expence required.

PRO-

PROBLEM II.

To find the Solidity of a Ring.

Multiply the axe by a section perpendicular to t, and theprod uct will be the solidity.

EXAMPLE.

Required the price of a ring of iron, whose dimensions are the same with that in the example to the last problem, at 4 pence a pound; a cubic inch of iron weighing 4·423 ounces averdupois.

Here the area of a section being $2^2 \times \cdot 785398 = 3\cdot 14159$, which is the same number as that expressing half the circumference, and the axe being the same as before, it is evident that the solidity will be expressed by half the surface in the last example, viz. the solidity $= 197\cdot 3925$ cubic inches; which multiplied by 4·423 give 873·065 ounces = 54·56657 pounds; which, at 4 pence each, come to £0 18 $2\frac{1}{4}$, the price required.

SCHOLIUM.

I omit any more examples, as the manner of operation is the same in all forms, with those for prisms, both with regard to the surfaces and solidities; for, since it is evident that any ring is equal to a prism whose altitude and end are respectively equal to the axe and section of the ring, both in surface and solidity, the rules for them both must be the same; and for this reason also any demonstration of the rules for rings was quite unnecessary in this place.

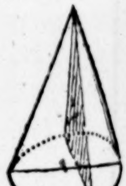
SECTION IV.

OF CONIC SECTIONS, AND OF THE FIGURES ARISING FROM, OR DEPENDING ON, THEM.

DEFINITIONS.

1. CONIC Sections are the plane figures formed by cutting a cone.

According to the different positions of the cutting plane there will arise five different figures or sections.



2. If the cutting plane pass through the vertex, and any part of the base, the section will be a triangle.

3. If the cone be cut parallel to the base, the section will be a circle.

4. The section is called an ellipsis, when the cone is cut obliquely through both sides.

5. The

5. The section is a parabola, when the cone is cut parallel to one of its sides.

6. The section is an hyperbola, when the cutting plane meets the opposite cone continued above the vertex, where it will make another section or hyperbola.

7. The vertices of any section, are the points where the cutting plane meets the opposite sides of the cone.

8. The transverse axis is the line between the two vertices. And the middle point of the transverse, is the center of the conic section.

9. The conjugate axis, is a line drawn through the center, and perpendicular to the transverse.

10. A diameter is any right line drawn through the center, and terminated on each side by the curve; the intersections of the diameter and curve being the

vertices of the diameter.—Hence every diameter of the ellipſe and hyperbola have two vertices; but of the parabola, only one; unleſs we conſider the other as infinitely diſtant. And hence, alſo, all the diameters of a parabola are parallel to each other, and infinite.

11. If a tangent to the curve be drawn through the vertex of any diameter, and another diameter be drawn parallel to the tangent, thoſe diameters are ſaid to be conjugates the one to the other.—The axes, or principal conjugate diameters, are perpendicular to each other.

12. An abſciſs is any part of a diameter, terminated at the vertex.

13. An ordinate to any diameter, is a line contained between the diameter and the curve, and is parallel to the conjugate diameter, or to the tangent at the vertex.—The ordinates to the axe are perpendicular to it. And in the ellipſe and hyperbola, every ordinate hath two abſciſſes, in the parabola only one.

14. The parameter of any diameter, is a third proportional to that diameter and its conjugate.

15. The focus is the point of interſection of the axe and an ordinate, to it, which is equal to half the parameter of the axe.—The ellipſe and hyperbola have each two foci, the parabola only one.

16. A ſpheroid, or ellipſoid, is a ſolid generated by the revolution of an ellipſe about one of its axes. It is a prolate one, when the revolution is made about the tranſverſe axis; and oblate, when about the conjugate.

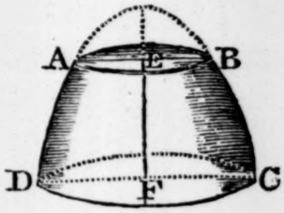
17. A

17. A conoid is a solid formed by the revolution of a parabola, or hyperbola, about the axis. And is accordingly called parabolic, or hyperbolic.—The parabolic conoid is also called a paraboloid; and the hyperbolic conoid, an hyperboloid.

18. A spindle is formed by any of the three sections revolving about a double ordinate, like the circular spindle.

19. A segment, of any of these figures, is a part cut off at the top, by a plane parallel to the base.

20. And a frustum is the part left next the base, after the segment is cut off.



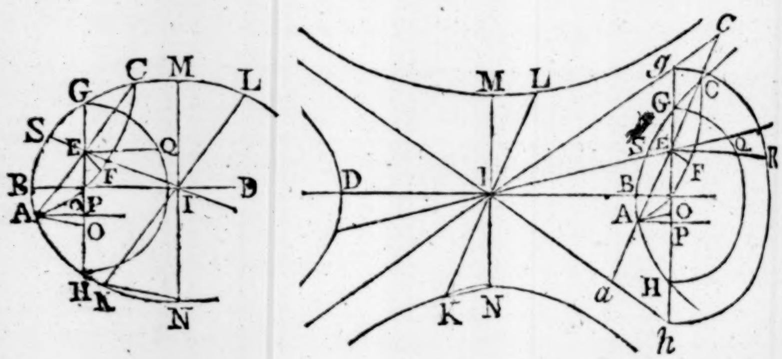
PROPOSITION I.

If any solid, formed by the revolution of a conic section about its axe, i. e. a spheroid, paraboloid, or hyperboloid, be cut by a plane in any position; the section will be some conic section, and all the parallel sections will be like and similar figures.

DEMONSTRATION.

Let ABC be the generating section, or a section of the given solid through its axe BD, and perpendicular to the proposed section AFC, their common intersection

interfection being AC; let GH be any other line meeting the generating fection in G and H, and cutting AC in E; and erect EF perpendicular to the plane ABC, and meeting the propofed plane in F.



Then, if AC and GH be conceived to be moved continually parallel to themfelves, will the rectangle $AE \times EC$ be to the rectangle $GE \times EH$, always in a conftant ratio; but if GH be perpendicular to BD, the points G, F, H will be in the circumference of a circle whofe diameter is GH, fo that $GE \times EH$ will be $= EF^2$; therefore $AE \times EC$ will be to EF^2, always in a conftant ratio; confequently AFC is a conic fection, and every fection parallel to AFC will be of the fame kind with, and fimilar to, it. *Q. E. D.*

Corol. 1. The above conftant ratio, in which $AE \times EC$ is to EF^2, is that of KI^2 to IN^2, the fquares of the diameters of the generating fection refpectively parallel to AC, GH; that is, the ratio of the fquare of the diameter parallel to the fection, to the fquare of the revolving axe of the generating plane.

This will appear by conceiving AC and GH to be moved into the pofitions KL, MN, interfecting in I, the center of the generating fection.

Corol.

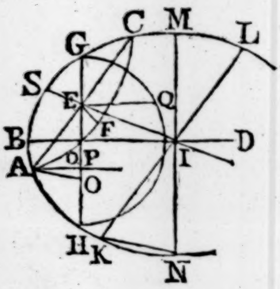
Corol. 2. And hence it appears that the axes AC and 2EF of the section, supposing E now to be the middle of AC, will be to each other, as the diameter KL is to the diameter MN of the generating section.

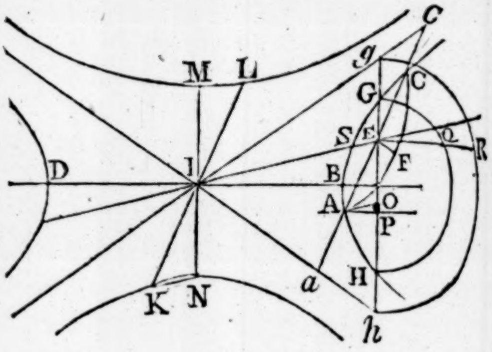
Corol. 3. If the section of the solid be made so as to return into itself, it will evidently be an ellipse. Which always happens in the spheroid, except when it is perpendicular to the axe; which position is also to be excepted in the other solids, the section being always then a circle: in the paraboloid the section is always an ellipse, excepting when it is parallel to the axe: and in the hyperboloid the section is always an ellipse, when its axe makes with the axe of the solid, an angle greater than that made by the said axe of the solid and the asymptote of the generating hyperbola; the section being an hyperbola in all other cases, but when those angles are equal, and then it is a parabola.

Corol. 4. But if the section be parallel to the fixed axe BD, it will be of the same kind with, and similar to, the generating plane ABC; that is, the section parallel to the axe, in a spheroid, is an ellipse similar to the generating ellipse; in the paraboloid, the section is a parabola similar to the generating parabola; and in an hyperboloid, it is an hyperbola similar to the generating hyperbola of the solid.

Corol. 5. In the spheroid, the section through the axe is the greatest of the parallel sections; but in the hyperboloid, it is the least; and in the paraboloid, all the sections parallel to the axe, are equal to one another.

For the axe is the greatest parallel chord line in the ellipse, but the least in the opposite hyperbolas, and all the diameters are equal in a parabola.





Corol. 6. If the extremities of the diameters KL, MN, be joined by the line KN, and AO be drawn parallel to KN, and meeting GEH in O, E being the middle of AC, or AE the ſemi-axe, and GH parallel to MN. Then EO will be equal to EF, the other ſemi-axe of the ſection.

For, by ſimilar triangles, KI : IN :: AE : EO.

Or upon GH as a diameter deſcribe a circle meeting EQ, perpendicular to GH, in Q; and it is evident that EQ will be equal to the ſemi-diameter EF.

Corol. 7. Draw AP parallel to the axe BD of the ſolid, and meeting the perpendicular GH in P. And it will be evident that, in the ſpheroid, the ſemi-axe EF = EO will be greater than EP; but in the hyperboloid, the ſemi-axe EF = EO, of the elliptic ſection, will be leſs than EP; and in the paraboloid, EF = EO is always equal to EP.

SCHOLIUM.

The analogy of the ſections of an hyperboloid to thoſe of the cone, are very remarkable, all the three conic ſections being formed by cutting an hyperboloid in the ſame poſitions as the cone is cut.

Thus,

Thus, let an hyperbola and its asymptote be revolved together about the transverse axe, the former describing an hyperboloid, and the latter a cone circumscribing it; then let them be supposed to be both cut by a plane in any position, and the two sections will be like, similar, and concentric figures: that is, if the plane cut both the sides of each, the sections will be concentric, similar ellipses; if the cutting plane be parallel to the asymptote, or to the side of the cone, the sections will be parabolas; and in all other positions, the sections will be similar and concentric hyperbolas.

That the sections are like figures, appears from the foregoing corollaries. That they are concentric, will be evident when we consider that Cc is = Aa, producing AC both ways to meet the asymptotes in a and c. And that they are similar, or have their transverse and conjugate axes proportional to each other, will appear thus: Produce GH both ways to meet the asymptotes in g and h; and on the diameters GH, gh, describe the semi-circles GQH, gRh, meeting EQR, drawn perpendicular to GH, in Q and R; EQ and ER being then evidently the semi-conjugate axes, and EC, Ec, the semi-transverse axes of the sections. Now if GH and AC be conceived to be moved parallel to themselves, AE × EC or CE^2 will be to GE × EH or EQ^2, in a constant ratio, or CE to EQ will be a constant ratio; and since cE is as Eg, and aE as Eh, aE × Ec or cE^2 will be to gE × Eh or ER^2, in a constant ratio, or cE to ER will be a constant ratio; but at an infinite distance from the vertex, C and c coincide, or EC = Ec, as also EG = Eg, consequently EQ = ER, and then CE to EQ will be = cE to ER; but as these ratios are constant, if they be equal to each other in one place, they must be always so; and consequently CE : Ec :: QE : ER.

And

And this analogy of the sections will not seem strange, when we consider that a cone is a species of the hyperboloid; or a triangle a species of the hyperbola, whose axes are infinitely little.

PROPOSITION II.

If SI be the semi-diameter belonging to the double ordinate AEC of the generating plane, AEC being the diameter of the section AFC, conceived to be moved continually parallel to itself; and if x denote any part of the diameter SI, intercepted by E the middle of AC, and any given fixed point taken in SI; then will the section AFC be always as $a + bx + cxx$; a, b, c, being constant quantities; b in some cases affirmative, and in others negative; c being affirmative in the hyperbola, negative in the ellipse, and nothing in the parabola; and a may always be supposed to denote the distance of the given fixed point from the vertex S.

DEMONSTRATION.

In any conic section, AC^2 is as $a + bx + cxx$; but all the parallel sections are like and similar figures, and similar plane figures are as the squares of their like dimensions; therefore the section AFC is as AC^2, that is, as $a + bx + cxx$. *Q. E. D*

Corollary. If the given fixed point, where x begins, coincide with the vertex S, then will a be equal to nothing, and the section will be as $bx \pm cxx$, or as $x \pm dxx$, in the hyperbola and ellipse; and as bx, or as x, in the parabola.

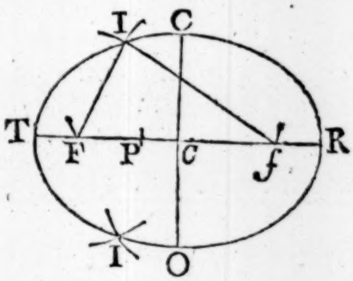
SEC-

SECTION V.

OF THE ELLIPSE, AND THE FIGURES GENERATED FROM IT.

PROBLEM I.

*To Describe an Ellipse.**



DRAW the transverse TR, and its conjugate CO, bisecting each other perpendicularly in the center *c*.

With the radius T*c* and center C, describe an arc cutting TR in the points F, *f*; which will be the two foci of the ellipse.

Take any point P in the transverse; then with the radii TP, PR, and centers F, *f*, describe two arcs intersecting in I; so will the point I be in the curve or circumference of the ellipse.

And thus, by assuming several points P in the transverse, there will be found as many points in the

* The truth of this construction will appear, by observing that the transverse axe is equal to the sum of two lines drawn from the foci to meet in any point in the curve.

the curve as we pleaſe. Through all which let the curve line be drawn.

Otherwiſe.

Having found the foci F, f, take a thread of the length of the tranſverſe, and faſten its ends with pins in the foci; then ſtretch the thread F I f and it will reach to I in the curve: and by moving a pencil round within the thread, keeping it always ſtretched, it will trace out the curve.

PROBLEM II.

In an Ellipſe, to find any two Conjugate Diameters, an Ordinate to one of them, and its Abſciſs, one from another; viz. having any Three of them given, to find the Fourth.

CASE I.

* *To find the Ordinate.*

As any diameter,
Is to its conjugate;
So is the mean proportional between the abſciſſes,
To the ordinate.

That is, $d : c :: \sqrt{x(d-x)} : \frac{c}{d}\sqrt{x(d-x)} = y$; putting d for the diameter, c its conjugate, and x an abſciſs to the ordinate y.

EX-

* The values of the ſeveral quantities in the four caſes of this problem, are found from the general equation expreſſing the property of the curve, viz. $dd : cc :: x(d-x) : yy$.

EXAMPLE.

If the tranſverſe be 35, the conjugate 25, and the abſciſs 7; what is the ordinate.

Here $\frac{c}{d}\sqrt{x \times (d-x)} = \frac{25}{35}\sqrt{7 \times 28} = \frac{5}{7}\sqrt{7^3 \times 2^2}$ $= \frac{5 \times 7 \times 2}{7} = 10$ the ordinate.

CASE II.

To find the Abſciſſes.

As the conjugate,
Is to the diameter;
So is the ſquare root of the difference of the ſquares of the ordinate and ſemi-conjugate,
To the diſtance between the ordinate and center.

Then that diſtance being added to, and ſubtracted from, the ſemi-diameter, will give the two abſciſſes.

That is, $x = \frac{1}{2}d \pm \frac{d}{c}\sqrt{\frac{1}{4}cc - yy}$.

EXAMPLE.

What are the two abſciſſes of the ordinate 10, the diameters being 35 and 25?

Here $\frac{1}{2}d \pm \frac{d}{c}\sqrt{\frac{1}{4}cc - yy} = \frac{35}{2} \pm \frac{35}{25}\sqrt{(\frac{25}{2})^2 - 10^2} =$ $\frac{35}{2} \pm \frac{7}{2}\sqrt{5^2 - 4^2} = \frac{35 \pm 21}{2} = 28$ and 7, the two abſciſſes.

CASE III.

To find the Conjugate.

As the mean proportional between the abſciſſes,
Is to the ordinate;
So is the diameter,
To its conjugate.

That is, as $\sqrt{x(d-x)} : y :: d : c = \frac{dy}{\sqrt{dx-xx}}$.

EXAMPLE.

What is the conjugate to the diameter 35; the abscifs to an ordinate of 10 being 7?

Here $\frac{dy}{\sqrt{(d-x)\times x}} = \frac{35\times 10}{\sqrt{28\times 7}} = \frac{350}{14} = 25$ the conjugate.

CASE IV.

To find the Diameter.

To or from the femi-conjugate, according as the lefs or greater abfcifs is ufed, add or fubtract the root of the difference of the fquares of the ordinate and femi-conjugate: Then, as the fquare of the ordinate, is to the product of the conjugate and abfcifs; fo is the fum or difference above found, to the diameter.

That is, $d = \frac{cx}{yy} \times (\frac{1}{2}c \mp \sqrt{\frac{1}{4}cc - yy})$.

EXAMPLE.

If an ordinate and its lefs abfcifs be 10 and 7; what is the diameter, fuppofing the conjugate to be 25?

Here $d = \frac{cx}{yy} \times (\frac{1}{2}c + \sqrt{\frac{1}{4}cc - yy}) = \frac{25\times 7}{10\times 10} \times (\frac{25}{2} + \sqrt{(\frac{25}{2})^2 - 10^2}) = \frac{7}{4} \times \frac{25+15}{2} = 7 \times 5 = 35$ the tranfverfe.

PRO-

PROBLEM III.

To find the Circumference of an Ellipse.

* RULE I.

Multiply the circumference of the circumscribing circle by the sum of the series

$$1 - \frac{d}{2.2} - \frac{3dd}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6} - \frac{3.3.5.5.7d^4}{2.2.4.4.6.6.8.8} \text{ \&c},$$

and the product will be the periphery of the ellipse sought; where d is the difference between an unit and the square of the quotient of the less axe divided by the greater.

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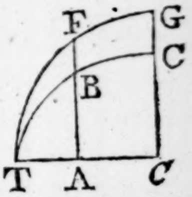
* DEMONSTRATION.



Let a be the semi-transverse axe TC,
c the semi-conjugate cC,
y the ordinate AB,
x its distance cA from the center,
z the arc BC.

Then, since by the nature of the curve, y is $= \frac{c}{a}\sqrt{aa - xx}$, $\dot{y}$ will be $= \frac{-cx\dot{x}}{a\sqrt{aa-xx}}$; and consequently $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}\sqrt{a^2 - \frac{a^2 - c^2}{aa} \times x^2}}{\sqrt{aa-xx}} = \frac{\dot{x}\sqrt{aa - dxx}}{\sqrt{aa-xx}}$, by writing d for $\frac{aa-cc}{aa}$ or $1 - \frac{cc}{aa}$, $= \frac{a\dot{x}\sqrt{1 - \frac{dxx}{aa}}}{\sqrt{aa-xx}} = \frac{a\dot{x}}{\sqrt{aa-xx}}$ $\times \left(1 - \frac{dx^2}{2a^2} - \frac{d^2x^4}{2.4a^4} - \frac{3d^3x^6}{2.4.6a^6}\right.$ &c) by throwing the numerator into a series.

EXAMPLE.

Required the periphery of an ellipse whose axes are 24 and 18.

Here $1 - (\frac{18}{24})^2 = 1 - (\frac{3}{4})^2 = \frac{7}{16} = \cdot 4375 = d.$

Then

But the fluent of $\frac{a\dot{x}}{\sqrt{aa-xx}}$ is $=$ the corresponding circular arc FG described with the center c and radius CT, which call A: then by page 66 *Cotesii Harmonia Mensurarum*, the fluent comes out $A - \frac{d}{2a^2}B - \frac{d^2}{2.4a^4}C - \frac{3d^3}{2.4.6a^6}D$ &c for the length of the arc BC; where $B = \frac{aaA - x\sqrt{aa-xx}}{2}$, $C = \frac{3aaB - x^3\sqrt{aa-xx}}{4}$, $D = \frac{5aaC - x^5\sqrt{aa-xx}}{6}$, &c.

But when A arrives at T, x is $= a$, and $\sqrt{aa-xx} = 0$; hence the values of the quantities B, C, D, &c, become barely

$$B = \frac{aa}{2}A, \quad C = \frac{3aa}{4}B = \frac{3a^4}{2.4}A, \quad D = \frac{5aa}{6}C = \frac{3.5a^6}{2.4.6}A, \text{ \&c.}$$

And consequently the series above becomes

$A \times (1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6}$ &c) for $\frac{1}{4}$ of the elliptic periphery; where A is $\frac{1}{4}$ of the periphery of the circle, or $=$ the whole circumference of the ellipse when A is that of the circle. *Q. E. D.*

Corol. 1. Hence the peripheries of similar ellipses are to each other as those of their circumscribed circles, or as any two similar diameters of those ellipses.

Corol. 2. If a be the less semi-axe, and c the greater, the fluxion of the arc will be $\dot{z} = \dot{x}\sqrt{\frac{aa+dxx}{aa-xx}}$; and if $a = 1$, and $c = \sqrt{2}$, the same fluxion $\dot{z}$ will be

$$= \dot{x}\sqrt{\frac{1+xx}{1-xx}} = \frac{1+xx}{\sqrt{1-x^4}}\dot{x} = \frac{\dot{x}}{\sqrt{1-x^4}} + \frac{x^2\dot{x}}{\sqrt{1-x^4}}.$$

And if we write v for xx, we shall have $\dot{z} = \frac{\dot{v}}{2\sqrt{v}}\sqrt{\frac{1+v}{1-v}}$, as Mr. *Landen* found in page 142 of his *Math. Lucub.*

3 Or,

Then the 2d term A $= \frac{d}{4} = \cdot 10938$

the 3d B $= \frac{3d}{4.4}$ A $= \cdot 00897$

the 4th C $= \frac{3.5d}{6.6}$ B $= \cdot 00164$

the 5th D $= \frac{5.7d}{8.8}$ C $= \cdot 00039$

the 6th E $= \frac{7.9d}{10.10}$ D $= \cdot 00011$

the 7th F $= \frac{9.11d}{12.12}$ E $= \cdot 00003$

the 8th G $= \frac{11.13d}{14.14}$ F $= \cdot 00001$

their sum is ·12053,
which taken from the first term 1·00000,
leaves ·87947

for the value of the series; and being drawn into 3·1416 × 24, the periphery of the circumscribing circle, gives 66·31056 for the curve of the ellipse required.

* RULE II.

Multiply the sum of the two axes by 1·5706, or the half of 3·1416, and the product will be the circumference nearly.

 That

Or, by writing w^n for x, we shall have universally $\dot{z} = nw^{n-1}\dot{w}\sqrt{\frac{1 + w^{2n}}{1 - w^{2n}}}$. Which theorems are of use in determining some fluents.

* DEMONSTRATION.

It will be evident that this rule is very near the truth, if it be considered that this arithmetical mean between the axes, exceeds

That is, $(t + c) \times \frac{1}{2}p =$ the circumference nearly; putting t for the tranſverſe, c for its conjugate, and p for 3·1416.

EXAMPLE.

Let there be here taken the ſame example as before, in which the axes are 24 and 18.

Then $(24 + 18) \times 1{\cdot}5706 = 42 \times 1{\cdot}5706 =$ 65·9736 is the circumference nearly.

RULE

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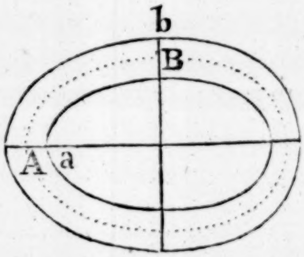
exceeds their geometrical mean; and that the geometrical mean (as will be proved hereafter) is the diameter of a circle equal in area to the ellipſe, which circle is of a leſs ambit than the ellipſe, or any other figure, of the ſame area.

Otherwiſe.

Let the dotted curve line repreſent the elliptic curve whoſe length is required. And let curve lines be deſcribed at ſmall and equal diſtances, within it, and without it, as in the annexed figure. Then theſe two curves will be very nearly ellipſes alſo; and the difference of their areas, computed as ellipſes, will be nearly the area of the elliptic ring, or ſpace contained between them, through the middle of which runs the ellipſe whoſe length is ſought; which area being alſo equal to the ſaid length of the ellipſe drawn into the breadth of the ſpace, or diſtance between the other two curves, it follows that the length of the elliptic curve will be nearly equal to the ſaid difference of areas divided by the ſaid breadth or diſtance. Therefore



let $t =$ the tranſverſe of the ellipſe,

$c =$ its conjugate,

$a = 3{\cdot}1416$, and d the ſmall ſemi-diſtance Aa or Bb.

Then $(t + 2d) \times (c + 2d) \times \frac{1}{4}a =$ area of the greateſt curve.

And $(t - 2d) \times (c - 2d) \times \frac{1}{4}a =$ area of the leaſt.

The dif. $(t + c) \times ad =$ the annular ſpace.

Theref. $(t + c) \times \frac{1}{2}a =$ the length, nearly.

*RULE III.

Multiply the square root of $\frac{1}{2}$ the sum of the squares of the two axes by 3·1416, and the product will be the circumference nearly.

That is, $p\sqrt{\frac{tt+cc}{2}}$ = the circumference nearly.

EXAMPLE.

Taking again the same example,

We have $\sqrt{\frac{24^2+18^2}{2}} \times 3{\cdot}1416 = 15\sqrt{2} \times 3{\cdot}1416 = 21{\cdot}2132 \times 3{\cdot}1416 = 66{\cdot}6433$ = the circumference nearly.

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* DEMONSTRATION.

Call the infinite series in rule 1, s;

$$\text{viz. } s = 1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \&c.$$

Then since $\sqrt{1-\frac{1}{2}d}$ is $= 1 - \frac{d}{2.2} - \frac{d^2}{2^3.4} - \frac{3d^3}{2^4.4.6}$ &c.

we shall have $s - \sqrt{1-\frac{1}{2}d} = \quad - \frac{d^2}{64} - \frac{3d^3}{256}$ &c.

Therefore, rejecting this last remaining series, on account of its smallness, we have $s = \sqrt{1-\frac{1}{2}d}$ nearly; and consequently the circumference of the ellipse $C = c\sqrt{1-\frac{1}{2}d}$ nearly, c being the circumference of the circumscribed circle.

Or $C = pt\sqrt{1-\frac{1}{2}d} = pt\sqrt{1-\frac{1}{2}+\frac{cc}{2tt}} = pt\sqrt{\frac{1}{2}+\frac{cc}{2tt}} = p\sqrt{\frac{tt+cc}{2}}$, which is the 3d rule, p being $= 3{\cdot}14159$ &c.

* RULE IV.

Find the ſquare root of $\frac{1}{2}$ the ſum of the ſquares of the axes as in the laſt rule, and call it A.

To 3 times the greater axe add its parameter, divide the ſum by 4, and call the quotient B.

Then multiply the difference between B and 3 times A, by 1·5708, and the product will be the periphery nearer than by the 3d rule.

That is, $\frac{1}{2}p \times (3\sqrt{\frac{tt + cc}{2}} - \frac{3t + p}{4}) =$ the periphery, p being the parameter.

EXAMPLE.

Taking the ſame example as in the former rules,

We have $A = \sqrt{\frac{24^2 + 18^2}{2}} = 15\sqrt{2} = 21{\cdot}2132$.

As 24 : 18 :: 18 : $13\frac{1}{2}$ = the parameter, by the definition.

And

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* Again, $s - 1 + \frac{1}{4}d = -\frac{3d^2}{64} - \frac{5d^3}{256}$ &c;

but $3s - 3\sqrt{1 - \frac{1}{2}d} = -\frac{3d^2}{64} - \frac{9d^3}{256}$ &c;

theref. $2s - 3\sqrt{1 - \frac{1}{2}d} + 1 - \frac{1}{4}d = -\frac{d^3}{64}$ &c.

hence $s = \frac{3\sqrt{1 - \frac{1}{2}d} - 1 + \frac{1}{4}d}{2}$ nearly,

and ſo $c = \frac{1}{2}pt \times (3\sqrt{1 - \frac{1}{2}d} - \frac{4 - d}{4}) =$

$\frac{1}{2}p \times (3\sqrt{\frac{tt + cc}{2}} - \frac{3tt + cc}{4t}) = \frac{1}{2}p \times (3\sqrt{\frac{tt + cc}{2}} - \frac{3t + p}{4})$

nearly, where p is the parameter of the axe. And this is the 4th rule.

And $\frac{3 \times 24 + 13\frac{1}{2}}{4} = 21\frac{3}{8} = 21 \cdot 375 = \text{B}$.

Then $(3\text{A} - \text{B}) \times 1 \cdot 5708 = 42 \cdot 2646 \times 1 \cdot 5708$ $= 66 \cdot 3892$ the circumference nearly.

RULE V.*

Find A and B as in the last rule.

Divide the difference of the squares of the axes by the square of the greater axe, multiply the square of the quotient by $\frac{1}{16}$ of the greater axe, and call the result C.

Then from 5 times A take 3 times B, to the remainder add C, multiply the sum by 1·5708, and the product will give the circumference of the ellipse still nearer.

Or,

From 5 times A, subtract $\frac{1}{16}$ of the sum of 35 times the greater axe and 14 times its parameter, to the remainder add $\frac{1}{16}$ of the product arising from the multiplication of the said parameter, by the quotient of the square of the less axe divided by that of the greater, and multiply the sum by 1·5708.

That

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* Farther, $\text{s} - 1 + \frac{d}{4} + \frac{3d^2}{64} = - \frac{5d^3}{256}$ &c,

$$\text{or } \frac{4}{5}\text{s} - \frac{4}{5} + \frac{d}{5} + \frac{3d^2}{80} = - \frac{d^3}{64},$$

which taken from the last approximation, and reduced, gives

$$\text{s} = \tfrac{1}{2} \times 5\sqrt{1 - \tfrac{1}{2}d} - 3 \times \frac{4 - d}{4} + \frac{dd}{16};$$

and consequently $\text{C} = \frac{1}{2}pt \times (5\sqrt{1 - \frac{1}{2}d} - 3 \times \frac{4 - d}{4} + \frac{dd}{16})$

$$= \tfrac{1}{2}p \times (5\sqrt{\frac{tt + cc}{2}} - 3 \times \frac{3t + \text{P}}{4} + (\frac{tt - cc}{tt})^2 \times \frac{t}{16})$$

$= \frac{1}{2}p \times (5\sqrt{\frac{tt + cc}{2}} - \frac{35t + 14\text{P}}{16} + \frac{\text{P}cc}{16tt})$. Which is the 5th rule.

And thus we may proceed to any degree of accuracy whatever.

That is, $\frac{1}{2}p \times [5\sqrt{\frac{tt+cc}{2}} - 3 \times \frac{3t+p}{4} + (\frac{tt-cc}{tt})^2 \times \frac{t}{16}]$, or $\frac{1}{2}p \times [5\sqrt{\frac{tt+cc}{2}} - \frac{35t+14p}{16} + \frac{pcc}{16tt}]$ = the periphery very nearly.

EXAMPLE.

Taking again the ſame example as before, we have

$A = 21{\cdot}2132$,
$B = 21{\cdot}375$,
$C = {\cdot}2871$.

Then $(5A - 3B + C) \times 1{\cdot}5708 = 42{\cdot}2281 \times 1{\cdot}5708 = 66{\cdot}3319$ = the circumference very near.

Or, to uſe the latter part of the rule,

$(5A - \frac{35 \times 24 - 14 \times 13\frac{1}{2}}{16} + \frac{13\frac{1}{2} \times 18^2}{16 \times 24^2}) \times 1{\cdot}5708 = 42{\cdot}2281 \times 1{\cdot}5708 = 66{\cdot}3319$, the ſame as before.

RULE VI.

Take $\frac{1}{2}$ the ſum of the quantities in the ſecond and third rules, and it will be the circumference very nearly.*

That is, $\frac{1}{2}p \times (\frac{t+c}{2} + \sqrt{\frac{tt+cc}{2}})$ = the periphery.

EXAMPLE.

Taking, ſtill, the ſame example,

The number found by the 2d rule is 65·9736,
And by the 3d rule is 66·6433,

their ſum is 132·6169,
the half of which is 66·3084,

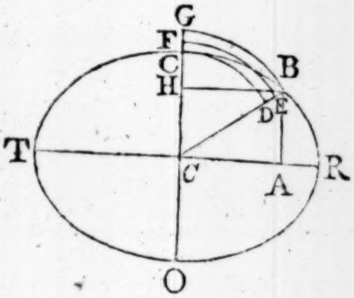
for the circumference, and is nearer to the true number found by rule 1 than any of the other approximations.

PRO-

* For the one being nearly as much too great as the other is too little, $\frac{1}{2}$ their ſum muſt be near the truth.

PROBLEM IV.

To find the Length of any Arc BC *of an Ellipse.*



* RULE I.

If a denote the semi-transverse Tc,
c the semi-conjugate cC,
and x the distance cA of the ordinate AB from the center,
then will the arc CB be $= x \times$
$(1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$ &c).

And the series will be the same if x be considered as an ordinate to either axe $2c$, the other axe being $2a$, the arc beginning at the vertex of the axe $2c$, and terminated by the ordinate x.

EX-

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* DEMONSTRATION.

By proceeding as in the investigation of problem 3, we have the fluxion of the curve $\dot{z} = \dot{x}\sqrt{\frac{aa - dxx}{aa - xx}} = \dot{x} \times$
$(1 + \frac{c^2}{2a^4}x^2 + \frac{4a^2c^2 - c^4}{8a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{16a^{12}}x^6$ &c),
restoring c and exterminating d; and the fluent is
$z = x \times (1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$ &c).

Corol.

EXAMPLE.

If the two axes be 24 and 18, and the distance *c*A 3; required the length of the arc BC.

Here $a = 12$, $c = 9$, and $x = 3$.

Then the 1st term $=$ — — 1·00000

2d term A $= \frac{c^2}{6a^4}x^2 =$ — — 0·00586

3d term B $= 3 \times \frac{4a^2 - c^2}{20a^4}x^2 \text{A} =$ 0·00019

4th term C $= \frac{5}{14a^4} \times \frac{8a^4 - 4a^2c^2 + c^4}{4a^2 - c^2}x^2 \text{B} =$ 0·00001

the sum 1·00606
mult. by 3

gives the arc BC 3·01818

* RULE II.

Find the length of the circular arc FE intercepted by *c*C, *c*B, and whose radius is half the sum of the lines

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Corol. 1. When x is $= a$, the rule becomes

$$a \times (1 + \frac{c^2}{6a^2} + \frac{4a^2c^2 - c^4}{40a^4} + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^6} \ \&\text{c})$$

for one fourth of the periphery of the ellipse. *Q. E. D.*

Corol. 2. By supposing $a = c$, the ellipse will become a circle, and the general series for the elliptic arc will become for the circular arc

$$x \times (1 + \frac{x^2}{6a^2} + \frac{3x^4}{40a^4} + \frac{5x^6}{112a^6} \ \&\text{c}), =$$

$$x \times (1 + \frac{x^2}{2.3a^2} + \frac{3x^4}{2.4.5a^4} + \frac{3.5x^6}{2.4.6.7a^6} \ \&\text{c}),$$

the same as was determined at page 88.

* For the circular arc FE is equal to the arithmetical mean between the circular arcs BG, CD, described with the radii *c*B, *c*C;

lines cc, cB; and it will be equal to the elliptic arc BC nearly.

EXAMPLE.

Taking here the same example,

We have AR $= 9$, and AT $= 15$.

Hence $Tc^2 : cc^2 :: TA \times AR : AB^2 = \frac{9^2 \times 9 \times 15}{12 \times 12} = \frac{9 \times 9 \times 15}{16}$,

and $cB = \sqrt{cA^2 + AB^2} = \sqrt{9 + \frac{9 \times 9 \times 15}{16}} = \frac{3}{4}\sqrt{151}$ $= 9{\cdot}21616$.

Then $\frac{1}{2}cc + \frac{1}{2}cB = 9{\cdot}10808 =$ the radius of the circular arc EF.

But HB $\div$ $cB = 3 \div 9{\cdot}21616 = {\cdot}325515$ is the sine of the angle ccB or arc FE to the radius 1, answering to $18{\cdot}9968$ degrees.

Therefore, by rule 1 prob. 6 sect. 1 part 2,

${\cdot}01745 \times 18{\cdot}9968 \times 9{\cdot}10808 = 3{\cdot}0192$ is the circular arc EF, or elliptic arc BC, nearly.

* RULE III.

Divide the difference of the squares of the semi-axes, by the square of the greater, and call the quotient q.

Then

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cc; to which it is evident the elliptic arc must be nearly equal; and so much the nearer, the less the arc itself is.

This rule will also be the more exact, the nearer the axes of the ellipse approach towards an equality.

Rule 2 of the last problem is a particular case of this general rule.

* DEMONSTRATION.

By the first rule the arc A is equal to

$$x \times (1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6 \text{ \&c}).$$

But

Then divide the difference between the ſquare of the greater ſemi-axe, and the product of $\frac{1}{3}$ of the ſquare of the diſtance *c*A of the ordinate from the center multiplied by *q*, by the difference between the ſquare of the ſaid greater ſemi-axe and $\frac{1}{3}$ of the ſquare of the ſaid diſtance; and the root of the quotient multiplied by the ſaid diſtance, will be the arc nearly.

That is, $x\sqrt{\frac{aa - \frac{1}{3}qxx}{aa - \frac{1}{3}xx}}$ = the arc;
where a = T*c* the greater ſemi-axe,
x = *c*A, and $q = \frac{aa - cc}{aa} = 1 - \frac{cc}{aa}$,
c being the leſs ſemi-axe *c*C.

EXAMPLE.

Taking again the ſame example,

We have $1 - \frac{cc}{aa} = 1 - (\frac{9}{12})^2 = 1 - (\frac{3}{4})^2 = 1 - \frac{9}{16} = \frac{7}{16} = q$.

And $x\sqrt{\frac{aa - \frac{1}{3}qxx}{aa - \frac{1}{3}xx}} = 3\sqrt{\frac{12^2 - \frac{1}{3} \times \frac{7}{16} \times 3^2}{12^2 - \frac{1}{3} \times 3^2}} = 3{\cdot}017898$ = the arc required nearly.

RULE

But in the inveſtigation of that rule it appears that $\sqrt{\frac{aa - dxx}{aa - xx}}$ is $= 1 + \frac{c^2}{2a^4}x^2 + \frac{4a^2c^2 - c^4}{8a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{16a^{12}}x^6$ &c; hence it is evident that $x\sqrt{\frac{aa - \frac{1}{3}dxx}{aa - \frac{1}{3}xx}}$ is $= x \times (1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{72a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{432a^{12}}x^6$ &c).

And conſequently A $= x\sqrt{\frac{aa - \frac{1}{3}dxx}{aa - \frac{1}{3}xx}}$ nearly. *Q. E. D.*

Corollary. When the two axes are equal to each other, the ellipſe will become a circle, *d* will be = o, and the rule will become $x\sqrt{\frac{aa}{aa - \frac{1}{3}xx}}$ for the circular arc.

* RULE IV.

Call the quantity found by the last rule B.

Multiply the distance cA by the less semi-axe, divide the product by the square of the greater, to $\frac{1}{6}$ of the square of the quotient add 1, multiply the sum by the distance cA, and call the product C.

Then from 9 times B take 4 times C, and $\frac{1}{5}$ of the remainder will be the arc very near.

That is,

$$A = \frac{9B \div 4C}{5} = \frac{1}{5} \times \left[9x\sqrt{\frac{aa - \frac{1}{5}qxx}{aa - \frac{1}{3}xx}} - \left(1 + \frac{c^2x^2}{6a^4}\right) \times 4x\right].$$

EXAMPLE.

Taking still the same example,

The Quantity found by the last rule is 3·017898 = B.

And

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* DEMONSTRATION.

For since $A = x \times \left(1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6 \&c\right)$,

and $B = x \times \left(1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{72a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{432a^{12}}x^6 \&c\right)$;

We have $A - B = x \times \left(\frac{4a^2c^2 - c^4}{90a^8}x^4 + \frac{40a^4c^2 - 20a^2c^4 + 5c^6}{756a^{12}} \&c\right)$.

But

$\frac{4}{5}\left[B - \left(1 + \frac{c^2}{6a^4}x^2\right)x\right] = x \times \left(\frac{4a^2c^2 - c^4}{90a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{540a^{12}}x^6 \&c\right)$.

Theref. $A - B - \frac{4}{5}\left[B - \left(1 + \frac{c^2x^2}{6a^4}\right)x\right] = x \times \left(\frac{8a^4c^2 - 4a^2c^4 + c^6}{210a^{12}}x^6 \&c\right)$.

And $A = \frac{1}{5}\left[9B - \left(1 + \frac{c^2x^2}{6a^4}\right) \times 4x\right] =$

$\frac{1}{5}x \times \left[9\sqrt{\frac{aa - \frac{1}{3}dxx}{aa - \frac{1}{3}xx}} - \left(1 + \frac{c^2x^2}{6a^4}\right)\right]$ nearly. *Q. E. D.*

Corollary. When $a = c$, the rule becomes

$$\frac{1}{5}x > 9\sqrt{\frac{aa}{aa - \frac{1}{3}xx}} - 4 \times \frac{6aa + xx}{6aa})$$

for the circular arc.

And $x \times (1 + \frac{c^2x^2}{6a^4}) = 3 \times (1 + \frac{9^2 \times 3^2}{6 \times 12^4}) =$
$3 \times (1 + \frac{3}{2 \times 4^4}) = 3{\cdot}017578 = \text{C}$.

Then $\frac{9\text{B} - 4\text{C}}{5} = \frac{27{\cdot}161082 - 12{\cdot}070212}{5} = \frac{15{\cdot}09077}{5} =$
$3{\cdot}018154 =$ the arc very nearly.

RULE V.

As the ſum of 15 times the parameter of the axe, at whoſe vertex the arc begins, and a third proportional to the ſaid axe, the abſciſs, and the difference between 9 times the ſaid axe and 21 times its parameter; is to the ſum of 15 times the parameter, and a third proportional to the axe, the abſciſs, and the difference between 19 times the axe and 21 times its parameter; ſo is the ordinate, to the length of the arc, very nearly.*

That

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* DEMONSTRATION.

Since y is $= c - \frac{c\sqrt{aa - xx}}{a} = \frac{cx^2}{2a^2} + \frac{cx^4}{8a^4}$ &c, it is evident that a fraction of this form $x \times \frac{\text{A} + (\text{B} + 1)y}{\text{A} + \text{B}y}$, when expanded in a ſeries, will produce a ſeries of the ſame form with that expreſſing the length of the arc in the firſt rule, and which being put equal to it, will afford equations for determining the values of A and B.

Thus $x \times \frac{\text{A} + (\text{B} + 1)y}{\text{A} + \text{B}y}$ is $= x \times \frac{\text{A} + (\text{B} + 1) \times (\frac{cx^2}{2a^2} + \frac{cx^4}{8a^4} \&c)}{\text{A} + \text{B} \times (\frac{cx^2}{2a^2} + \frac{cx^4}{8a^4} \&c)}$

$= x \times (1 + \frac{cx^2}{2\text{A}a^2} + \frac{\text{A}c - 2\text{B}c^2}{8\text{A}^2a^4}x^4$ &c), which put

$= x \times (1 + \frac{c^2x^2}{6a^4} + \frac{4a^2c^2 - c^4}{8a^8}x^4$ &c).

Then

That is, $\dfrac{15p + \dfrac{19\text{C} - 21p}{\text{C}} \times y}{15p + \dfrac{9\text{C} - 21p}{\text{C}} \times y} \times x =$ the arc CB,

where $\text{C} =$ the whole axe CO where the arc begins,
$p =$ its parameter $= \text{TR}^2 \div \text{CO}$,
$y =$ the abscifs CH,
$x =$ the ordinate HB.

EXAMPLE.

Taking, ftill, the fame example,
In which $c = 9$, $a = 12$, $\text{C} = 18$, $p = 12 \times 24 \div 9 = 32$, and $x = 3$.

U Hence

Then, by equating the correfponding terms, we obtain $\dfrac{c}{2\text{A}a^2} = \dfrac{c^2}{6a^4}$, or $\text{A} = \dfrac{3a^2}{c} = \frac{3}{2}p$; and $\dfrac{\text{A}c - 2\text{B}c^2}{8\text{A}^2a^4} = \dfrac{4a^2c^2 - c^4}{8a^8}$,

hence $\text{B} = \dfrac{9c^2 - 21a^2}{10cc} = \dfrac{9c - \dfrac{21aa}{c}}{10c} = \dfrac{9\text{C} - 21p}{10\text{C}}$,

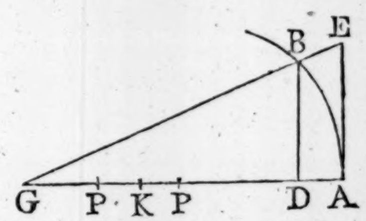
putting $\text{C} = 2c$, and $p =$ its parameter $\dfrac{2aa}{c}$.

Confequently

$$x \times \frac{\text{A}+(\text{B}+1)y}{\text{A}+\text{B}y} \text{ is} = x \times \frac{\frac{3}{2}p + \dfrac{19\text{C}-21p}{10\text{C}}y}{\frac{3}{2}p + \dfrac{9\text{C}-21p}{10\text{C}}y} = x \times \frac{15p+\dfrac{19\text{C}-21p}{\text{C}}y}{15p+\dfrac{9\text{C}-21p}{\text{C}}y}$$

Q. E. D.

Corol. 1. Hence in the ellipfe AB, whofe vertex is A, either axe AK, and latus rectum AP; if there be drawn the ordinate BD, and GP be taken $= \frac{1}{2}\text{PA} + \dfrac{19\text{AK} - 21\text{AP}}{10\text{AK}} \times \text{AD}$, and there be drawn the right line GBE meeting the tangent AE in E; then will AE be = the arc AB, very nearly.



For,

Hence $y = c - \frac{c\sqrt{aa - xx}}{a} = 9 - \frac{9\sqrt{12^2 - 3^2}}{12}$ $= 9 - \frac{9\sqrt{15}}{4} = \cdot 2857874715$.

Then

$$x \times \frac{15p + \frac{19c - 21p}{c}y}{15p + \frac{9c - 21p}{c}y} = 3 \times \frac{15 \times 32 + \frac{19 \times 18 - 21 \times 32}{18}y}{15 \times 32 + \frac{9 \times 18 - 21 \times 32}{18}y}$$

$= 3 \times 1\cdot006056 = 3\cdot018168 =$ the length of the arc very nearly.

PRO-

For, by similar triangles, GD : DB :: GA : AE $= \frac{AG \times BD}{DG} =$

$$\frac{AP + PG}{AP + PG - AD} \times DB = (\text{by the conft.}) \frac{\frac{3}{2}AP + \frac{19AK - 21AP}{10AK} \times AD}{\frac{3}{2}AP + \frac{9AK - 21AP}{10AK} \times AD} \times DB;$$

which is the arc, nearly, by the above demonftration.

And this proves the truth of the conftruction given by *Sir I. Newton*, in his letter of June 13, 1676, to Mr. *Oldenburg*. But Mr. *Jones*, in publifhing this letter, has printed AP inftead of AD.

Or the fame rule may be otherwife expreffed, like rule 3 for the hyperbolic arc.

Corol. 2. When the ellipfe becomes a circle, then AK = AP, and the above rule will become $\frac{15AP - 2AD}{15AP - 12AD} \times DB$ for the length of the circular arc, whofe diameter is AP, verfed fine AD, and right fine DB. And this is nearer the truth than any of the other approximations, before given for the circular arc, except that in rule 6 for that purpofe; for the example in that rule, calculated by this approximation, gives 6·1170598 for the length of the arc. And thus, by making AP = AK, all the rules for the ellipfe will be adapted to the circle.

Corol. 3. The fame rules for the ellipfe may alfo be eafily adapted to the other two conic fections, the hyperbola and parabola; namely, by only changing the fign of the axe C or AK for the hyperbola, and by making that axe infinite for the parabola.

PROBLEM V.

*To find the Area of an Ellipse.**

RULE I.

Multiply continually together the two axes and the number ·7854, for the area of the ellipse.

That is, $ntc =$ the area; putting $t =$ the transverse, $c =$ the conjugate, and $n = \cdot 7854$. By corollary 4.

U 2 EX-

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* GENERAL INVESTIGATION.

Let AB, DE, be any two conjugate diameters of the ellipse ADBEA, GH a double ordinate to the diameter AB, AKBL a circle whose diameter is AB, KL a double ordinate of the circle to the diameter AB, and AI perpendicular to GH.

Now whilst KL, by flowing, generates the circular segment KAL, GH will describe the elliptic segment GAH; but the velocity of KL is to that of GH, as AF to AI; and by the property of all conjugate diameters, AC : CD :: ($\sqrt{AF \times FB} =$, by the nature of the circle,) KF : FG; therefore as the circular segment KAL is to the elliptic segment GAH, so is AC × AF to CD × AI, and so is AC × radius to CD × sine ∠AFI, and so is AC to CD × sine ∠C; putting 1 for the radius.

Corol. 1. The whole ellipse and circle, described upon any diameter of it, are to each other as the corresponding segments cut off them by their particular double ordinates GH, KL, passing through the same point of that diameter.

Corol. 2. As radius is to the sine of the angle made by any two conjugate diameters, so is a mean proportional between the two circles described upon those diameters, to the ellipse,

For $\left\{\begin{array}{l} r \times d : s \times c :: \odot d : \bigcirc \\ r \times c : s \times d :: \odot c : \bigcirc \end{array}\right\}$ by cor. 1.

Hence $r^2 cd : s^2 cd :: r^2 : s^2 :: \odot d \times \odot c : \bigcirc^2$,

And $r : s :: \sqrt{\odot d \times \odot c} : \bigcirc$.

Where d and c are the diameters, s the sine of the angle made by them to the radius r, $\odot$ denotes a circle, and $\bigcirc$ an ellipse.

Corol.

EXAMPLE.

If the axes of an ellipſe be 35 and 25, what is the area?

$\cdot 7854 \times 35 \times 25 = 687\cdot 225 =$ the area required.

RULE II.

Multiply continually together any two conjugate diameters, the natural ſine of their included angle, and the number $\cdot 7854$.

That is, $dcns =$ the area;
putting d and c = any two conjugate diameters, s = ſine of their included angle, and $n = \cdot 7854$.

EX-

Corol. 3. If the conjugate diameters be equal to each other, it will follow that, As radius is to the ſine of the angle made by the equal conjugate diameters; ſo is the circle deſcribed on one of thoſe diameters, to the ellipſe.

Corol. 4. The ellipſe is a mean proportional between the two circles deſcribed on the two axes.—For they make with each other a right angle, whoſe ſine is = to the radius.

Corol. 5. As radius is to the ſine of the angle made by any two conjugate diameters; ſo is the circle whoſe diameter is a mean proportional between the conjugate diameters, to the ellipſe.—This follows from corollary 2.

Corol. 6. The ellipſe is equal to a circle whoſe diameter is a mean proportional between the two axes.——From corollary 4.

Corol. 7. As an ellipſe : is to the rectangle of its two axes, or to the rectangle of any two conjugate diameters drawn into the ſine of their included angle, the radius being 1 :: ſo is any circle : to the ſquare of its diameter.——Any two like ſegments or zones of the ellipſe and circle are alſo in the ſame proportion.

Corol. 8. Ellipſes, and their like ſegments, are to one another as the rectangles of their axes, or as the rectangles of any conjugate diameters forming the ſame angle in each.

Corol. 9. Similar ellipſes are to one another as the ſquares of their like diameters.

Corol.

EXAMPLE.

If two conjugate diameters of an ellipse be 28 and 32, and their included angle 77° $34\frac{1}{4}'$; required its area.

The sine of 77° $34\frac{1}{4}'$ is ·9765625; therefore $\cdot9765625 \times 32 \times 28 \times \cdot7854 = 687\cdot225 =$ the area.

PROBLEM VI.

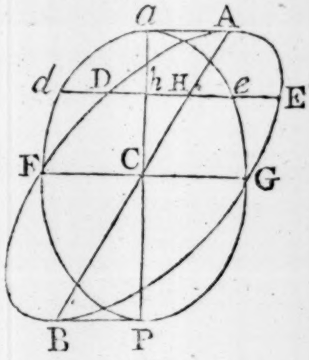
To find the Area of the Segment of an Ellipse cut off by a Double Ordinate to either Axe, that is, by a Line Perpendicular to that Axe.

RULE I.

Find the corresponding segment of the circle described upon the same axe to which the cutting line, or base of the segment, is perpendicular.

 Then

Corol. 10. From corollary 7 comes also the following construction.



Let ADE be an oblique segment of the ellipse AFBGA, cut off by a double ordinate to the diameter AB, FG being the conjugate. Through the center C draw aP perpendicular to FG and meeting Aa and BP, in a and P, both parallel to FG; then about the axes aP, FG, describe the ellipse $aFPGa$, meeting the ordinate produced in e and d. Then will the right elliptic segment dae be equal to the oblique segment DAE, as well as the whole ellipse $aFPGa$ equal to the ellipse AFBGA; moreover their corresponding ordinates de, DE, parallel to the common diameter FG, are every where equal, as are the like parts or zones contained between any two of such ordinates. And the same may be said of all ellipses contained between the parallels Aa, BP, infinitely produced: in which property they resemble parallelograms of the same base and between the same parallels.

Then as this axe : is to the other axe : : ſo is the circular ſegment : to the elliptic ſegment.

This follows from corollary 7 to the laſt problem.

RULE II.

Find the tabular circular ſegment whoſe verſed ſine or height is equal to the quotient of the height of the elliptic ſegment divided by its axe. Then multiply continually together this ſegment and the two axes of the ellipſe, for the area of the ſegment required.

This rule follows from the former.

EXAMPLE.

What is the area of an elliptic ſegment cut off by a line parallel to, and at the diſtance of, $7\frac{1}{2}$ from the leſs axe, the axes being 35 and 25?

Here $17\frac{1}{2} - 7\frac{1}{2} = 10 =$ the height of the ſegment.

And $10 \div 35 = 2 \div 7 = \cdot 2857\frac{1}{7} =$ the tabular verſed ſine; whoſe ſegment is $\cdot 1851669$.

Then $\cdot 1851669 \times 35 \times 25 = 162\cdot 0210375 =$ the area of the leſs ſegment.

If the greater ſegment had been required,

Then $\cdot 78539816 - \cdot 1851669 = \cdot 60023126$.

And $\cdot 60023126 \times 25 \times 35 = 525\cdot 2023525 =$ the area of the greater ſegment.

EXAMPLE II.

What is the area of the elliptic ſegment cut off by a double ordinate perpendicular to the conjugate axe at the diſtance of $7\frac{1}{2}$ from the center, the axes being 35 and 25?

Here $12\frac{1}{2} - 7\frac{1}{2} = 5 =$ the altitude of the ſegment.

And $5 \div 25 = 1 \div 5 = \cdot 2 =$ the tabular verſed ſine; whoſe correſponding ſegment is $\cdot 1118238$.

Hence $\cdot 1118238 \times 25 \times 35 = 97\cdot 845825 =$ the area of the leſs ſegment.

Again,

Again, $\cdot 78539816 - \cdot 1118238 = \cdot 67357436$.
And $\cdot 67357436 \times 25 \times 35 = 589\cdot 377565$
= the greater ſegment.

PROBLEM VII.

To find the Area of an Elliptic Segment cut off by a Double Ordinate to any Diameter; that is, by a Line Oblique to the Axes.

Divide the abſciſs AF (fig. to prob. 5) of the double ordinate, by its diameter AB, and find the tabular circular ſegment whoſe verſed ſine is the quotient.—Then multiply continually together the tabular area and the two axes. Or the tabular area, the diameter AB to which the baſe of the ſegment is a double ordinate, its conjugate diameter GH, and the ſine of their included angle, for the area of the elliptic ſegment required.*

EXAMPLE.

The axes of an ellipſe being 35 and 25, it is required to find the area of a ſegment whoſe baſe is a double ordinate to a diameter whoſe length is 33, it being divided by the double ordinate into the two abſciſſes 7 and 26.

Here $FA \div AB = 7 \div 33 = \cdot 2121\frac{7}{33}$ = the tabular verſed ſine; to which correſponds the area $\cdot 12162869$.

 Hence

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* DEMONSTRATION.

For, by cor. 7 prob. 5, as AB^2 : to circular ſegment LAK :: ſo is $AB \times DE \times s. \angle C$ = rectangle of the two axes : to elliptic ſegment GAH; but circular ſegment $LAK = AB^2 \times$ tabular circular ſegment s whoſe verſed ſine is $FA \div AB$; therefore $AB^2 : AB^2 \times s :: 1 : s ::$ rectangle of the axes to elliptic ſegment. *Q.E.D.*

Hence $\cdot 12162869 \times 25 \times 35 = 106\cdot 425104 =$ the ſegment GAH.

Moreover, $\cdot 78539816 - \cdot 12162869 = \cdot 66376947$. And $\cdot 66376947 \times 25 \times 35 = 580\cdot 79828625 =$ the greater ſegment GBH.

PROBLEM VIII.

To find the Trilineal Area ABQ, *included by either Axe, a Line drawn from any Point in it, and their Intercepted Arc.*

Draw the ordinate DB meeting the circle deſcribed upon the ſaid axe in C, and draw AC, OC.*

Then

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* DEMONSTRATION.

For, by the inveſtigation and corollaries to prob. 5,

As the axe A : to the axe a :: DC : DB

:: circular ſegment DCQ : elliptic ſegment DBQ

:: (becauſe of the common baſe AD) triangle ACD : triangle ABD

:: (by compoſition) the trilineal ACQ : trilineal ABQ. *Q. E. D.*

Corol. 1. Since, by rule 2 prob. 6 ſect. 1 part 2, putting OQ $= r$, and DC $= y$,

the circ. arc QC is $= y \times (1 + \frac{y^2}{3\cdot 2r^2} + \frac{3y^4}{5\cdot 2\cdot 4r^4} + \frac{3\cdot 5y^6}{7\cdot 2\cdot 4\cdot 6r^6}$ &c),

we have the ſect. OCQ $= \frac{1}{2}ry \times (1 + \frac{y^2}{3\cdot 2r^2} + \frac{3y^4}{5\cdot 2\cdot 4r^4} + \frac{3\cdot 5y^6}{7\cdot 2\cdot 4\cdot 6r^6}$ &c);

which being increaſed or diminiſhed by $\frac{1}{2}$OA $\times$ DC = the triangle ACO, we ſhall have

$$\tfrac{1}{2}y \times (\text{AQ} + \frac{y^2}{3\cdot 2r} + \frac{3y^4}{5\cdot 2\cdot 4r^3} + \frac{3\cdot 5y^6}{7\cdot 2\cdot 4\cdot 6r^5} \text{ \&c}),$$

for the general value of the trilineal ACQ; and conſequently

$$\frac{cy}{2r} \times (\text{AQ} + \frac{y^2}{3\cdot 2r} + \frac{3y^4}{5\cdot 2\cdot 4r^3} + \frac{3\cdot 5y^6}{7\cdot 2\cdot 4\cdot 6r^5} \text{ \&c})$$

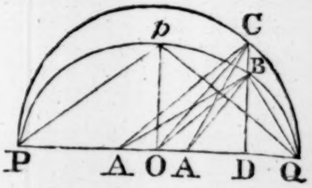
for the elliptic trilineal ABQ; r being the radius of the circle, or ſemi-axe OQ of the ellipſe, and c the other ſemi-axe.

And ſince $r : c :: y :$ BD $= z$, we ſhall have $y = \frac{rz}{c}$, and conſequently the value of the elliptic trilineal ABQ, expreſſed in terms of its own ordinate z or BD, and ſemi-axes r, c, will be

$$\tfrac{1}{2}z \times (\text{AQ} + \frac{rz^2}{3\cdot 2c^2} + \frac{3rz^4}{5\cdot 2\cdot 4c^4} + \frac{3\cdot 5rz^6}{7\cdot 2\cdot 4\cdot 6c^6} \text{ \&c}).$$

Corol.

Then, As the said axe :
Is to the other axe : :
So the circ. trilineal ACQ :
To the elliptic trilin. ABQ.



Note. It is evident that the triangle ACO added to, or subtracted from, the circular sector OCQ, will give the circular trilineal ACQ, according as AQ is greater or less than half the axe.

Or the semi-segment CDQ increased or diminished by the triangle ACD, will give the same.

EX-

Corol. 2. If A coincide with O, the triangle AOB will vanish, and the trilineal become barely a sector OBQ, whose value will be $\frac{1}{2}rz \times (1 + \frac{z^2}{3\cdot2c^2} + \frac{3z^4}{5\cdot2\cdot4c^4} + \frac{5\cdot3z^6}{7\cdot2\cdot4\cdot6c^6}$ &c).

Corol. 3. When $z = c$, then the quadrant OpQ will be $\frac{1}{2}rc \times (1 + \frac{1}{3\cdot2} + \frac{3}{5\cdot2\cdot4} + \frac{5\cdot3}{7\cdot2\cdot4\cdot6}$ &c).

Corol. 4. If A coincide with P, then AQ $=$ QP $= 2r$, and the trilineal BPQ will be $= \frac{1}{2}rz \times (2 + \frac{z^2}{3\cdot2c^2} + \frac{3z^4}{5\cdot2\cdot4c^4} + \frac{5\cdot3z^6}{7\cdot2\cdot4\cdot6c^6}$ &c).

And when B coincides with p, then will the trilineal PpQ be $= \frac{1}{2}rc \times (2 + \frac{1}{3\cdot2} + \frac{3}{5\cdot2\cdot4} + \frac{5\cdot3}{7\cdot2\cdot4\cdot6}$ &c).

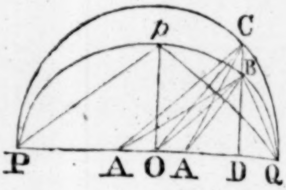
Corol. 5. If from the double of the quadrant in corollary 3, be taken the trilineal in corol. 4, there will remain the seg. PpB $=$ $r \times (\frac{c - z}{1} + \frac{2c^3 - z^3}{6\cdot2c^2} + 3\,\frac{2c^5 - z^5}{10\cdot2\cdot4c^4} + 5\cdot3\,\frac{2c^7 - z^7}{14\cdot2\cdot4\cdot6c^6}$ &c).

Corol. 6. If from the value of the sector OBQ in corollary 2, be taken that of the triangle OBQ $= \frac{1}{2}rz$, there will remain the segment BQ $= \frac{1}{2}rz \times (\frac{z^2}{3\cdot2c^2} + \frac{3z^4}{5\cdot2\cdot4c^4} + \frac{5\cdot3z}{7\cdot2\cdot4\cdot6c^6}$ &c).

Corol.

EXAMPLE.

Given the axes 35 and 25 of an ellipse, and an ordinate to the transverse 10; required the area included by the transverse and a line drawn from the top of the ordinate to cut the transverse in an angle of 40 degrees.



Here, as $20p = 25 :$ PQ $=$ $35 ::$ DB $= 10 :$ DC $= 14$.

As rad. : tang. $\angle$B $(50°) ::$ DB : DA $= 11{\cdot}917536$.

Hence AD $\times \frac{1}{2}$DC $=$ $83{\cdot}422752 =$ the $\triangle$ADC.

As

Corol. 7. When B coincides with p, the series in each of the two last corollaries becomes, for the value of each of the segments Pp, pQ, $\frac{1}{2}rc \times (\frac{1}{3\cdot2} + \frac{3}{5\cdot2\cdot4} + \frac{5\cdot3}{7\cdot2\cdot4\cdot6}$ &c).

Corol. 8. If A be the focus of the ellipse, then AQ $= r \pm \sqrt{rr - cc}$, and the trilineal

$$\text{ABQ} = \tfrac{1}{2}z \times (r \pm \sqrt{rr - cc} + \frac{rz^2}{3.2c^2} + \frac{3rz^4}{5.2.4c^4} + \frac{3.5rz^6}{7.2.4.6c^6} \ \&c),$$

$$= \tfrac{1}{2}rz \times (1 \pm \sqrt{1 - \frac{cc}{rr}} + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4} + \frac{3.5z^6}{2.4.6.7c^6} \ \&c).$$

Corol. 9. Putting $\text{T} = \text{ABQ} = \frac{1}{2}rz \times (a + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4}$ &c)

a being $= 1 \pm \sqrt{1 - \frac{cc}{rr}}$, and $q = \frac{2\text{T}}{r} = z \times (a + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4}$ &c) we shall have, by reversion,

$$z = \frac{q}{a} - \frac{q^3}{6c^2a^4} + \frac{10 - 9a}{120c^4a^7}q^5 - \frac{280 - 504a + 225a^2}{5040c^6a^{10}}q^7 \ \&c,$$

$$= \frac{2\text{T}}{ra} \times (1 - \frac{1}{6a^3}p^2 + \frac{10 - 9a}{120a^6}p^4 - \frac{280 - 504a + 225a^2}{5040a^9}p^6 \ \&c)$$

where $p = \frac{2\text{T}}{rc}$; which series generally converges very fast, and affords a good direct solution of *Kepler*'s problem.

Suppose, for example, it were required to find the true anomaly of the earth, and its distance from the sun, answering to the

As $25 : 35 :: \sqrt{12 \cdot 5^2 - 10^2} : 10 \cdot 5 = $ OD;

And hence OQ — OD $= 7 =$ DQ. Then, by the table of circular segments, the semi-segment CDQ will come out 68·4920775.

The triangle ADC + segment CDQ = 151·9143395 = the trilineal ACQ.

Theref. as $35 : 25 :: 151 \cdot 9143395 : 108 \cdot 5102425$ = the area of the elliptic trilineal ABQ required.

PRO-

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the mean anomaly of 1 sign or 30 degrees, the excentricity being ·01681 to the mean distance 1.

Here PO $= r = 1$, AO $= \cdot 01681$, $op = c = \sqrt{1 - \cdot 01681^2} =$ ·9998587, the area of a quadrant of the ellipse = ·785398 &c $\times rc = \cdot 7852872$; then, as 3 signs : 1 sign :: ·7852872 : ·2617624 = T; hence $\frac{2\text{T}}{r} = 2\text{T} = \cdot 5235248$, and $\frac{2\text{T}}{rc} = \frac{2\text{T}}{c} = \frac{2}{3} \times$ ·78539 &c = ·5235988 $= p$:

Also A $= 1 \cdot 01681$, and $\frac{p^2}{a^3} = 2589869$.

$$\text{Then, 1st term A} = \frac{1}{a} = \quad — \quad — \quad — \quad + \cdot 98346790$$

$$\text{2d} — \text{B} = \tfrac{1}{6}\text{A} \times \frac{p^2}{a^3} = \quad — \quad — \quad - \cdot 04245088$$

$$\text{3d} — \text{C} = \frac{10 - 9a}{20}\text{B} \times \frac{p^2}{a^3} = \quad — \quad + \cdot 00046655$$

$$\text{4th} — \text{D} = \frac{280 - 504a + 225a^2}{42}\text{C} \times \frac{p^2}{a^3} = - \cdot 00000044$$

$$\text{the sum} = \quad \cdot 94148313$$

which drawn into $\frac{2\text{T}}{r} = 2\text{T} = \cdot 5235248$, gives ·49288971 = DB $= z$.

But

PROBLEM IX.

To find the Surface of a Spheroid.

* RULE I.

For Both Spheroids.

Let f denote the fixed axe, or the axe about which the ellipse is conceived to revolve, r the revolving axe, $p = 3{\cdot}14159$, and $q = \frac{ff \backsim rr}{ff}$: Then

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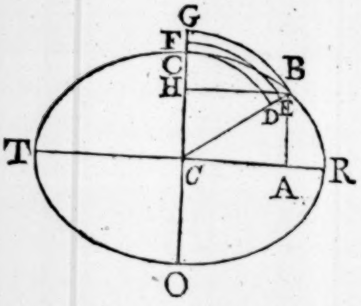
But OD $= \frac{r}{c}\sqrt{cc - zz} = {\cdot}87005233$; hence AD $=$ AO $+$ OD $= {\cdot}88686233$, and as AD : DB :: rad. : tang. $\angle$DAB $= 29{\cdot}063904^\circ = 29^\circ\ 3'\ 50''\ 3'''$ the true anomaly.

Also, as rad. : sec. $\angle$DAB :: AD : AB $= 1{\cdot}0146255$, the distance of the earth from the sun, to the mean distance 1.

* DEMONSTRATION.



Put the fixed semi-axe Tc $=$ cR $= a$, Cc $=$ cO $= b$, cA $= x$, AB $= y$, arc CB $= z$, and $3{\cdot}14159 = p$.

Then, $a : b :: \sqrt{aa - xx} : \frac{b}{a}\sqrt{aa - xx} = y$; hence $\dot{y} = -\frac{bx\dot{x}}{a\sqrt{aa - xx}}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{x}^2 + \frac{b^2x^2\dot{x}^2}{a^2(a^2 - x^2)}}$ $= \frac{\dot{x}}{a}\sqrt{\frac{a^4 - x^2(aa - bb)}{aa - xx}}$; and consequently the fluxion of the surface $\dot{s} = 2py\dot{z} = \frac{2pb\dot{x}}{aa}\sqrt{a^4 - x^2(aa - bb)}$

=

Then will the surface s of the spheroid be expressed by either of the following series; using the upper or under signs, according as it is an oblong or oblate spheroid; viz.

$$s = prf \times (1 \mp \frac{1}{2.3}q - \frac{1}{2.4.5}q^2 \mp \frac{3}{2.4.6.7}q^3 - \frac{3.5}{2.4.6.8.9}q^4 \text{ \&c}),$$

or

$$s = prf \times (1 \mp \frac{A}{2.3}q - \frac{3B}{4.5}q \mp \frac{3.5C}{6.7}q - \frac{5.7D}{8.9}q \text{ \&c}).$$

Where A, B, C, &c, are the several preceding terms.

E X-

$= \frac{2p\dot{b}x}{aa}\sqrt{a^4 - c^2x^2}$ (putting $cc = aa - bb$) $= \frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{cc} - xx}$

$= \frac{2pbc\dot{x}}{aa} \times (\frac{a^2}{c} - \frac{cx^2}{2a^2} - \frac{c^3x^4}{2.4a^5} - \frac{3c^5x^6}{2.4.6a^{10}}$ &c). And the fluent is

$$s = 2pbx \times (1 - \frac{c^2x^2}{2.3a^4} - \frac{c^4x^4}{2.4.5a^8} - \frac{3c^6x^6}{2.4.6.7a^{12}} - \frac{3.5c^8x^8}{2.4.6.8.9a^{16}} \text{ \&c}).$$

And if for $\frac{cc}{aa}$ be written dd, the series will become

$$s = 2pbx \times (1 - \frac{d^2x^2}{2.3a^2} - \frac{d^4x^4}{2.4.5a^4} - \frac{3d^6x^6}{2.4.6.7a^6} - \frac{3.5d^8x^8}{2.4.6.8.9a^8} \text{ \&c}).$$

and when $x = a$, it is

$$= 2pab \times (1 - \frac{d^2}{2.3} - \frac{d^4}{2.4.5} - \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9} \text{ \&c})$$

$$= 2pab \times (1 - \frac{A}{2.3}d^2 - \frac{3B}{4.5}d^2 - \frac{3.5C}{6.7}d^2 - \frac{5.7D}{8.9}d^2 \text{ \&c}).$$

where, when cc or $aa - bb$ is affirmative, viz. in the case of an oblong spheroid, all the signs after the first will be negative; but for an oblate spheroid, cc being negative, the signs after the first will become alternately *plus* and *minus*. *Q. E. D.*

Corol. 1. The series above will not always converge for an oblate spheroid; viz. when d is greater than 1, or when $\frac{bb}{aa}$ is greater than 2, the series will not converge.

Corol. 2. Hence the surfaces of similar spheroids, and also of their like parts, are to each other as the rectangles of their axes, or as the squares of their like dimensions.

EXAMPLE.

If the axes be 50 and 40, required the surface of each spheroid.

1. For the Oblong Spheroid.

Here $f = 50$, $r = 40$, and $\frac{ff - rr}{ff} = \frac{30^2}{50^2} = \frac{9}{25} = \cdot 36 = q$.

Hence 1st term $A = \text{—} \quad \text{—} \quad 1\cdot$

2d term $B = \frac{A}{2 \cdot 3} q = -0\cdot 06$

3d term $C = \frac{3B}{4 \cdot 5} q = -0\cdot 00324$

4th term $D = \frac{3 \cdot 5C}{6 \cdot 7} q = -0\cdot 0004166-$

5th term $E = \frac{5 \cdot 7D}{8 \cdot 9} q = -0\cdot 000072900$

6th term $F = \frac{7 \cdot 9E}{10 \cdot 11} q = -0\cdot 0000150+$

7th term $G = \frac{9 \cdot 11F}{12 \cdot 13} q = -0\cdot 0000035-$

8th term $H = \frac{11 \cdot 13G}{14 \cdot 15} q = -0\cdot 0000009-$

the sum of the negative terms $= -0\cdot 0637489$, which taken from the 1st term 1, leaves $\cdot 9362511$ for the sum of the series; and this being drawn into $prf = 3\cdot 14159265 \times 50 \times 40 = 6283\cdot 18531$, will produce $5882\cdot 6385$ for the surface required.

2. For the Oblate Spheroid.

Here $f = 40$, $r = 50$, and $\frac{rr - ff}{ff} = \frac{30^2}{40^2} = \frac{9}{16} = q$. Then by increasing the values of the terms, found above for the oblong spheroid, in proportion to such power of the ratio of the value of q in this case, to its value in the former, whose exponent is respectively

1 less than the number of each term; viz. in the proportion of the 0, 1, 2, 3, &c power of $\frac{9}{25}$ to $\frac{9}{16}$, or of 16 to 25, or of 64 to 100; we shall have the

1st term	$= 1 \times \frac{100^0}{64^0}$		$= 1 \times 1 =$	$+1\cdot$
2d term	$= \cdot06$	$\times$	$\frac{100}{64} =$	$+0\cdot09375$
3d term	$= \cdot00324$	$\times$	$\frac{100^2}{64^2} =$	$-0\cdot00791$
4th term	$= \cdot0004166$	$\times$	$\frac{100^3}{64^3} =$	$+0\cdot00159$
5th term	$= \cdot0000729$	$\times$	$\frac{100^4}{64^4} =$	$-0\cdot00044$
6th term	$= \cdot0000150$	$\times$	$\frac{100^5}{64^5} =$	$+0\cdot00014$
7th term	$= \cdot0000035$	$\times$	$\frac{100^6}{64^5} =$	$-0\cdot00005$
8th term	$= \cdot0000009$	$\times$	$\frac{100^7}{64^7} =$	$+0\cdot00002$
sum of the affirmative terms				$+1\cdot09550$
sum of the negative terms				$-0\cdot00840$
the difference				$1\cdot0871$;

then $6283\cdot18531 \times 1\cdot0871 = 6830\cdot4507 =$ the surface of the oblate spheroid required.

RULE II.

For Both Spheroids.

Divide the revolving axe by the fixed axe, and call the quotient q. Find the difference between 1 and the square of q, and call the square root of that difference s.

For an oblong spheroid, multiply $\cdot01745329$ by the degrees in the arc whose sine is s, and call the product p.

For

For an oblate ſpheroid, multiply 2·302585 by the logarithm of the ſum of s and q, and call the product P, alſo.

Multiply P by the fixed axe, and divide the product by s; to the quotient add the revolving axe; then the ſum multiplied by $\frac{1}{2}$ the circumference of the greateſt circle, viz. by 3·14159 and by $\frac{1}{2}$ the revolving axe, will give the ſurface required.

That is,

putting f = the fixed axe,
r = the revolving axe,
$q = r \div f$,
$s = \sqrt{1 \backsim qq}$,
p = 3·14159 &c,
P = ·01745329 × degrees to the ſine s, in the oblong ſpheroid,
and P = 2·302585 × log. of $s + q$ in the oblate ſpheroid;

Then $(\frac{Pf}{s} + r) \times \frac{1}{2}pr$ = the ſurf. of the ſpheroid.*

EXAMPLE.

The axes of a ſpheroid being 50 and 30, required the ſurface.

1. For the Oblong Spheroid.

Here $f = 50$, $r = 30$, $q = \frac{30}{50} = ·6$, $s = \sqrt{1 - qq}$ $= \sqrt{1 - ·36} = \sqrt{·64} = ·8$, which is the ſine of 53·130105 degrees.

Then

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* DEMONSTRATION.

For the fluent of $\frac{2phx}{aa}\sqrt{a^4 - c^2x^2}$, the fluxion of the ſurface in the demonſtration of the firſt rule, when $x = a$, is $pb \times (b + \frac{aa}{c}P)$,

where

Then $53{\cdot}130105 \times {\cdot}01745329 = {\cdot}92729513 =$ p, and $(\frac{50 \times {\cdot}92729513}{{\cdot}8} + 30) \times \frac{30 \times 3{\cdot}14159}{2} =$ $4144{\cdot}8263$ = the surface of the oblong spheroid.

2. For the Oblate Spheroid.

Here $f = 30$, $r = 50$, $q = \frac{50}{30} = \frac{5}{3}$, and $s = \sqrt{1 \backsim \frac{25}{9}}$ $= \sqrt{\frac{16}{9}} = \frac{4}{3}$.

Hence $s + q = \frac{5}{3} + \frac{4}{3} = \frac{9}{3} = 3$, whose log. is $\cdot4771213$.
And $\text{P} = {\cdot}4771213 \times 2{\cdot}302585 = 1{\cdot}0986124$.
Then $(\frac{30 \times 1{\cdot}0986124}{\frac{4}{3}} + 50) \times \frac{50 \times 3{\cdot}14159}{2} =$ $5868{\cdot}39918$ = the surface of the oblate spheroid.

RULE III.

For the Oblong Spheroid.

Find q and s as in the last rule.

Find also the area of the circular middle zone whose radius is half the quotient of the fixed axe divided by s, and the distance of each end from the center is half the fixed axe; and call this zone z.

Then multiply continually together z, q, s, and $3{\cdot}14159$ &c, for the surface of the oblong spheroid.

That is, $pqsz$ = the surface of the oblong spheroid ABDP; where z is equal to the middle zone EFGH of the circle whose radius is $\frac{\frac{1}{2}f}{s}$.*

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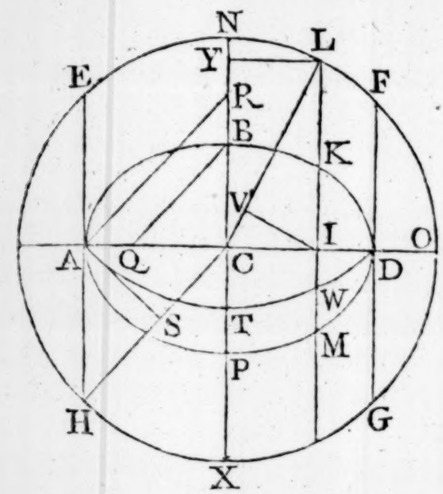
where $\text{P} = {\cdot}017453 \times$ deg. in arc to sine $\frac{c}{a}$, when a is greater than b;

or $\text{P} = 2{\cdot}3025$ &c $\times$ log. of $\frac{b+c}{a}$, when a is less than b;

c being $= \sqrt{aa \backsim bb}$. *Q. E. D.*

* DEMONSTRATION.

For since the fluxion of the surface BKMP is $\frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{cc} - xx}$, and



EXAMPLE.

Taking again the same example,

We have $\frac{\frac{1}{2}f}{s} = \frac{25}{\cdot 8} = 31\cdot 25 =$ the radius, or $62\cdot 5$ = the diameter of the circle.

Hence CO − CD $= 31\cdot 25 - 25 = 6\cdot 25 =$ DO.

And DO ÷ 2CO $= 6\cdot 25 \div 62\cdot 5 = \cdot 1 =$ the tabular versed sine, to which belongs the area $\cdot 04087528$; and this taken from $\cdot 39269908$, the tabular semi-circle, leaves $\cdot 3518238$ for half the tabular zone.

Whence

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and that of the circular area NLIC $= \dot{x}\sqrt{\frac{a^4}{cc} - xx}$, and being to the former in the constant ratio of 1 to $\frac{2pbc}{aa}$, their fluents must be continually in the same ratio; and consequently the surface EKMP $= \frac{2pbc}{aa} \times$ CNLI $= pqs \times$ 2CNLI $= p \times \frac{\text{BC}}{\text{CN}} \times$ 2CNLI.

And the whole surface ABDPA $= pqs \times$ EFGH. *Q. E. D.*

Whence $\cdot 3518238 \times 2 \times 62\cdot 5 \times 62\cdot 5 = 2748\cdot 6234375$ = the middle zone EFGH = z.

Then $pqsz = 3\cdot 14159 \times \cdot 6 \times \cdot 8 \times 2748\cdot 6234375 = 4144\cdot 8265$ the surface, nearly the same as before.

RULE IV.

For the Oblong Spheroid.

Between the semi-conjugate BC and the sum of the conjugate BP and circular arc EF, find a mean proportional, and it will be the radius of a circle equal to the surface of the spheroid; the circle being described as in the last rule.

That is, multiply continually together BC, BP + EF, and $3\cdot 14159$, for the surface.*

* DEMONSTRATION.

Apply BQ = AC, and draw AR parallel to BQ.

Then is AR = the radius CN; for, by the construction,

CN is $= \dfrac{aa}{\sqrt{aa - bb}} = \dfrac{AC^2}{\sqrt{QB^2 - BC^2}} = \dfrac{AC^2}{CQ} = \dfrac{AC \times BQ}{QC} =$ AR by similar triangles. Draw CH, and AS perpendicular to it.

Then, by the last rule, the surface BAP is $= p \times \dfrac{BC}{CN} \times 2CNEA$ or $2CXHA = p \times BC \times \dfrac{2 \text{ sector } CXH + 2\triangle HAC}{CH} = p \times BC \times (NE + AS)$.

But, by similar triangles, $\begin{cases} HC : AC :: AH : AS, \\ AR : QB :: CR : CB. \end{cases}$

And since HC = AR, and AC = QB, by the construction; as also AH = CR, by reason of the equal hypotenuses HC, AR, and common base AC; the fourth terms AS, CB, must also be equal to each other. And consequently the surface BAP = $p \times$ BC $\times$ (EN + BC) = $p \times$ BC $\times$ (AT + BC); and that of the whole spheroid = $p \times$ BC $\times$ (ATD + BP) = $p \times$ BC $\times$ (ENF + BP), the

EXAMPLE.

Taking still the same example,

We have the radius CN $= \frac{\frac{1}{4}f}{s} = \frac{25}{\cdot 8} = 31\frac{1}{4}$.

Then as $31\frac{1}{4} : 25 =$ AC the sine of the arc EN :: $1 : \frac{4}{5} = \cdot 8 =$ the sine of the similar arc to the radius

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the arc ATD being described with the center R and radius RA; viz. equal to a circle whose radius is a mean proportional between BC and ATD + BP. *Q. E. D.*

SCHOLIUM.

This construction, for finding a circle equal to the surface of an oblong spheroid, was first given by Mr. *Huygens* in his *Horologium Oscillitorium*, *prop.* 9, but without demonstration.—The construction is here rendered general for any part or zone BKMP of the spheroid, viz. that its curve surface is equal to a circle whose radius is a mean proportional between BC and NL + IV or TW + IV, IV being perpendicular to CL. For in the same manner as $p \times$ BC $\times$ (EN + AS) was found for the surface BAP, may the surface of any zone BKMP be found to be $p \times$ BC $\times$ (NL + IV).

In imitation of the two rules above for expressing the surface of the oblong spheroid, by means of a circular arc and a circular or elliptic area, I shall here shew the investigation of others for that of the oblate spheroid, by means of the parabolic arc, or hyperbolic area.

Thus if x be the abscis, and y the ordinate, of a parabola whose parameter is n; since $x = \frac{yy}{n}$, and $\dot{x} = \frac{2y\dot{y}}{n}$, the fluxion of the arc will be

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{4y^2\dot{y}^2}{nn} + \dot{y}^2} = \frac{\dot{y}}{n}\sqrt{nn + 4yy} = \frac{\dot{y}}{m}\sqrt{mm + yy},$$

putting $m = \frac{1}{2}n = \frac{1}{2}$ the parameter; which being to that of the oblate surface, viz.

$$\frac{2pb\dot{x}}{aa}\sqrt{a^4 + c^2x^2} = \frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{cc} + x^2} = \frac{2pb\dot{x}}{m}\sqrt{mm + xx},$$

(putting $m = \frac{a^2}{c}$, and supposing $x = y$), in the constant ratio of

dius 1; to which belong 53·130105 degrees; hence 53·130105 × ·01745329 × $31\frac{1}{4}$ = 28·977972 = the arc EN.

Then BP × (BC + EN) × 3·14159 = 4144·8262 = the surface as before, nearly.

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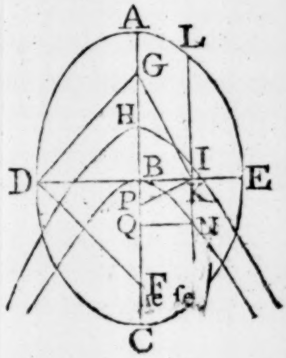
of 1 to $2bp$, their fluents must be in the same ratio; and consequently if P be put to denote the parabolic arc, then will $2pb$P express the value of the oblate surface, which therefore is equal to a circle whose radius is a mean proportional between b and 2P, or between $2b$ and P.—And hence this construction.—Let DE be the less or fixed axe of the ellipse, AC the greater axe, and F the focus; join DF, and draw DG perpendicular to DF; bisect BG in H; with the vertex H and focus B, describe a parabola HI: Then draw any ordinate LK to the axe DE, meeting the parabola in I; and the surface generated by the arc AL will be equal to a circle whose radius is a mean proportional between AC and the parabolic arc HI. Or the surface generated by AL, will be equal to the curve surface of a cylinder whose diameter is AC and height HI.



Again, since P $= \frac{2y}{m}\sqrt{mm+yy} - \frac{A}{b}$, A being the area of an hyperbola whose semi-axes are b and m, and y the ordinate; see cor. 7 rule 1 prob. 6 sect 5; the surface generated by AL will be equal to $2p \times (\frac{2by}{m}\sqrt{mm+yy} - A)$; and hence this construction is evident. Let LK, produced, meet, in N, the hyperbola BN, whose vertex is B, center A, and semi-conjugate axe BG; draw GK, and KP perpendicular to it, and draw the ordinate NQ: Then will the difference between the rectangle AC × KP and the hyperbolic area BQN, be to the surface generated by AL, as the radius of a circle, is to its circumference.

RULE V.

For Both Spheroids.

Find q as in the first rule; then to or from 1, add or subtract $\frac{1}{3}$ of q, according as it is the oblate or oblong spheroid; and call the square root of the sum or difference A. Or $A = \sqrt{1 \pm \frac{1}{3}q}$.

Multiply the product of the two axes by 3·14159, and call the product P.

Then the product of A and P will be the surface nearly.

That is, $pfr\sqrt{1 \pm \frac{1}{3}q}$.*

EXAMPLE.

Let nere be taken here the same example as at rule 1, in which the axes are 50 and 40.

1. For the Oblong Spheroid.

Here $q = \frac{9}{25}$, and $\sqrt{1 - \frac{9}{3 \times 25}} = \frac{1}{5}\sqrt{22} = \cdot 93808 = A$,

But $3 \cdot 14159 \times 50 \times 40 = 6283 \cdot 1853 = P$.

Then $A \times P = 5893 \cdot 6278$ the surface nearly.

2. For

* For since, by rule 1, the value of the surface is

$$s = 4abp \times (1 \pm \frac{d^2}{2.3} - \frac{d^4}{2.4.5} \pm \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9} \text{ \&c}), \text{ and}$$

$$4abp\sqrt{1 \pm \tfrac{1}{3}d^2} = 4abp \times (1 \pm \frac{d^2}{2.3} - \frac{d^4}{2.4.9} \pm \frac{3d^6}{2.4.6.27} - \frac{3.5d^8}{2.4.6.8.81} \text{ \&c})$$

we shall have, by taking the latter of these quantities from the former,

$$s - 4abp\sqrt{1 \pm \tfrac{1}{3}d^2} = -\frac{d^4}{2.5.9} \pm \frac{5d^6}{4.7.27} - \frac{5d^8}{2.8.81} \text{ \&c, and}$$

consequently $s = 4abp\sqrt{1 \pm \frac{1}{3}d^2}$ nearly, which is rule 5.

2. For the Oblate Spheroid.

Here $q = \frac{9}{16}$, and $\sqrt{1 + \frac{9}{3 \times 16}} = \frac{1}{4}\sqrt{19} =$ $1\cdot089725 = A$.

Then $A \times P = 1\cdot0897 \times 6283\cdot1853 = 6846\cdot7869$ the surface of the oblate spheroid nearly.

RULE VI.

For Both Spheroids.

To or from 1, add or subtract, according as it is for the oblate or oblong spheroid, $\frac{1}{6}$ of q, and call $\frac{4}{9}$ of the sum or difference B.

Then multiply P by the difference between A and B, and $\frac{9}{5}$ of the product will be the surface nearly.

That is,

$\frac{9}{5}P \times (A - B) = \frac{9}{5} \times pfr \times (\sqrt{1 \pm \frac{1}{3}q} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6}q})$ = the surface nearly.*

X 4 EX-

* Again, by transposing the two first terms of the series in the value of s, we shall have

$s - 4abp \times (1 \pm \frac{d^2}{6}) = 4abp \times (-\frac{d^4}{2.4.5} \pm \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9}$ &c),

or $\frac{4}{9} \times [s - (1 \pm \frac{d^2}{6}) \times 4abp] = 4abp \times (-\frac{d^4}{2.5.9} \pm \frac{d^6}{4.7.9} - \frac{5d^8}{4.8.81}$ &c);

which taken from

$s - 4abp\sqrt{1 \pm \frac{1}{3}d^2} = -\frac{d^4}{2.5.9} \pm \frac{5d^6}{4.7.27} - \frac{5d^8}{2.8.81}$ &c,

the remainder above found, it leaves

$\frac{5}{9}s - 4abp \times (\sqrt{1 \pm \frac{1}{3}d^2} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6}d^2}) = \pm \frac{d^6}{2.7.27} - \frac{5d^8}{4.8.81}$ &c,

and consequently $s = \frac{9}{5} \times 4abp \times [\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9} \times (1 \pm \frac{1}{6}dd)]$ $= \frac{4}{5}abp \times [9\sqrt{1 \pm \frac{1}{3}dd} - 4(1 \pm \frac{1}{6}dd)]$ more nearly, which is rule 6.

EXAMPLE.

Taking the same example, we shall have

1. For the Oblong Spheroid.

$B = \frac{4}{9} \times (1 - \frac{1}{6}q) = \frac{4}{9} \times (1 - \frac{9}{6 \times 25}) = \frac{4}{9} \times \frac{47}{50} =$ $\cdot 417777$ &c.

Then $A - B = \cdot 93808 - \cdot 417777 = \cdot 5203$.

And $\frac{9}{5} \times \cdot 5203 \times 6283 \cdot 1853 = 5884 \cdot 2029 =$ the surface of the oblong spheroid very nearly.

2. For the Oblate Spheroid.

$B = \frac{4}{9} \times (1 + \frac{1}{6}q) = \frac{4}{9} \times (1 + \frac{9}{6 \times 16}) = \frac{4}{9} \times \frac{35}{32} =$ $\cdot 48611$ &c.

Then $A - B = 1 \cdot 0897 - \cdot 4861 = \cdot 6036$.

And $\frac{9}{5} \times \cdot 6036 \times 6283 \cdot 1853 = 6826 \cdot 6806 =$ the surface of the oblate spheroid nearly.

RULE VII.

For Both Spheroids.

From the sum or difference of 1 and $\frac{1}{6}$ of q, according as it is for the oblate or oblong spheroid, take $\frac{1}{45}$ of the square of q, and call $\frac{8}{27}$ of the remainder c.*

Then

* Farther, by transposing the first three terms of the original series, and reducing the result, to make the first term of it equal to that of the last remainder, we shall have

$\frac{8}{27} \times [s - 4abp(1 \pm \frac{1}{6}d^2 - \frac{1}{45}d^4)] = 4abp \times (\pm \frac{d^5}{2.7.27} - \frac{5d^8}{6.8.81}$ &c), which

Then from A subtract the sum of B and C, multiply the remainder by P, and $\frac{27}{7}$ of the product will be the surface of the spheroid, still nearer.

That is, $\frac{27}{7}P \times (A - B - C) =$ the surface nearly; where $C = \frac{8}{27} \times (1 \pm \frac{1}{6}q - \frac{1}{40}qq)$, and the value of the other letters as before.

EXAMPLE.

Taking still the same example, in which the axes are 40 and 50;

1. For the Oblong Spheroid.

Here again $q = \frac{9}{25} = \cdot 36$, $P = 6283\cdot 1853$, $A = \cdot 93808$, $B = \cdot 417777$ &c, and $C = \frac{8}{27} \times (1 - \frac{1}{6}q - \frac{1}{40}qq) = \cdot 277558$.

Then $\frac{27}{7}P \times (A - B - C) = 6283\cdot 1853 \times \cdot 93628 = 5882\cdot 8206$ the surface of the oblong spheroid very near.

2. For the Oblate Spheroid.

Here $q = \frac{9}{16} = \cdot 5625$, $P = 6283\cdot 1853$, $A = 1\cdot 089725$, $B = \cdot 486111$, and $C = \frac{8}{27} \times (1 + \frac{1}{6}q - \frac{1}{40}qq) = \cdot 32173$.

Then $\frac{27}{7} \times (A - B - C) \times P = 1\cdot 0872 \times 6283\cdot 1853 = 6831\cdot 079 =$ the surface of the oblate spheroid very nearly.

PRO-

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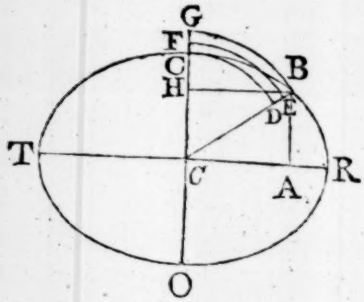
which being taken from the last remainder, $\frac{5}{9}s - 4abp \times [\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9}(1 \pm \frac{1}{6}dd)] = \pm\frac{d^6}{2.7.27} - \frac{5d^8}{4.8.81}$ &c, leaves $\frac{7}{27}s - 4abp \times [\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9}(1 \pm \frac{1}{6}dd) - \frac{8}{27}(1 \pm \frac{1}{6}d^2 - \frac{1}{40}d^4)] = -\frac{5d^8}{8.12.81}$ &c, and consequently $s = \frac{27}{7} \times 4abp \times [\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9}(1 \pm \frac{1}{6}dd) - \frac{8}{27}(1 \pm \frac{1}{6}d^2 - \frac{1}{40}d^4)]$ $= \frac{4}{7}abp \times [27\sqrt{1 \pm \frac{1}{3}dd}) - 12(1 \pm \frac{1}{6}dd - 8(1 \pm \frac{1}{6}d^2 - \frac{1}{40}d^4)]$, nearer still; and this is rule 7.

PROBLEM X.

To find the Curve Surface of the Frustum of a Spheroid, contained between two Planes cutting the Spheroid Perpendicular to the Fixed Axe, and one of them passing through the Center.



RULE I.

For Both Spheroids.

Let f denote the fixed, and r the revolving axe; put $p = 3{\cdot}14159$, and $q = \frac{ff \backsim rr}{ff}$, as in the last problem; also $h =$ the height CA of the frustum, and $z = \frac{4qhh}{ff}$; then will the value of the surface be expressed by

$$prh \times (1 \pm \frac{1}{2.3}z - \frac{1}{2.4.5}z^2 \pm \frac{3}{2.4.6.7}z^3 - \frac{3.5}{2.4.6.8.9}z^4 \ \&c),$$

or $prh \times (1 \pm \frac{1}{2.3}\text{A}z - \frac{3}{4.5}\text{B}z \pm \frac{3.5}{6.7}\text{C}z - \frac{5.7}{8.9}\text{D}z \ \&c).$ Where A, B, C, &c, are the several preceding terms, and the upper or under signs to be used, according as it is the oblate or oblong spheroid.*

EX-

* See the investigation of rule 1 of the last problem.

EXAMPLE I.

What is the surface of the frustum of a spheroid, whose height is 15; the diameter of the greater end being 40, and that of the less 32?

Here CO $=$ 40 $=$ the revolving axe, 2AB $=$ 32, and HB $=$ cA $=$ 15; hence CH $=$ Cc $-$ AB $=$ 4, and HO $=$ Cc $+$ AB $=$ 36; and consequently, by case 3 prob. 2, we have as

$\sqrt{\text{CH} \times \text{HO}} = 12 : \text{HB} = 15 :: \text{CO} = 40 : \text{TR} = 50$ the fixed axe; which shews that the frustum is that of an oblong spheroid.

Then, as in the example to rule 1 of the last problem, $f = 50$, $r = 40$, $q = \frac{9}{25} = \cdot 36$. But $h = 15$, and $z = \frac{4qhh}{ff} = \frac{4 \times 9 \times 15 \times 15}{25 \times 50 \times 50} = \cdot 1296$.

Hence the 1st term A $=$ — $+1\cdot$

$$\begin{array}{lllll}
\text{2d} & \text{—} & \text{B} = \frac{1}{2.3}\text{A}z & = & -0\cdot0216 \\
\text{3d} & \text{—} & \text{C} = \frac{3}{4.5}\text{B}z & = & -0\cdot0004199+ \\
\text{4th} & \text{—} & \text{D} = \frac{3.5}{6.7}\text{C}z & = & -0\cdot0000195- \\
\text{5th} & \text{—} & \text{E} = \frac{5.7}{8.9}\text{D}z & = & -0\cdot0000012+ \\
\text{6th} & \text{—} & \text{F} = \frac{7.9}{10.11}\text{E}z & = & -0\cdot0000001-
\end{array}$$

the sum of the negative terms is $-0\cdot0220407$,
which taken from the first term $1\cdot0000000$,
leaves $0\cdot9779593$
for the value of the infinite series; which being drawn into $prb = 3\cdot14159 \times 40 \times 15 = 1884\cdot955592$, produces $1843\cdot4098512$ for the surface required.

EX

EXAMPLE II.

It is required to find the curve surface of the frustum 2cABC of a spheroid, whose height is 16; the diameter of the greater end being 50, and that of the less 30.

Here the revolving axe TR = 50, cH = AB = 16, and HB = cA = 15; hence AR = cR − cA = 10, and TA = Tc + cA = 40; then, by case 3 prob. 2, we shall have as

$\sqrt{AR \times AT} = 20 : AB = 16 :: TR = 50 : CO = 40$ = the fixed axe; which indicates the frustum to be that of an oblate spheroid.

Wherefore $f = 40$, $r = 50$, $q = \frac{9}{16}$, $h = 16$, and $z = \frac{4qhh}{ff} = \frac{4 \times 9 \times 16 \times 16}{16 \times 40 \times 40} = \frac{36}{100} = \cdot 36$, which being the same as the converging quantity q in the example to rule 1 of the last prob. the several terms of the series must be the same as there found, viz.

the 1st term	A	=	+ 1
2d	— B	=	+ 0·06
3d	— C	=	− 0·00324
4th	— D	=	+ 0·0004166 −
5th	— E	=	− 0·000072900
6th	— F	=	+ 0·0000150 +
7th	— G	=	− 0·0000035 −
8th	— H	=	+ 0·0000009 −
9th	— I	=	− 0·0000002 +

sum of the affir. terms = + 1·0604325
sum of the neg. terms = − 0·0033166

their difference = 1·0571159 the value of the series; which being drawn into $prb = 3{\cdot}14159 \times 50 \times 16 = 2513{\cdot}27412287$, will produce 2656·822036 for the surface of the oblate frustum required.

RULE

RULE II.

For Both Spheroids.

1. Multiply the ſquare root of the difference of the ſquares of the axes, by the height of the fruſtum; divide the product by the ſquare of the fixed axe; and call double the quotient q.

That is, $q = \frac{2h\sqrt{ff \backsim rr}}{ff}$, h being the height of the fruſtum, f the fixed, and r the revolving axe.

2. In the oblong ſpheroid, call the ſquare root of the difference between 1 and the ſquare of q, A. But in the oblate ſpheroid, let A be the root of the ſum of 1 and the ſquare of q.

That is, $A = \begin{cases} \sqrt{1 - qq} \text{ in the oblong,} \\ \sqrt{1 + qq} \text{ in the oblate.} \end{cases}$

3. In the oblong ſpheroid, let the product of ·01745329 and the degrees whoſe ſine is q, be called B. But in the oblate ſpheroid, let B be the product of 2·30258509 and the logarithm of the ſum of q and the root of the ſum of 1 and the ſquare of q.

That is, B $=$
$\begin{cases} \cdot 0174 \text{ \&c} \times \text{degrees whoſe ſine is } q \text{ in the oblong,} \\ 2\cdot 302 \text{ \&c} \times \text{log. of } q + \sqrt{1 + qq} \text{ in the oblate.} \end{cases}$

4. Divide B by q, to the quotient add A, multiply the ſum by the continual product of the height, revolving axe, and the number 3·14159, and half the laſt product will be the ſurface required.

That is, $\frac{1}{2}prh \times (A + \frac{B}{q}) =$ the ſurface.*

EX-

* For this is the fluent of the fluxion of the ſurface in the inveſtigation of the rules in the laſt problem.

EXAMPLE I.

Let there be taken here the first example to the last rule, in which f is $= 50$, $r = 40$, and $h = 15$.

Then $\frac{2h\sqrt{ff-rr}}{ff} = \frac{30^2}{50^2} = \frac{9}{25} = \cdot 36 = q$.

And $A = \sqrt{1-qq} = \sqrt{1-\frac{9^2}{25^2}} = \frac{4\sqrt{34}}{25} = \cdot 9329523$.

Also the sine $q = \cdot 36$ answers to $21\cdot 1001965$ degrees, which being drawn into $\cdot 01745329$, will produce $\cdot 3682678 = B$.

Hence $(A + \frac{B}{q}) \times \frac{1}{2}prh = \cdot 9779593 \times 1884\cdot 955592$ $= 1843\cdot 4098512 =$ the surface, the same as before.

EXAMPLE II.

Let there be taken the oblate frustum, in the second example to the preceding rule, in which the axes are 50 and 40, and the height 16.

Here then $f = 40$, $r = 50$, and $h = 16$.

Hence $\frac{2h\sqrt{rr-ff}}{ff} = \frac{32 \times 30}{40 \times 40} = \frac{3}{5} = \cdot 6 = q$.

And $\sqrt{1+qq} = \sqrt{1+\frac{3^2}{5^2}} = \frac{1}{5}\sqrt{34} = 1\cdot 16619038$ $= A$.

Likewise $q + \sqrt{1+qq} = 1\cdot 76619038$, whose logarithm is $\cdot 24703757$, which being drawn into $2\cdot 30258509$ will produce $\cdot 56882503 = B$.

Then $(A + \frac{B}{q}) \times \frac{1}{2}prh = 1\cdot 057116 \times 2513\cdot 27412287$ $= 2656\cdot 822287 =$ the surface, nearly the same as before.

RULE III.

For the Oblong Spheroid.

1. Divide the square of the fixed axe, by the root of the difference of the squares of the axes; and call the quotient d.

That is, $d = \frac{ff}{\sqrt{ff - rr}}$.

2. Find the area of the circular zone whose diameter is d, and its height from the center equal to the height h of the frustum of the spheroid; and call the area of that zone z.

3. Then as the diameter of the circle, is to the circular zone; so is the circumference of the greatest circle of the spheroid, to the surface of its frustum.

That is, $d : z :: pr : \frac{prz}{d} =$ the surface of the frustum.*

EXAMPLE.

It is required to find the surface of the frustum of an oblong spheroid whose height is 15, the axes being 50 and 40.

Here $\frac{ff}{\sqrt{ff - rr}} = \frac{50^2}{30} = \frac{250}{3} = 83\frac{1}{3} = d$ the diam. of the circle.

And, by the table of circular areas, the area of the zone will be found to be $\cdot 17603268 \times \frac{250^2}{3^2}$.

Also $pr = 3\cdot 14159 \times 40 =$ the greatest circumference of the spheroid.

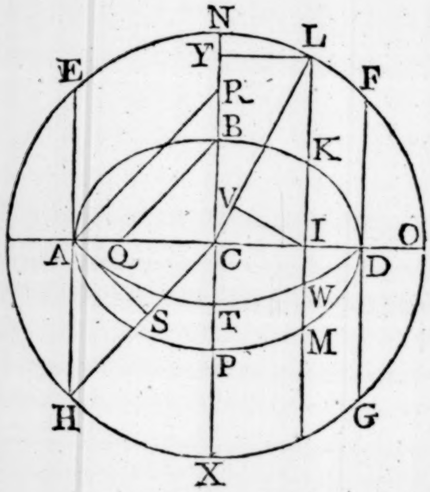
Then,

* This is the same with rule 3 to the last problem, only expressed in other terms.

Then, as $\frac{250}{3}$: $\cdot17603268 \times \frac{250^2}{3^2}$ :: $3\cdot14159 \times 40$: $\cdot17603268 \times 3\cdot14159 \times 40 \times \frac{250}{3} = 1843\cdot4099$ the surface, nearly the same as before.

RULE IV.

For the Oblong Spheroid.



Let BKMP be the frustum, whose surface is sought, NLO a quadrant of the circle mentioned in the last problem; draw CL, and IV perpendicular to it.

Then the surface of the frustum will be equal to the circle whose radius is a mean proportional between BC and NL + IV.

Or the surface will be equal to the continual product of NL + IV, BC, and 3·14159.*

EXAMPLE.

Let there be taken here the same example as before, in which the axes are 50 and 40, and the height 15.

Then

* This is proved at rule 4 to the last problem.

Then ſince, by the example to the laſt rule, $CL = CN = \frac{125}{3}$, we ſhall have as $CL = \frac{125}{3}$: $YL = 15$ :: $1 =$ radius : $\frac{9}{25} = \cdot 36 =$ the ſine of $21\cdot1001965$ degrees, which are thoſe contained in the arc NL; and therefore $\cdot01745329 \times 21\cdot1001965 \times \frac{125}{3} = 15\cdot3444958 =$ NL. And, by ſimilar triangles, ſince $IL = \sqrt{CL^2 - CI^2} = \sqrt{\frac{125^2}{3^2} - 15^2} = \frac{20\sqrt{34}}{3} = 38\cdot8730126$, as $CL = \frac{125}{3}$: LI :: $CI = 15$: $IV = 13\cdot9942845$.

And hence $(NL + IV) \times 3\cdot14159 \times BC = 29\cdot3387803 \times 3\cdot14159 \times 20 = 1843\cdot409933 =$ the ſurface, nearly the ſame as before.

RULE V.

For Both Spheroids.

The 5th rule to the laſt problem will ſerve, if inſtead of q we uſe z, as found in rule 1 of this problem, and h the height inſtead of f the fixed axe.

That is, $prh\sqrt{1 \pm \frac{1}{3}z} =$ the ſurface nearly.

EXAMPLE I.

Taking the ſame example as in the laſt rule; we have $r = 40$, $h = 15$, and z, found in the firſt example to rule 1, $= \cdot1296$.

Conſequently $prh\sqrt{1 - \frac{1}{3}z} = 1884\cdot955592 \times \cdot97816 = 1843\cdot7879 =$ the ſurface nearly.

EXAMPLE II.

If we take the oblate fruſtum, whoſe height is 16, and the axes 40 and 50; we ſhall have $r = 50$,

 $h =$

$b = 16$, and z, found in the second example to rule 1, $= \cdot 36$.

Then $prb\sqrt{1+\frac{1}{3}z} = 2513\cdot 27412287 \times 1\cdot 0583$ $= 2659\cdot 797 =$ the surface nearly.

RULE VI.

For Both Spheroids.

This rule will be the same as the 6th rule to the last problem, using z instead of q, and b for f, as in the last rule. That is,

$\frac{9}{5}\text{P} \times (\text{A} - \text{B}) = \frac{9}{5}prb \times [\sqrt{1 \pm \frac{1}{3}z} - \frac{4}{9}(1 \pm \frac{1}{6}z)]$ $=$ the surface nearly.

EXAMPLE I.

Taking the 1st example to the last rule; we have $\text{P} = 1884\cdot 955592$, $\text{A} = \sqrt{1 - \frac{1}{3}z} = \cdot 97816157$, and $\text{B} = \frac{4}{9} \times (1 - \frac{1}{6}z) = \cdot 43484444$.

Then $\frac{9}{5}\text{P} \times (\text{A} - \text{B}) = 1884\cdot 955592 \times \cdot 97797$ $= 1843\cdot 4298 =$ the surface nearly.

EXAMPLE II.

Taking here the 2d example to the last rule; we have $\text{P} = prb = 2513\cdot 27412287$, $\text{A} = \sqrt{1+\frac{1}{3}z}$ $= 1\cdot 0583005$, and $\text{B} = \frac{4}{9} \times (1 + \frac{1}{6}z) = \cdot 47\frac{1}{9}$.

Then $\frac{9}{5}\text{P}\times(\text{A}-\text{B}) = 2513\cdot 27412287 \times 1\cdot 0569409$ $= 2656\cdot 3821 =$ the surface nearly.

RULE

RULE VII.

For Both Spheroids.

This rule also will be the same with the 7th to the last problem, using z and b instead of q and f.

That is, $\frac{27}{7}\text{P} \times (\text{A} - \text{B} - \text{C}) = \frac{27}{7}prb \times (\sqrt{1 \pm \frac{1}{3}z} - \frac{4}{9}(1 \pm \frac{1}{6}z) - \frac{7}{27}(1 \pm \frac{1}{6}z - \frac{1}{40}zz)$ = the surface very nearly.

EXAMPLE I.

Taking, still, the same example of the oblong spheroid; we shall have, as before, $\text{P} = 1884{\cdot}955592$, $\text{A} = {\cdot}97816157$, and $\text{B} = {\cdot}43484444$; also $\text{C} = \frac{8}{27} \times (1 - \frac{1}{6}z - \frac{1}{40}zz) = {\cdot}28977188$.

Then $\frac{27}{7}\text{P} \times (\text{A}-\text{B}-\text{C}) = 1884{\cdot}955592 \times {\cdot}97796 = 1843{\cdot}411$ = the surface very nearly.

EXAMPLE II.

Let there be taken, again, the oblate frustum, in which, as before, $\text{P} = 2513{\cdot}27412287$, $\text{A} = 1{\cdot}0583005$, and $\text{B} = {\cdot}47\frac{1}{9}$; also $\text{C} = \frac{8}{27} \times (1 + \frac{1}{6}z - \frac{1}{40}zz) = {\cdot}31311407$.

Then $\frac{27}{7}\text{P} \times (\text{A}-\text{B}-\text{C}) = 2513{\cdot}27412287 \times 1{\cdot}05714 = 2656{\cdot}8825$ = the surface very nearly.

SCHOLIUM.

It is evident that the double of the frustum will give the middle zone; and that the frustum being added to, or taken from, half the spheroid, will give the greater or less segment.

PROBLEM XI.

To find the Solidity of a Spheroid.

RULE I.

Multiply continually together the fixed axe, the ſquare of the revolving axe, and the number $\cdot 52359877$ or $\frac{1}{6}$ of $3\cdot 14159$, and the laſt product will be the ſolidity.*

That

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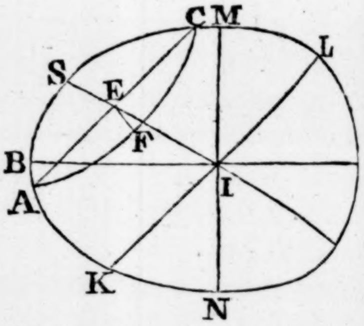
* DEMONSTRATION.



Put $f =$ BI the fixed ſemi-axe, $r =$ IM the revolving ſemi-axe of the ſpheroid, $a =$ SI any ſemi-diameter of the ſection NBM, $b =$ IK its ſemi-conjugate, $y =$ AE an ordinate to the diameter SI, or a ſemi-axe of the elliptic ſection AFC parallel to KL, and $z =$ EF its other ſemi-axe, alſo $x =$ EI, $s =$ the ſine of the angle AES, or of the angle KIS, to the radius 1, and $p = 3\cdot 14159$.

Then, by the property of the ellipſe KSL, $aa : bb :: aa - xx : bb \times \frac{aa - xx}{aa} = yy$; and, by prop. 1 ſect. 2, $b : r :: y : \frac{ry}{b} = z$.

But the fluxion of the ſolid KACL is $psyz\dot{x} = \frac{psryy\dot{x}}{b}$ by writing for z its value $\frac{ry}{b}$, $= pbsr\dot{x} \times \frac{aa - xx}{aa}$ by ſubſtituting for yy its value $bb \times \frac{aa - xx}{aa}$, $= pfrr\dot{x} \times \frac{aa - xx}{aaa}$ by putting for abs its value rf; and hence the fluent $pfrrx \times \frac{aa - \frac{1}{3}xx}{aaa}$ or $\frac{1}{3}pfrr \times \frac{3aa - xx}{aaa}$ will be the value of the fruſtum KACL; which, when EI or x becomes SI or a, gives $\frac{2}{3}pfrr$ for the value of

That is, $\frac{1}{6}pttc$ = the oblate, and $\frac{1}{6}ptcc$ = the oblong spheroid, where $p = 3{\cdot}14159$, t = the transverse, and c = the conjugate axe of the generating ellipse.

of the semi-spheroid KSL; or the whole spheroid $= \frac{1}{6}p$FRR, putting F and R for the whole fixed and revolving axes. *Q.E.D.*

Corol. 1. From the foregoing demonstration it appears that the value of the general frustum KAECL is expressed by $\frac{1}{3}pfrrx \times \frac{3aa - xx}{aaa}$.

And if for fr be substituted its value abs, the same frustum will also be expressed by $\frac{1}{3}pbrsx \times \frac{3aa - xx}{aa}$.

Also, if for aa be put its value $\frac{bbxx}{bb - yy}$, the last expression will become $\frac{1}{3}prsx \times \frac{2bb + yy}{b}$ or $\frac{1}{3}psx \times (2br + \frac{ryy}{b})$;

which, by writing z instead of its value $\frac{ry}{b}$, gives $\frac{1}{3}psx \times (2br + yz)$ for the value of the frustum, viz. The sum of the area of the less end and twice that of the greater, drawn into one-third of the altitude or distance of the ends.

And out of this last expression may be expunged any one of the four quantities b, r, y, z, by means of the proportion $b : r :: y : z$.

When the ends of the frustum are perpendicular to the fixed axe; then a is $= f$, and the value of the frustum becomes $\frac{1}{3}prrx \times \frac{3ff - xx}{ff}$ for the value of the frustum whose ends are perpendicular to the fixed axe, its altitude being x.

And when the ends of the frustum are parallel to the fixed axe, a is $= r$, and the expression for such a frustum becomes $\frac{1}{3}pfx \times \frac{3rr - xx}{r}$.

Corol. 2. If to or from $\frac{2}{3}pfrr$, the value of the semi-spheroid, be added or subtracted $\frac{1}{3}pfrrx \times \frac{3aa - xx}{aaa}$, the value of the general

RULE II.

Multiply the area of the generating ellipſe, by $\frac{2}{3}$ of the revolving axe, and the product will be the content of the ſpheroid.

That is, $\frac{2}{3}t\text{A}$ = the oblate, and $\frac{2}{3}c\text{A}$ = the oblong ſpheroid; where A is the area of the ellipſe.

And this rule is evidently taken from the former.

EX-

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general fruſtum KACL, there will reſult $\frac{1}{3}pfrrhh \times \frac{3a-h}{aaa}$ for the value of a general ſegment, either greater or leſs than the ſemi-ſpheroid, whoſe height, taken upon the diameter paſſing through its vertex and center of its baſe, is $h = a \pm x$.

When a coincides with f, the above expreſſion becomes $\frac{1}{3}prrhh \times \frac{3f-h}{ff}$ for the value of a ſegment whoſe baſe is perpendicular to the fixed axe.—And here if we put R for the radius of the ſegment's baſe, and for rr its value $\frac{\text{RR}ff}{2fh-hh}$, the ſaid ſegment will become $\frac{1}{3}p\text{RR}h \times \frac{3f-h}{2f-h}$.

And when a coincides with r, the general expreſſion will become $\frac{1}{3}pfhh \times \frac{3r-h}{r}$ for the value of the ſegment whoſe baſe is parallel to the fixed axe.—And if we put F, R, for the two ſemi-axes, of the elliptic baſe of this ſegment, reſpectively correſponding or parallel to f, r, the ſemi-axes of the generating ellipſe, when parallel to the baſe of the ſegment, and for $\frac{f}{r}$ and r ſubſtitute their values $\frac{\text{F}}{\text{R}}$ and $\frac{\text{RR}+hh}{2h}$, the ſaid fruſtum will be expreſſed by $\frac{1}{3}p\text{F}h \times \frac{3\text{RR}+hh}{2\text{R}}$ in which the dimenſions of itſelf only are concerned.

Corol. 3. A ſemi-ſpheroid is equal to $\frac{2}{3}$ of a cylinder, or to double a cone, of the ſame baſe and height; or they are in proportion as the numbers 3, 2, 1. For the cylinder is $= 4nfrr = \frac{12}{3}nfrr$, the ſemi-ſpheroid $= \frac{8}{3}nfrr$, and the cone $= \frac{4}{3}nfrr$.

Corol. 4. When $f = r$, the ſpheroid becomes a ſphere, and the expreſſion $\frac{8}{3}fnrr$ for the ſemi-ſpheroid becomes $\frac{8}{3}nr^3$ for the ſemi-

EXAMPLE.

Required the content of an oblate, and of an oblong, ſpheroid; the axes being 50 and 30.

Firſt, $50 \times 30 \times \cdot 78539816 = 1178 \cdot 09724 =$ the area of the ellipſe.

Then $1178 \cdot 09724 \times \frac{2}{3} \times 30 = 23561 \cdot 9448 =$ the oblong ſpheroid.

And $1178 \cdot 09724 \times \frac{2}{3} \times 50 = 39269 \cdot 908 =$ the oblate one.

ſemi-ſphere, as in prob. 24 ſect. 1.—And in like manner f and r being ſuppoſed equal to each other in the values of the fruſtums and ſegments of a ſpheroid, in the preceding corollaries, will give the values of the like parts of a ſphere.

Corol. 5. All ſpheres and ſpheroids are to each other as the fixed axes drawn into the ſquares of the revolving axes.

Corol. 6. Any ſpheroids, and ſpheres, of the ſame revolving axe, as alſo their like or correſponding parts cut off by planes perpendicular to the ſaid common axe, are to one another as their other or fixed axes. This follows from the foregoing corollaries.

Corol. 7. But if their fixed axes be equal, and their revolving axes unequal, the ſpheroids and ſpheres, with their like parts terminated by planes perpendicular to the common fixed axe, will be to each other as the ſquares of their revolving axes.

Corol. 8. An oblate ſpheroid is to an oblong ſpheroid, generated from the ſame ellipſe, as the longer axe of the ellipſe is to the ſhorter. For, if T be the tranſverſe axe, and C the conjugate; the oblate ſpheroid will be $= \frac{2}{3} n T^2 C$, and the oblong $= \frac{2}{3} n C^2 T$; and theſe quantities are in the ratio of T to C.

Corol. 9. And if about the two axes of an ellipſe, be generated two ſpheres and two ſpheroids, the four ſolids will be continual proportionals, and the common ratio will be that of the two axes of the ellipſe; that is, as the greater ſphere, or the ſphere upon the greater axe, is to the oblate ſpheroid, ſo is the oblate ſpheroid to the oblong ſpheroid, ſo is the oblong ſpheroid to the leſs ſphere, and ſo is the tranſverſe axe to the conjugate. For theſe four bodies will be as T^3, T^2C, TC^2, C^3, where each term is to the conſequent one, as T to C.

PROBLEM XII.

To find the Content of the Frustum of a Spheroid; its Ends being Perpendicular to one of the Axes, and one of them passing through the Center.

RULE I.

To the area of the less end, add twice that of the greater; multiply the sum by the altitude of the frustum, and $\frac{1}{3}$ of the product will be the content.— By corollary 1 to the last problem.

That is, $(2D^2 + d^2) \times \frac{1}{3}an =$ the frustum whose ends are perpendicular to the fixed axe. Where D is the diameter of the greater end, d that of the less, a the altitude, and $n = \cdot 785398$.

And $(2TC + tc) \times \frac{1}{3}an =$ the frustum whose ends are parallel to the fixed axe. Where T and C are the transverse and conjugate axes of the greater end, and t and c those of the less end.

Note. It is evident that the double of the frustum will give the content of the zone, or spheroidal cask.

EXAMPLE I.

There is a cask in the form of the middle frustum, or zone, of an oblong spheroid; the bung diameter is 30, the head diameter 18, and the length of the cask 40 inches; what is the content in ale and wine gallons?

Here $D = 30$, $d = 18$, and $a = 40$.

Therefore $(2D^2 + d^2) \times \frac{1}{3}an = (2 \times 30^2 + 18^2) \times \cdot 2618 \times 40 = 2124 \times 10\cdot 472 = 22242\cdot 528 =$ the content in inches.

Then, since the gallon ale measure contains 282 cubic inches, and the wine gallon 231, we have $22242\cdot 528 \div 282 = 78\cdot 874$ the ale gallons.

And $22242\cdot 528 \div 231 = 96\cdot 288$ the wine gallons.

EXAMPLE II.

If a veſſel, in the form of the middle fruſtum of an oblate ſpheroid, have the diameter of each end 40, in the middle 50, and its length 18 inches; what is its content in ale and wine gallons?

Here $(2D^2 + d^2) \times \frac{1}{3}an = (2 \times 50^2 + 40^2) \times \cdot 2618 \times 18 = 118800 \times \cdot 2618 = 31101 \cdot 84$ cubic inches.

Then $31101 \cdot 84 \div 282 = 110 \cdot 29$ ale gallons.

And $31101 \cdot 84 \div 231 = 134 \cdot 64$ wine gallons.

EXAMPLE III.

In the fruſtum of an oblong ſpheroid, the greater end is the generating ellipſe, whoſe axes are 50 and 30, the axes of its leſs end 40 and 24, and its height 9 inches; required the content in Wincheſter buſhels.

Here $(2TC + tc) \times \frac{1}{3}an = (2 \times 50 \times 30 + 40 \times 24) \times \cdot 2618 \times 9 = 9330 \cdot 552$ cubic inches.

But $268 \cdot 8$ inches = a gal. or $2150 \cdot 4$ a corn buſhel.

And therefore $9330 \cdot 552 \div 2150 \cdot 4 = 4 \cdot 339$ buſhels.

EXAMPLE IV.

In the fruſtum of an oblate ſpheroid, the greater end is the generating ellipſe, whoſe axes are 50 and 30, and the height is 20 inches; required the ſolidity.

Here, by the nature of the ellipſe, as $50 : 30 :: \sqrt{(25 + 20) \times (25 - 20)} = \sqrt{45 \times 5} = 15 : 9$ = the ſemi-conjugate axe of the leſs end.

And,

And, by prop. 1 sect. 2, as 30 : 50 :: 9 : 15 = the semi-transverse axe of the less end.

Then $(2TC + tc) \times \frac{1}{3}na = (2 \times 50 \times 30 + 30 \times 18) \times \cdot 2618 \times 20 = 118 \times 30 \times 20 \times \cdot 2618 = 70800 \times \cdot 2618 = 18535 \cdot 44$ cubic inches.

RULE II.

From 3 times the square of the semi-axe perpendicular to the ends of the frustum, subtract the square of the height of the frustum; then multiply the difference by $\frac{1}{3}$ of the height, and the product by 3·14159 &c; and call the last product P.

Then

1. *If the ends be parallel to the fixed axe,*

As the revolving axe is to the fixed axe, so will P be to the content of the frustum.

2. *When the ends are perpendicular to the fixed axe,*

As the square of the fixed axe is to the square of the revolving axe, so is P to the content of the frustum.

That is, $\frac{3ff - hh}{3ff} \times phrr$ will be the frustum whose ends are perpendicular to the fixed axe. And $\frac{3rr - hh}{3r} \times phf$ the frustum whose ends are parallel to the fixed axe: f being the fixed, and r the revolving semi-axe, h the height, and p = 3·14159.

EXAMPLE I.

If the axes of an oblong spheroid be 50 and 30, required the content of a frustum whose ends are perpendicular to the fixed axe, and one of them passing through the center, the height of the frustum being 20 inches.

Here

Here $\frac{3ff - hh}{3ff} \times phrr = \frac{3 \times 25^2 - 20^2}{3 \times 25^2} \times 3{\cdot}14159 \times 20 \times 15^2 = 1475 \times 3{\cdot}14159 \times \frac{12}{5} = 11121{\cdot}23799$ = the content required.

EXAMPLE II.

If from the same spheroid be cut a frustum whose height is 9, the ends being parallel to the fixed axe, and one of them passing through the center, what will be its content?

Here $\frac{3rr - hh}{3r} \times phf = \frac{3 \times 15^2 - 9^2}{3 \times 15} \times 9 \times 25 \times 3{\cdot}14159 = 594 \times 5 \times 3{\cdot}14159 = 9330{\cdot}53018$ = the content required.

EXAMPLE III.

If from an oblate spheroid, whose axes are 50 and 30, a frustum, whose height is 9, and its ends perpendicular to the fixed axe, be cut; what will be its solidity?

Here $\frac{3ff - hh}{3ff} \times phrr = \frac{3 \times 15^2 - 9^2}{3 \times 15^2} \times 9 \times 25^2 \times 3{\cdot}14159 = 594 \times \frac{25}{3} \times 3{\cdot}14159 = 15550{\cdot}88363$ the content required.

EXAMPLE IV.

In the same spheroid, it is required to find the content of a frustum whose height is 20, the ends being parallel to the fixed axe.

Here $\frac{3rr - hh}{3r} \times phf = \frac{3 \times 25^2 - 20^2}{3 \times 25} \times 20 \times 15 \times 3{\cdot}14159 = 1475 \times 4 \times 3{\cdot}14159 = 18535{\cdot}39665$ the content required.

PRO-

PROBLEM XIII.

To find the Content of a Spheroidal Cask, not full, standing upon its End, the Axe being Perpendicular to the Horizon. That is, of the Frustum of an Oblong Spheroid, the Ends being Perpendicular to the Axe, but neither of them passing through the Center.

Multiply the difference of the squares of the diameters of the ends, by 4 times the square of the difference between the height of the liquor and half the length of the cask, and divide the product by the square of the length of the cask; subtract the quotient from 3 times the square of the bung diameter, and multiply the remainder by the aforesaid difference between the height of the liquor and $\frac{1}{2}$ the length of the cask; then the product multiplied by ·261799 will give the quantity by which the cask is more or less than half full; and which, therefore, being added to, or taken from, half the content of the cask, will give the content of the part filled.

That is, $[(2b^2 + h^2)\,\frac{1}{2}l \pm d(3b^2 - \frac{4dd}{ll}(b^2 - h^2)]\times\frac{1}{3}n$ = the content of the part filled, called the ullage; using the upper or under signs, according as the cask is more or less than half full; where b and h = the bung and head diameters, l = the length of the cask, n = ·785398, and $d = \frac{1}{2}l \sim w$, w being the wet part of l, or the height of the liquor in the cask.*

EX-

* DEMONSTRATION.

For, if D be the diameter at the surface of the liquor, by the property of the ellipse $b^2 - h^2 : \frac{1}{4}l^2 :: b^2 - D^2 : d^2$; and hence $D^2 = b^2 - \frac{4dd}{ll} \times (b^2 - h^2)$.

But, by the last problem, $(2b^2 + h^2) \times \frac{1}{6}ln$ = half the content

EXAMPLE I.

If a spheroidal cask, whose head and bung diameters are 18 and 30 inches, and length 40 inches, be filled to the height of 30 inches; how many ale and wine gallons are in it?

Here $b = 30$, $b = 18$, $l = 40$, $w = 30$, and $d = \frac{1}{2}l \backsim w = 30 - 20 = 10$.

Then $[3b^2 - \frac{4dd}{ll}(b^2 - b^2)] \times d = (2700 - \frac{400}{1600} \times 48 \times 12) \times 10 = (2700 - 144) \times 10 = 25560$.

And $(2b^2 + b^2) \times \frac{1}{2}l = (1800 + 18 \times 18) \times 20 = 42480$.

Consequently $(42480 + 25560) \times \frac{1}{3}n = 68040 \times \cdot 2617993878 = 17812 \cdot 8303458$ is the content in inches; and being divided by 282 and 231, gives 63·166065 ale, and 77·111819 wine gallons.

EXAMPLE II.

If the height of the liquor in the same cask be only 10 inches, required the ullage.

Here $d = \frac{1}{2}l \backsim w = 20 - 10 = 10$ the same as before, and therefore the part which in the last example was added, must here be subtracted; so that $(42480 - 25560) \times \frac{1}{3}n = 16920 \times \cdot 2617993878 = 4429 \cdot 6456415$ is the content in inches $= 15 \cdot 707963$ ale, and 19·175955 wine gallons.

PRO-

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tent of the cask, and $[3b^2 - \frac{4dd}{ll}(b^2 - b^2)] \times \frac{1}{3}dn =$ the frustum by which the part filled is more or less than half the cask.

Consequently $[(2b^2 + b^2)\frac{1}{2}l \pm (3b^2 - \frac{4dd}{ll} \times (b^2 - b^2)d)] \times \frac{1}{3}n =$ the part filled.

PROBLEM XIV.

To find the Solidity of the Segment of a Spheroid whose Base is Perpendicular to one of the Axes.

RULE I.

From 3 times the semi-axe perpendicular to the base of the segment, take the height of the segment, multiply the remainder by $\frac{1}{3}$ of the square of the height; then multiply the product by 3·14159, and call the last product Q.

Divide the axe which is parallel to the base by the other axe, and call the quotient q.

Then for the segment whose base is perpendicular to the fixed axe, multiply Q by the square of q; and for the other segment, multiply Q by q.

That is, $Qqq = \frac{3f-h}{3ff} \times prrhh =$ the segment whose base is perpendicular to the fixed axe.

And $Qq = \frac{3r-h}{3r} \times pfhh =$ the other segment, whose base is parallel to the fixed axe.

Where h is the height of the segment, and the other symbols as in the 12th problem.

EXAMPLE. I.

If from an oblong spheroid, whose axes are 50 and 30, be cut a segment whose base is perpendicular to the fixed axe, its height being 5; required the content of it.

Here $\frac{3f-h}{3ff} \times prrhh = \frac{3\times 25-5}{3\times 25^2} \times 15^2 \times 5^2 \times 3{\cdot}14159$ $= 70 \times 3 \times 3{\cdot}14159 = 659{\cdot}73445 =$ the content required.

EX-

EXAMPLE II.

If from the ſame ſpheroid be cut a ſegment whoſe height is 6, what will its content be, ſuppoſing its baſe to be parallel to the fixed axe?

Here $\frac{3r-h}{3r} \times pfhh = \frac{3 \times 15 - 6}{3 \times 15} \times 25 \times 6^2 \times 3{\cdot}14159$ $= 39 \times 20 \times 3{\cdot}14159 = 2450{\cdot}44226 =$ the content required.

EXAMPLE III.

Required the ſolidity of the ſegment of an oblate ſpheroid, whoſe axes are 50 and 30; the height of the ſegment being 6, and its baſe perpendicular to the fixed axe.

Here $\frac{3f-h}{3ff} \times prrhh = \frac{3 \times 15 - 6}{3 \times 15 \times 15} \times 25 \times 25 \times 6 \times 6 \times 3{\cdot}14159 = 1300 \times 3{\cdot}14159 = 4084{\cdot}07044$ $=$ the content required.

EXAMPLE IV.

To find the content of a ſegment of the ſame ſpheroid, its baſe being parallel to the fixed axe, and its height 5.

Here $\frac{3r-h}{3r} \times pfhh = \frac{3 \times 25 - 5}{3 \times 25} \times 15 \times 5 \times 5 \times 3{\cdot}14159$ $= 70 \times 5 \times 3{\cdot}14159 = 1099{\cdot}55742 =$ the content required.

RULE II.

For the Segment whoſe Baſe is Parallel to the Fixed Axe; having given its Height and the Diameters of its End.

To the ſquare of the height add 3 times the ſquare of the leſs or greater ſemi-axe of the baſe, according

according as it is the ſegment of an oblong or oblate ſpheroid; divide the ſum by the ſame ſemi-axe, and multiply the quotient by the other ſemi-axe, the product by the height, and this product by ·52359, will give the content of the ſegment.

That is, $\frac{3A^2 + b^2}{6A} \times pBb =$ the content of the ſegment.

Where b is its height, A the leſs or greater ſemi-diameter of the baſe, according as the ſegment is that of an oblong or oblate ſpheroid; B is the other ſemi-diameter, and $p = 3·14159$.

EXAMPLE I.

It is required to find the content of the ſegment of an oblong ſpheroid, whoſe baſe is parallel to the fixed axe; its height being 6, and the axes of its elliptic baſe 40 and 24.

Here $\frac{3AA + bb}{6A} \times pBb = \frac{3 \times 12^2 + 6^2}{6 \times 12} \times 20 \times 6 \times 3·14159$ $= 13 \times 60 \times 3·14159 = 2450·44226$ the content required.

EXAMPLE II.

Required the content of the ſegment of an oblate ſpheroid, whoſe baſe is parallel to the fixed axe; its height being 5, and the diameters of its baſe 18 and 30.

Here $\frac{3A^2 + b^2}{6A} \times pBb = \frac{3 \times 15^2 + 5^2}{6 \times 15} \times 9 \times 5 \times 3·14159$ $= 70 \times 5 \times 3·14159 = 1099·55742$ the content required.

RULE

RULE III.

For a Segment whose Base is Perpendicular to the Fixed Axe; having given the Height, the Diameter of its Base, and a Diameter in the Middle between its Base and Vertex.

To the square of the diameter of the base, add 4 times the square of the diameter in the middle, or the square of twice this diameter; multiply the sum by the height, and the product again by ·1308996 for the content.

That is, $(D^2 + 4d^2) \times \frac{1}{6}nh =$ the content of the segment; D being the diameter of the base, d the diameter in the middle, h the height, and $n =$ ·785398.*

Z EX-

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* DEMONSTRATION.

By rule I the segment is $= 4nrrhh \times \frac{3f-h}{3ff}$. But, by the nature of the ellipse, D is $= \frac{2r\sqrt{2fh-hh}}{f}$, and $d = \frac{2r\sqrt{fh-\frac{1}{4}hh}}{f}$; hence $f = h \times \frac{4dd - DD}{8dd - 4DD}$, and $\frac{r}{f} = \frac{\sqrt{4dd - 2DD}}{2h}$; which values being substituted in the above value of the segment, gives $\frac{1}{6}nh \times (DD + 4dd)$ for the value of the segment in terms of D, d, and h.

SCHOLIUM.

This theorem I have investigated in order to express the value of this segment independent of the axes of the spheroid. For this purpose I was under a necessity of introducing another dimension of the segment, because it is not determinable from its base and height alone, as the other segment was.

EXAMPLE I.

What is the content of the ſegment of an oblong ſpheroid, whoſe baſe is perpendicular to the fixed axe; its height being 5, the diameter of its baſe 18, and its middle diameter $3\sqrt{19}$?

Here $(DD + 4dd) \times \frac{1}{6}nh = [18^2 + (6\sqrt{19})^2] \times \frac{5}{6} \times \cdot785398 = (3^2 + 19) \times 30 \times \cdot785398 = 210 \times 3\cdot14159 = 659\cdot73445$ the content required.

EXAMPLE II.

Required the content of the ſpheroidal ſegment whoſe height is 6, its baſe diameter 40, and diameter in the middle 30; the baſe being perpendicular to the fixed axe of the ſpheroid.

Here $(DD + 4dd) \times \frac{1}{6}hp = (40^2 + 60^2) \times \frac{6}{6} \times \cdot785398 = (2^2 + 3^2) \times 20^2 \times \cdot785398 = 1300 \times 3\cdot14159 = 4084\cdot07044$ the content required.

PROBLEM XV.

To find the Content of the Second Segment of a Spheroid.

As a ſphere is to the ſpheroid, ſo is any part of the ſphere to the like part of the ſpheroid.*

PRO-

* DEMONSTRATION.

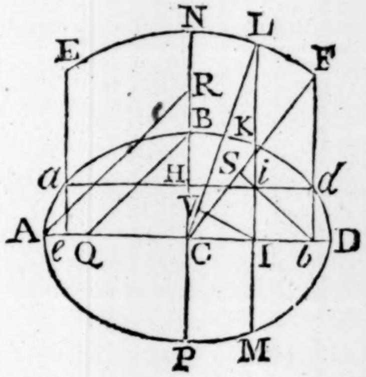
For the like parts of any quantities are as the wholes. And that the ſecond ſegments, FGH, *fgh*, are like parts of the ſphere and ſpheroid, is evident from the nature of the figures.

Corol.

PROBLEM XVI.

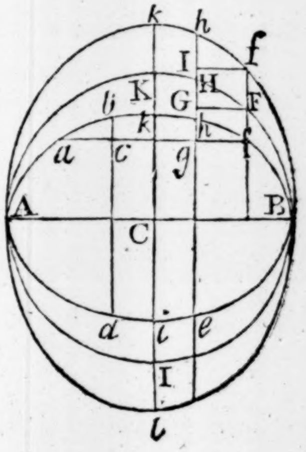
To find the Surface of an Elliptic Spindle; or of any Frustum or Segment of it.

GENERAL RULE.



From the spheroidal surface, generated by any arc BK of the ellipse, take the product of the same arc BK, and the circumference of the circle whose diameter is equal to twice CH, the distance of the centers of the spheroid and spindle; and the remainder will be

 the



Corol. 1. And if the sphere AKBI and spheroid A*k*B*i* have one axe AB common; then the whole solids and the second segments FGH, *fgh*, will be to each other, as the squares of the other axes KI, *ki*, if the common axe AB be the fixed one. Or they will be to each other, as the other axes KI, *ki*, simply, if the common axe be the revolving one. For, if A be the axe of the sphere, F and R the fixed and revolving axes of the spheroid; the bodies will be to each other as A^3 to FR^2; hence if $A = F$, they will be as A^2 to R^2; but if $A = R$, they will be as A to F.

Corol. 2. Hence may be found the true quantity of liquor in a spheroidal cask, not full, whose axe is parallel to the horizon.

For if from the segment *akf* be taken the double of the second segment *fgh*, there will remain the part *bhgc*; which taken from the whole cask *bhed* will leave the part *cdeg*.

And after the same manner may be found the quantity of liquor in a spheroidal cask partly filled, and standing a-tilt, with its axe inclined to the horizon.

the ſurface of the part of the ſpindle generated by the arc BK about ad parallel to AD the fixed axe of the ſpheroid.*

EXAMPLE I.

Given the axes of an ellipſe 50 and 40, to find the ſurface of the ſpindle generated from an arc of that ellipſe, the length of the ſpindle being 30.

Here $AD = 50$, $BP = 40$, $ad = eb = 30$, $Ae = AC - eC = 25 - 15 = 10$, and $eD = CD + Cb =$

25

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* DEMONSTRATION.

For the fluxion $\dot{s}$ of the ſpindular ſurface is $= \dot{z}$ the fluxion of the arc BK drawn into $iK \times 2p = 2p\dot{z} \times (IK - Ii)$; and the fluent is equal to the ſpheroidal ſurface $BKMP - 2pz \times Ii$ or CH.

Corol. 1. The ſpindular ſurface generated by EK, is equal to the ſpheroidal ſurface BKMP — the ſurface of a ſphere whoſe axe is a mean proportional between BK and 2CH. For $2pz \times CH$ = that ſpheric ſurface.

Or $2pz \times CH$ = a circle whoſe radius is $\sqrt{2CH \times BK}$.

Corol. 2. When CH is equal to nothing, the ſpindle becomes barely a ſpheroid: And when H falls below C, the ſurface of the ſphere muſt be added.

What has been hitherto done, anſwers to the ſurface generated by an arc about a line parallel to either axe of the ellipſe.

Corol. 3. From B the extremity of the leſs axe of the ellipſe, apply BQ = AC the ſemi-tranſverſe; and parallel thereto draw AR meeting CB, produced in R; with the center C and radius AR deſcribe the arc EF meeting the parallels aE, CB, IK, bd in E, N, L, F: Draw CL and CF, and perpendicular to them IV and bS.

Then the circle whoſe radius is a mean proportional between the ſum and difference of two lines, of which the one is a mean proportional

$25+15 = 40$. Then as AC : CB :: $\sqrt{\text{A}e \times e\text{D}}$: ea = CH = 16, the distance of the centers of the ellipse and spindle.

By example 1 to problem 10, the spheroidal surface generated by the arc ad, is 3686·8197024; and by problem 4, the length of the elliptic arc ad is 30·852.

Therefore $3686{\cdot}8197 - 30{\cdot}852 \times 3{\cdot}1416 \times 32 = 3686{\cdot}8197 - 3101{\cdot}5872 = 585{\cdot}2325$ is the surface of the spindle.

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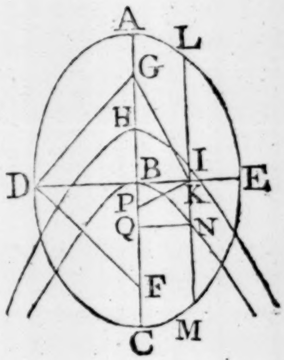
proportional between BC and IV + LN, and the other a mean proportional between BK and 2CH, will be equal to the spindular surface generated by BK about ad parallel to the transverse axe. For the spheroidal surface is equal to a circle whose radius is a mean proportional between BC and IV + LN, by rule 4 to prob. 9; and $2p \times$ BK $\times$ CH is equal to a circle whose radius is a mean proportional between BK and 2CH; but the spindular surface is equal to the difference of those two quantities; and the difference of two circles is equal to a circle whose radius is a mean proportional between the sum and difference of the radii of the two circles; therefore, &c.

And, in the same manner, the surface of half the spindle, generated by the arc Bd, is equal to a circle whose radius is a mean proportional between the sum and difference of two lines, the one of which is a mean proportional between BC and bs + NF, and the other a mean proportional between Bd and 2CH.

Corol. 4. In like manner, for the surface of the spindle whose axe is parallel to the less axe of the ellipse: having constructed the annexed figure, as in page 309, the spindular surface generated by the arc CM, about NQ, will be equal to a circle whose radius is a mean proportional between the sum and difference of two lines, of which the one is a mean proportional between AC and HI, and the other a mean proportional between CM and 2BQ.



This, like the last corollary, will be evident by comparing what is done above, with what is done in page 309.

EXAMPLE II.

Required the ſurface of the fruſtum of a ſpindle generated from an arc of the ſame ellipſe as in the laſt example; the height of the fruſtum being 15, and the central diſtance 10.

Here the arc generating the fruſtum, is equal to half the arc in the laſt; and conſequently that arc is = 15·426, and the ſpheroidal fruſtum = 1843·40985.

Then 1843·4098 − 15·426 × 3·1416 × 20 = 1843·4098 − 969·246 = 874·163 the ſurface of the fruſtum required.

PROBLEM XVII.

To find the Solidity of an Elliptic Spindle.

RULE I.

1. Divide the ſquare of the perpendicular axe DE by 3 times the ſquare of the parallel axe AB, and multiply the quotient by the cube of FG the axe or length of the ſpindle; and call the product P.

2. Find the area of the elliptic ſegment FDG from which the ſpindle is generated; multiply this area by 4 times CH the central diſtance, and call the product Q.

3. Multiply 1·57079 by the difference between P and Q, and the product will be the content of the ſpindle FDGN.

That

That is, $2n \times (\frac{aa}{3bb}l^3 - 4cs) =$ the ſpindle FDGN. Where $n = \cdot 78539$, $a =$ DE, $b =$ AB, $l =$ FG, $c =$ CH, and s $=$ the ſegment FDGF.*

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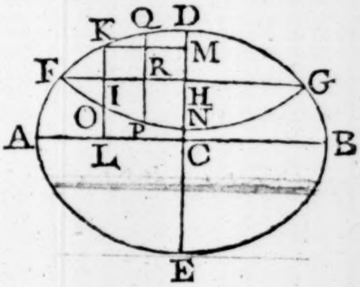
* DEMONSTRATION.



Let FDGN be a ſpindle generated by the arc FDG of the ellipſe ADBE. Put $a =$ FH $=$ half the axe of the ſpindle; $b =$ CH $=$ LI $=$ the central diſtance, or diſtance of the centers of the ellipſe and ſpindle; $c =$ AC $=$ the one ſemi-axe of the ellipſe, and $d =$ CD $=$ the other ſemi-axe; $x =$ HI $=$ CL; and $y =$ IK $=$ HM.

By the property of the ellipſe, $c : d :: \sqrt{cc-xx} : \frac{d}{c}\sqrt{cc-xx} =$ LK, hence IK $=$ KL $-$ LI $= \frac{d\sqrt{cc-xx}}{c} - b = y$, and the fluxion of the ſolid $\dot{s} = pyy\dot{x} = p\dot{x} \times (dd - \frac{ddxx}{cc} - \frac{2bd\sqrt{cc-xx}}{c} + bb)$ $= p\dot{x} \times (dd - bb - \frac{ddxx}{cc} - 2b \times \frac{d\sqrt{cc-xx}}{c} - 2b^2) = pdd\dot{x} \times \frac{aa-xx}{cc} - 2pby\dot{x}$; and the fluent is s $= pddx \times \frac{3aa-xx}{3cc} - 2bp \times$ area HIKD $=$ the fruſtum DKON.

When x is $= a$, the above theorem will become $\frac{2pdd}{3cc} \times a^3 - 2bp \times$ area FDH $=$ DFN the half of the ſpindle.

And if from the ſemi-ſpindle be taken the fruſtum, there will remain $pddee \times \frac{3a-e}{3cc} - 2bp \times$ area FKI $=$ the ſegment FKO of the ſpindle, e being the height FI of the ſegment.

Corol. 1. If d be ſuppoſed $= c$, the ellipſe will become a circle, and accordingly the theorems above given will become the ſame with thoſe before found for the circular ſpindle.

EXAMPLE.

The axes of an ellipse being 50 and 30, required the solidity of a spindle generated from an arc of it, about

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Corol. 2. If H coincide with C, b will vanish, a will be = c, and the theorems will become the same with those before found for the spheroid.

Corol. 3. Putting D = DN the greatest, and d = KO the least diameter of the frustum, h = HI its height, C = CH the central distance, s = the elliptic semi-segment KDM, and n = ·78539 &c. Then, in the foregoing theorems, $h = x$, $b =$ C, $d =$ C + $\frac{1}{2}$D, area HIKD = $s + \frac{1}{2}dh$, $c = h \times \frac{C + \frac{1}{2}D}{\sqrt{\frac{1}{4}DD + DC - \frac{1}{4}dd - dC}}$, and $a = h\sqrt{\frac{\frac{1}{4}DD + DC}{\frac{1}{4}DD + DC - \frac{1}{4}dd - dC}}$; which values being substituted in the theorems above, give $\frac{1}{3}nh \times [2DD + dd - 8C(-D + d + \frac{3s}{h})]$ for the value of the frustum, or half a cask in the form of the middle zone of an elliptic spindle; and $\frac{1}{3}nl \times [2DD - 8C(-D + \frac{3s}{l})]$ for half the spindle when s = the area DFH, and l = HF.

Corol. 4. But in real practice, such as cask gauging, none of these rules can be used, because we have not given either the axes of the ellipse or the central distance; and to accommodate rules to that purpose, we must introduce another dimension of a frustum, besides its length and greatest and least diameters. Thus, putting m = PQ the diameter through R the middle of the length HI, and the other letters as before. Then, by the property of the ellipse, $(C + \frac{1}{2}D)^2 - (C + \frac{1}{2}d)^2 : 4 :: (C + \frac{1}{2}D)^2 - (C + \frac{1}{2}m)^2 : 1$, hence $4(C + \frac{1}{2}m)^2 - 3(C + \frac{1}{2}D)^2 = (C + \frac{1}{2}d)^2$, and $C = \frac{1}{4} \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}$. And the last two theorems become $\frac{1}{3}nh \times [2D^2 + d^2 - 2 \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}(-D + d + \frac{3s}{h})]$ for the frust. DKON, and $\frac{1}{3}nl \times [2D^2 - 2 \times \frac{3D^2 - 4m^2}{-3D + 4m}(-D + \frac{3s}{l})]$ for the semi-spindle DFN.

about its chord parallel to, and at the distance of, 9 from the transverse axe.

Here $15 - 9 = 6 =$ DH the height of the generating segment; and, by the property of the ellipse, DC : AC :: $2\sqrt{\text{DH} \times \text{HE}}$: FG $= 40$, the base of the elliptic segment, or length of the spindle; also, by prob. 6, the area of the segment is 167·7345.

Then $\frac{3\cdot14159}{2} \times (\frac{30 \times 30}{3 \times 50 \times 50} \times 40^3 - 4 \times 9 \times 167\cdot7345)$ $= 18 \times 3\cdot14159 \times (213\frac{1}{3} - 167\cdot7345) = 2578\cdot55$ = the content required.

RULE II.

Divide 3 times the generating segment by the length of the spindle; from the quotient subtract the greatest diameter of the spindle; multiply the remainder by 4 times the central distance, and subtract the product from the square of the greatest diameter; then the difference multiplied by the length of the spindle, and the product by ·5236 will give the content of the spindle.

That is, $\frac{2}{3}nl \times [\text{D}^2 - 4c(-\text{D} + \frac{3s}{l})] =$ the spindle.

Where D = DN the greatest diameter, and the rest of the symbols as in the last rule.—By corollary 3.

EXAMPLE.

Required the solidity of an elliptic spindle, whose length is 40, greatest diameter 12, and the central distance 9.

Here $\frac{12}{2} = 6 =$ the height of the generating segment, which is therefore the same as before, the area being 167·7345.

Then

Then

$$3.14159 \times \frac{40}{6} \times [12^2 - 4 \times 9 \times (-12 + \frac{3 \times 167.7345}{40})]$$

$= 3.14159 \times 80 \times (12 - 1.74024) = 2578.56$ the content required.

RULE III.

From three times the ſquare of the greateſt diameter, take 4 times the ſquare of the diameter in the middle between the greateſt diameter and the end; and from 4 times the ſaid middle diameter, take 3 times the ſaid greateſt diameter: Divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central diſtance.

That is, $\frac{1}{4} \times \frac{3DD - 4mm}{4m - 3D} =$ the central diſtance; D being the greateſt diameter, and m the middle diameter.

Then proceed as in the laſt rule.

This is proved in corollary 4.

EXAMPLE.

If, as in the example to the laſt rule, the greateſt diameter be 12, and the length 40, required the content, ſuppoſing the diameter at $\frac{1}{4}$ of the length to be $6 \times (\sqrt{21} - 3)$ or 9.49546.

Here $\frac{1}{4} \times \frac{3DD - 4mm}{4m - 3D} = \frac{1}{4} \times \frac{3 \times 12^2 - 4 \times 6^2(\sqrt{21} - 3)^2}{4 \times 6(\sqrt{21} - 3) - 3 \times 12}$ $= \frac{12}{4} \times \frac{3 - 21 + 6\sqrt{21} - 9}{2\sqrt{21} - 6 - 3} = 3 \times \frac{6\sqrt{21} - 27}{2\sqrt{21} - 9} = 3 \times 3 = 9$ $=$ the central diſtance, the ſame as in the laſt example; and therefore the content will be 2578.56, as before.

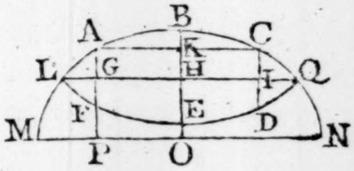
PRO-

PROBLEM XVIII.

To find the Solidity of the Frustum of an Elliptic Spindle, or the Content of a Cask in form of the Middle Zone of such a Spindle.

RULE I.

1. Divide the square of the less axe of the ellipse by the square of the greater; multiply the quotient by the length of the frustum, or half the length of the cask; then multiply the product by the difference between the square of the said half length of the cask, and 3 times the square of half the length of the whole spindle; and call $\frac{1}{3}$ of the product P.



That is, $P = ccl \times \frac{3LL - ll}{3tt}$, t being the transverse MN, and c the conjugate axe 2BO of the ellipse, l = GH half the length of the cask, and L = LH half the length of the whole spindle.

2. Multiply the central distance OH by the generating area ABHG; and call double the product Q.

That is, Q = 2OH × ABHG.

3. Then the difference between P and Q multiplied by 3·14159, will give the content of the frustum ABEF, or half the cask ACDF.

That is, (P − Q) × p = ABEF = $\frac{1}{2}$ACDF.

This follows from the demonstration of the last problem.

Note. It is evident that the sum or difference of two frustums, will give the ullage of a cask standing with its axe perpendicular to the horizon.

EX-

EXAMPLE.

The axes of an ellipse being 40 and $66\frac{2}{3}$; it is required to find the content of the frustum of a spindle generated by an arc of the ellipse; the length of the frustum being 20, and the central distance 4.

Here, as $40 : 66\frac{2}{3} :: \sqrt{16 \times 24} : 13\frac{1}{3}\sqrt{6} =$ LH half the length of the spindle.

And $ccl \times \frac{3\text{LL} - ll}{3tt} = 40 \times 40 \times 20 \times \frac{3 \times 6 \times 13\frac{1}{3} \times 13\frac{1}{3} - 20 \times 20}{3 \times 66\frac{2}{3} \times 66\frac{2}{3}} = 24 \times 280 = 6720 =$ P.

But the elliptic segment ABK is $= 54\frac{1}{2}$, to which adding the rectangle AH $= 20 \times 12 = 240$, makes $294\frac{1}{2} =$ the generating area ABHG; which being multiplied by 2OH or 8, produces $2356 =$ Q.

Then $(\text{P} - \text{Q}) \times p = (6720 - 2356) \times 3{\cdot}14159 = 13709{\cdot}91033$ the content required.

RULE II.

Divide 3 times the elliptic segment whose chord is the length of the cask, by the said length; to the quotient add the least or head diameter; and from the product subtract the greatest or bung diameter; and multiply the remainder by 8 times the central distance; then take the product from the sum of the square of the head diameter, and double the square of the bung diameter; multiply the difference by the length, and the product again by $\cdot 2618$ for the content of the cask.

That is, $[2\text{DD} + dd - 8c(-\text{D} + d + \frac{3s}{l})] \times \frac{1}{3}nl =$ the content of the cask; where D = BE the bung diameter, $d =$ AF the head diameter, $c =$ HO the central

central diſtance, l = GI the length, s = the ſegment ABC, and $n = \cdot 785398$.

This is proved at corollary 3 to the laſt problem.

EXAMPLE.

The bung and head diameters of a caſk, which is the middle zone of an elliptic ſpindle, being 32 and 24, and its length 40 inches; required the content in ale and wine gallons, ſuppoſing the diſtance of the centers of the ellipſe and ſpindle to be 4 inches.

Here D = 32, $d = 24$, $l = 40$, $c = 4$, and the ſegment ABC = s = 109.

Then $(2\text{DD} + dd - 8c(-\text{D} + d + \frac{3s}{l})] \times \frac{1}{3}nl =$ $(2624 - 5\cdot6) \times 40 \times \cdot261799 = 27419\cdot8219$ cubic inches.

Which being divided by 282, and 231, we have 97·2334 ale gallons, and 118·7005 wine gallons, for the content required.

RULE III.

From the ſum of the ſquare of the leaſt or head diameter, and 3 times the ſquare of the greateſt or bung diameter, take 4 times the ſquare of the diameter equidiſtant from the two former; and from 4 times the ſaid middle diameter, take the ſum of the ſaid leaſt and 3 times the greateſt diameter; then divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central diſtance; with which proceed as in the 2d rule.

That is, $\frac{1}{4} \times \frac{3\text{DD} + dd - 4mm}{-3\text{D} - d + 4m}$ is the central diſtance; m being the middle diameter, and D and d the other diameters, as before.

By corollary 4 to the laſt problem.

EXAMPLE.

If the bung and head diameters be 32 and 24, and the length 40 inches, as in the last example; required the content, supposing the middle diameter to be $4\sqrt{91} - 8$.

Here $\frac{3DD + dd - 4mm}{-3D - d + 4m} \times \frac{1}{4} = \frac{3 \times 22^2 + 24^2 - 64(\sqrt{91} - 2)^2}{-3 \times 32 - 24 + 16(\sqrt{91} - 2)} \times \frac{1}{4} = \frac{-76 + 8\sqrt{91}}{-19 + 2\sqrt{91}} = 4 = c$ the central distance, the same as in the last example; and since the other parts are all the same, the content must likewise be the same.

PROBLEM XIX.

To find the Content of the Segment of an Elliptic Spindle.

1. Multiply the square of the altitude of the segment by the square of the less axe of the ellipse, and divide the product by the square of the greater axe; multiply the quotient by the difference between $\frac{1}{2}$ the length of the whole spindle, and $\frac{1}{3}$ of the altitude of the segment, and call the product P.

That is, $P = aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{tt}$; where c is the conjugate, and t the transverse axe of the ellipse, a the altitude of the segment, and l the length of the whole spindle.

2. Multiply double the generating area by the distance of the centers of the ellipse and spindle, and call the product Q.

That is, $Q = 2CA$, where $C = OH$, and $A =$ the area LAG.

3. Then

3. Then the difference between P and Q drawn into 3·14159, will be the content of the segment LAF of the spindle.

That is, $(P - Q) \times p = (aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{tt} - 2AC) \times p$ = the content of the segment.

By the demonstration of problem 17.

EXAMPLE.

The axes of an ellipse being 50 and 30, required the content of the segment of a spindle, whose height is 10, and the central distance 9.

Here $t = 50$, $c = 30$, $a = 10$, and $C = 9$.

Then, by the nature of the ellipse, as $30 : 50 :: \sqrt{(15 + 9) \times (15 - 9)} = \sqrt{24 \times 6} = 12 : 20 = \frac{1}{2}l =$ LH half the length of the spindle.

And, by the same, as $50 : 30 :: \sqrt{MP \times PN} = \sqrt{(25 + 10) \times (25 - 10)} = \sqrt{35 \times 15} = 5\sqrt{21} : 3\sqrt{21} = 13·71495534 =$ AP. Hence BO — AP $= 1·285 =$ BK.

But $\frac{BK}{2BO} = \frac{1·285}{30} = ·0428\frac{1}{3}$, and $\frac{BH}{BO} = \frac{6}{30} = \frac{1}{5} = ·2$; the tabular numbers answering to which two quotients, in the table of circular segments, are ·01166678 and ·1118238; hence $50 \times 30 \times (·1118238 - ·01166678) = 1500 \times ·10015702 = 150·23553 =$ the area LBQ; from the half of which taking the rectangle AH = AG × GH = $4·714955 \times 10 = 47·149$, leaves $27·968765 = A$ for the generating area LAG.

Then

Then $(aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{tt} - 2\text{AC}) \times p =$ $(100 \times 900 \times \frac{20 - 3\frac{1}{3}}{2500} - 18 \times 27{\cdot}968765) \times 3{\cdot}14159$ $= (36 \times \frac{50}{3} - 18 \times 27{\cdot}968765) \times 3{\cdot}14159 = 18$ $\times 5{\cdot}364568 \times 3{\cdot}14159 = 303{\cdot}35917 =$ the content of the ſegment LAF required.

PROBLEM XX.

To find the Content of an Univerſal Spheroid, or a Solid conceived to be Generated by the Revolution of a Semi-Ellipſe about its Diameter, whether that Diameter be one of the Axes of the Ellipſe or not.

RULE I.

Divide the ſquare of the product of the axes of the ellipſe, by the axe of the ſolid, or the diameter about which the ſemi-ellipſe is conceived to revolve; multiply the quotient by ·5236, and the product will be the content required.

That is, $\frac{\text{T}^2\text{C}^2}{d} \times {\cdot}5236 =$ the content; T and C being the tranſverſe and conjugate axes of the ellipſe, and d the axe of the ſolid.

RULE II.

The continual product of ·5236, the diameter about which the revolution is made, the ſquare of its conjugate diameter, and the ſquare of the ſine of the angle made by thoſe diameters, the radius being 1, will be the content.

That

That is, $dccss \times \cdot 5236 =$ the content; c being the conjugate diameter to d, and s the fine of the angle made by the diameters.*

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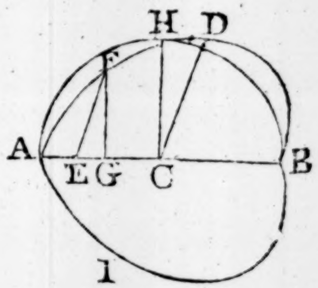
* DEMONSTRATION.



Let ADBIA be a fection of the folid through its axe AB; draw DC the femi-conjugate diameter to AB, as alfo the ordinate FE parallel to it, and let fall the perpendicular FG.

Put T and C for the tranfverfe and conjugate axes of the ellipfe, $d =$ the diameter AB, or axe of the folid, $c = 2\text{CD} = \sqrt{\text{T}^2 + \text{C}^2 - d^2}$ its conjugate diameter, $a = \dfrac{\text{TC}}{dc} =$ fine of the $\angle$DCB or $\angle$FEG, and $b =$ its cofine, $x =$ AE, and $p =$ 3·14159.

Then $d : c :: \sqrt{dx - xx} : \dfrac{c\sqrt{dx - xx}}{d} =$ EF, hence

$\text{EF} \times a = \dfrac{ac\sqrt{dx - xx}}{d} = \text{FG}$, and $\text{EF} \times b = \dfrac{bc\sqrt{dx - xx}}{d} = \text{GE}$;

then $\text{AG} = \text{AE} + \text{EG} = x + \dfrac{bc\sqrt{dx - xx}}{d}$, confeq. $p \times \text{FG}^2 \times \text{GA} =$

$\dfrac{paacc\dot{x}}{dd} \times (dx - xx + \dfrac{\frac{1}{2}d - x}{d} \times bc\sqrt{dx - xx}) =$ the fluxion of the folid; the fluent of which is

$\dfrac{paacc}{dd} \times [\frac{1}{2}dx^2 - \frac{1}{3}x^3 + \dfrac{bc}{3d} \times (dx - xx)^{\frac{3}{2}}] =$ the meafure of the part generated by AFG. And when $x = d$, it becomes $\frac{1}{6}pda^2c^2 = \dfrac{p\text{T}^2\text{C}^2}{6d}$ for the value of the whole folid.

Corol. 1. If $d = \text{T}$, the rule becomes $\frac{1}{6}p\text{TC}^2$ for the oblong fpheroid. And if $d = \text{C}$, it will be $\frac{1}{6}p\text{CT}^2$ for the oblate fpheroid. Alfo if T, C, and d, be all equal, the rule will be $\frac{1}{6}pd^3$ for the fphere. Which are the fame with the rules before found for the fame bodies.

Corol.

EXAMPLE I.

If the axes of an ellipſe be 50 and 30, and it be cut in two by a diameter whoſe length is 40; required the content of the ſolid generated by one of the halves about that diameter.

By rule 1, $\frac{50 \times 50 \times 30 \times 30}{40} \times \cdot 5236 = 225000 \times \cdot 1309$ $= 29452 \cdot 5$ is the content required.

EXAMPLE II.

The diameter of an ellipſe about which it revolves being 40, its conjugate diameter $30\sqrt{2}$, and the ſine of the angle made by thoſe diameters $\frac{5}{8}\sqrt{2}$; required the content of the ſolid formed by the revolution of the ellipſe.

By rule 2, we have $40 \times 30 \times 30 \times 2 \times \frac{5}{8} \times \frac{5}{8} \times 2 \times \cdot 5236 = 225000 \times \cdot 1309 = 29452 \cdot 5$ the content, the ſame as before.

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Corol. 2. Draw CH perpendicular, and DH parallel to AB, and about the axes AB and 2CH deſcribe the ſemi-ellipſe AHB; then the ſpheroid generated by the revolution of the ſemi-ellipſe AHB, about AB, will be equal to the ſolid generated by the ſemi-ellipſe AFDB about the ſame axe AB.

For $2CH = ac$, and therefore the ſolid AFBI, or $\frac{1}{8}pda^2c^2$, is $= \frac{1}{8}pd \times (2CH)^2 =$ the ſpheroid whoſe axes are d and 2CH.

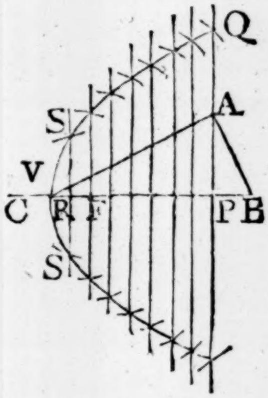
And ſo the ſolids generated by all ſemi-ellipſes upon the ſame baſe and between the ſame parallels, are all equal to each other.

SECTION VI.

OF PARABOLIC LINES, AREAS, SURFACES, AND SOLIDITIES.

PROBLEM I.

To Construct a Parabola; having given any Ordinate PQ *to the Axe, and its Abscissa* VP.



1. FIND the focus F thus: Bisect PQ in A; draw AV, and AB perpendicular to it; take VF = PB, and F will be the focus.

Arithmetically. Divide the square of the ordinate by 4 times the abscissa, and the quotient will be the focal distance VF.

2. In the axe, produced without the vertex V, take VC = VF; draw several double ordinates SRS; then with the radii CR, and center F, describe arcs cutting the corresponding ordinates in the points S.

Draw a curve through all the points of intersection, and it will be the parabola required.

PROBLEM II.

Of any Abſciſs x, *its Ordinate* y, *and Parameter* p; *having Two given, to find the Third.*

CASE I.

To find the Parameter.

Divide the ſquare of the ordinate by its abſciſs, and the quotient will be the parameter.

Or, take a third proportional to the abſciſſa and ordinate, for the parameter.

That is, $p = yy \div x$.

EXAMPLE.

If the abſciſs be 9, and its ordinate 6; required the parameter.

Here $6 \times 6 \div 9 = 36 \div 9 = 4 =$ the parameter.

CASE II.

To find the Abſciſs.

Divide the ſquare of the ordinate by the parameter, and the quotient will be the abſciſs.

That is, $x = yy \div p$.

EXAMPLE.

If the ordinate be 6, and the parameter 4; required the abſciſs.

Here $6 \times 6 \div 4 = 36 \div 4 = 9 =$ the abſciſs.

CASE III.

To find the Ordinate.

Multiply the parameter by the abſciſs, and the ſquare root of the product will be the ordinate.

That is, $y = \sqrt{px}$.

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EXAMPLE.

The abſciſs being 9, and the parameter 4; required the ordinate.

Here $\sqrt{9 \times 4} = \sqrt{36} = 6 =$ the ordinate.

PROBLEM III.

Of any Two Abſciſſes A, B, *taken upon the ſame Diameter, and their Two Ordinates* a, b; *having any Three given, to find the Fourth.*

The abſciſſes are to one another as the ſquares of their ordinates. That is, As any abſciſs is to the ſquare of its ordinate, ſo is any other abſciſs to the ſquare of its ordinate; and the contrary. Or, as the root of an abſciſs is to its ordinate, ſo is the root of another abſciſs to its ordinate.

$$\text{Hence} \begin{cases} \sqrt{A} : \sqrt{B} :: a : a\sqrt{\frac{B}{A}} = \frac{a\sqrt{AB}}{A} = b. \\ \sqrt{B} : \sqrt{A} :: b : b\sqrt{\frac{A}{B}} = \frac{b\sqrt{AB}}{B} = a. \end{cases}$$

$$\text{And} \begin{cases} aa : bb :: A : \frac{Abb}{aa} = B. \\ bb : aa :: B : \frac{Baa}{bb} = A. \end{cases}$$

EXAMPLE I.

If an abſciſs of 9 correſpond to an ordinate of 6, required the ordinate whoſe abſciſs is 16.

Here $\sqrt{9} : \sqrt{16} :: 6 : \frac{6 \times 4}{3} = 8$ the ordinate.

EXAMPLE II.

Required the abſciſs correſponding to the ordinate 6, the ordinate belonging to the abſciſs 16 being 8.

Here $8^2 : 6^2 :: 16 : 9 =$ the abſciſs.

SCHOLIUM.

The demonstration of the three preceding problems are omitted here, as they properly belong to, and are to be found in, all treatises of conic sections.

PROBLEM IV.

To find the Length of the Curve or Arc of a Parabola, cut off by a Double Ordinate to the Axe.

RULE I.*

Divide the double ordinate by the parameter, and call the quotient q.

Add

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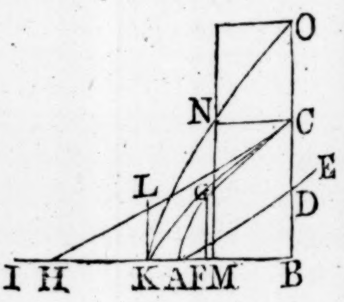
* DEMONSTRATION.

Putting z = any curve beginning at the vertex, y = the ordinate to the axe at the extremity of the curve, x = its absciss, and $a = \frac{1}{2}$ the parameter of the axe, the equation to the curve is $2ax = yy$; hence $2a\dot{x} = 2y\dot{y}$, and $\dot{x}^2 = \frac{y^2\dot{y}^2}{aa}$; consequently $\dot{z} = \sqrt{\dot{y}\dot{y} + \dot{x}\dot{x}} = \sqrt{\dot{y}^2 + \frac{y^2\dot{y}^2}{aa}} = \frac{\dot{y}\sqrt{aa+yy}}{a}$; and the corrected fluents give $z = \frac{y\sqrt{aa+yy}}{2a} + \frac{1}{2}a \times$ hyp. log. of $\frac{y+\sqrt{aa+yy}}{a} = \frac{1}{2}aq\sqrt{1+qq} + \frac{1}{2}a \times$ hyp. log. of $(q + \sqrt{1+qq})$, writing q for $\frac{y}{a}$. And the double of this quantity will give the value of the double curve, as in the rule.

Corol. 1. If for a be substituted its value $\frac{yy}{2x}$, there will be obtained $z = \sqrt{xx + \frac{1}{4}yy} + \frac{1}{2}a \times$ hyp. log. of $\frac{x + \sqrt{xx + \frac{1}{4}yy}}{\frac{1}{2}y}$. And this proves the truth of the construction in COTES's *Harmonia Mensurarum*, page 22, viz. that if F be the focus,



and

Add 1 to the square of q, and call the root of the sum s.

To the product of q and s add the hyperbolic logarithm of their sum; then the last sum multiplied by half the parameter, will be the length of the whole curve, on both sides of the axe.

That is, the curve $c = a \times (qs + \text{hyp. log. of } q + s)$; where $q =$ the quotient of the double ordinate divided by the parameter, $s = \sqrt{1 + qq}$ and $a = \frac{1}{2}$ the parameter.

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and if AD, drawn to bisect BC in D, be produced till DE be $=$ AF $\times$ hyp. log. of $\frac{AB + AD}{BD}$; then AE will be equal the curve AC.—For $\frac{1}{2}a =$ AF, and $\sqrt{xx + \frac{1}{4}yy} =$ AD.

Corol. 2. From sect. 5 it will appear that the area of an hyperbola whose semi-transverse axe is d, semi-conjugate a, and ordinate y, is $A = \frac{dy\sqrt{aa+yy}}{2a} - \frac{1}{2}ad \times \text{hyp. log. of} \frac{y + \sqrt{aa+yy}}{a}$; but the parabolic curve is $c = \frac{y\sqrt{aa+yy}}{2a} + \frac{1}{2}a \times \text{hyp. log. of}$ $\frac{y + \sqrt{aa+yy}}{a}$; hence $\frac{A}{d} + c = \frac{y\sqrt{aa+yy}}{a} = \sqrt{y^2 + \frac{y^4}{aa}} =$ $\sqrt{yy + 4xx} = 2AD =$ HC the tangent to the point C, meeting the axe BA produced in H: Consequently $c =$ HC $- \frac{A}{d}$. Where the semi-transverse d may be taken at pleasure.

If there be taken $d = y$, we shall obtain $c =$ HC $- \frac{A}{y}$; and to find the distance of the ordinate y from the center of the hyperbola, we shall have $a : y$ or $d :: \sqrt{aa+yy} : \frac{y\sqrt{aa+yy}}{a} =$ the said distance, and which, therefore, is $=$ HC. Whence this construction.—In the axe produced take BI $=$ HC; with the center I, semi-transverse IK $=$ the ordinate BC, and semi-conjugate

Note 1. If the common logarithm of any number be multiplied by 2·302585093 the product will be the hyperbolic logarithm of the same number.

2. If the value of s run into decimals, it will be much easiest found by a trigonometrical table; for s is the secant of the arc whose tangent is q, the radius being 1.

EXAMPLE.

Required the length of the curve of a parabola cut off by a double ordinate, to the axe, whose length is 12, the abscis being 2.

Here

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gate KL = the semi parameter FG of the parabola, describe the hyperbola KC, which will pass through C; and let the rectangle CBMN be equal the hyperbolic area KCB: Then will IM be equal to the parabolic curve AC.

When the abscis and transverse axe of an hyperbola are given, or constant; not only the ordinate, but the area also, is as the conjugate axe; and therefore the quotient arising from the division of the area by the ordinate, is a constant quantity; and consequently the parabolic curve AC = C = HC $- \frac{B}{z}$, where B is the area, and z the ordinate of any hyperbola, whose center is I, vertex K, and abscis KE. From hence arises the following general construction, given by Mr. HUYGENS in his *Horolog. Oscillat.* but without demonstration.—With the center I and vertex K, as before, and any conjugate taken at pleasure, describe the hyperbola KO meeting BC, produced, if necessary, in O; and making the rectangle OM = the hyperbolic area KOE, MI will be equal to the parabolic curve AC, as before.

Corol. 3. When $y = a =$ FG, then $z = \frac{1}{2}a \times (\sqrt{2} +$ hyp. log. of $1 + \sqrt{2}) = \frac{1}{2}a \times 2{\cdot}2955871 = 2{\cdot}2955871 \times$ AF $=$ $1{\cdot}1477935 \times$ FG = the curve AG; F being the focus.

Corol. 4. The lengths of similar parts of parabolic curves, are as their parameters, or ordinates, or abscisses. For q is the same in each.

Here $x = 2$, and $y = 6$.

Hence $a = \frac{yy}{2x} = \frac{36}{4} = 9$, and $q = \frac{y}{a} = \frac{6}{9} = \frac{2}{3}$, and $s = \sqrt{1+qq} = \sqrt{1+\frac{4}{9}} = \sqrt{\frac{13}{9}} = \frac{1}{3}\sqrt{13} = 1{\cdot}2018504$. Or, by the table of tangents and ſecants, the ſecant correſponding to the tangent $\frac{2}{3}$ or ·666 &c, is $1{\cdot}2018504 = s$.

Then, $\frac{2}{3} + 1{\cdot}2018504 = 1{\cdot}868517$, whoſe common logarithm is ·271497, which being multiplied by 2·302585093, produces ·6251449 for its hyperbolic logarithm; alſo $\frac{2}{3} \times 1{\cdot}2018504 = {\cdot}8012336$; and the ſum of theſe two is 1·4263785.

Therefore $9 \times 1{\cdot}4263785 = 12{\cdot}8374065$ is the length of the curve required.

RULE II.*

Putting y to denote the ordinate, and q the quotient ariſing from the diviſion of the double ordinate by

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* DEMONSTRATION.

By the laſt, the fluxion of the curve was $\dot{z} = \dot{y}\sqrt{1 + \frac{yy}{aa}} =$, by extracting the root, $\dot{y} \times (1 + \frac{y^2}{2a^2} - \frac{y^4}{2.4a^4} + \frac{3y^6}{2.4.6a^6}$ &c); and, by taking the fluents and writing q for $\frac{y}{a}$, we obtain

$$z = y \times (1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3q^6}{2.4.6.7} \text{ \&c})$$

$$= y \times (1 + \frac{1.1}{2.3}q^2\text{A} - \frac{1.3}{4.5}q^2\text{B} + \frac{3.5}{6.7}q^2\text{C} \text{ \&c}) =$$ the length of the curve from the vertex to the ordinate; the double of which will be that of the double curve. *Q. E. D.*

Corollary.

by the parameter, or from the division of double the abscifs by the ordinate; the length of the double curve will be denoted by the infinite feries

$2y \times (1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3q^6}{2.4.6.7} - \frac{3.5q^8}{2.4.6.8.9}$ &c); or by

$2y \times (1 + \frac{1}{2.3}q^2\text{A} - \frac{1.3}{4.5}q^2\text{B} + \frac{3.5}{6.7}q^2\text{C} - \frac{5.7}{8.9}q^2\text{D}$ &c).

Where A, B, C, &c, denote the 1ft, 2d, 3d, &c, terms.

Note. This feries will converge no longer than till $q = 1$, that is, when the ordinate to the curve, whofe length is required, meets the axe in the focus; for if the ordinate y be beyond the focus, it will be greater than the femi-parameter, confequently q will be greater than 1, and the feries will diverge.

EXAMPLE.

Let there be taken the fame example as before, in which the abfcifs is 2, and the ordinate 6.

Then

Corollary. The hyperbolic logarithm of $q + \sqrt{1 + qq}$ is $=$

$$q \times (1 - \frac{q^2}{2.3} + \frac{3q^4}{2.4.5} - \frac{3.5q^6}{2.4.6.7} \text{ \&c}).$$

For, by the laft rule, $\frac{2z}{a}$ — hyp. log. of $(q + \sqrt{1+qq})$ is $= q\sqrt{1+qq}$

$= q \times (1 + \frac{q^2}{2} - \frac{q^4}{2.4} + \frac{3q^6}{2.4.6}$ &c); and by this rule,

$\frac{2z}{a}$ is $= q \times (1 + \frac{q^2}{3} - \frac{q^4}{4.5} + \frac{3q^6}{4.6.7}$ &c); but by taking the former of thefe from the latter, we have

$q \times (1 - \frac{q^2}{2.3} + \frac{3q^4}{2.4.5} - \frac{3.5q^6}{2.4.6.7} + \frac{3.5.7q^8}{2.4.6.8.9}$ &c) $=$ hyp. log. of $(q + \sqrt{1 + qq})$.

Then $\frac{2 \times 2}{6} = \frac{2}{3} = q$, which being used for it in the general series, and the affirmative and negative terms collected, they will appear as below:

A =	1·00000000	C =	0·00493827
B =	0·07407407	E =	16935
D =	78385	G =	1215
F =	4311	I =	117
H =	368	L =	13
K =	38		
			— 0·0051211

+ 1·0749051 sum of the affirmative terms
— 0·0051211 sum of the negative terms

dif. 1·069784 sum of the whole series
12 = $2y$

12·837408 = length of the curve, nearly the same as before.

RULE III.*

To the square of the ordinate add $\frac{4}{3}$ of the square of the abscis, and the root of the sum will be the length of the single curve nearly; the double of which will be that of the curve on both sides of the abscis nearly.

That is, $\sqrt{yy + \frac{4}{3}xx} = c$ the length of the single curve nearly; y being the ordinate, and x the abscis.

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* DEMONSTRATION OF THIS AND RULE IV.

By the last rule, the curve is

$$c = y \times (1 + \frac{1}{2.3}q^2 - \frac{1}{2.4.5}q^4 + \frac{3}{2.4.6.7}q^6 \text{ \&c}),$$

but $\sqrt{1 + \frac{1}{3}qq} = 1 + \frac{1}{2.3}q^2 - \frac{1}{2.4.9}q^4 + \frac{3}{2.4.6.27}q^6$ &c),

hence

EXAMPLE.

Taking again the same example, in which $x = 2$, and $y = 6$, we shall have $c = \sqrt{yy + \frac{4}{3}xx} = \sqrt{36 + \frac{16}{3}}$ $= 6\cdot4291$ the single curve; the double of which is $12\cdot8582$ the length of the curve nearly.

RULE IV.*

To the square of the ordinate add $\frac{2}{3}$ of the square of the abscis, and divide the sum by the ordinate; then subtract 4 times this quotient from 9 times the length of the single curve, as found from the last rule, and $\frac{1}{5}$ of the remainder will be the length of the single curve very nearly.

That is, $c = \frac{1}{5}(9\sqrt{yy + \frac{4}{3}xx} - 4 \times \frac{yy + \frac{2}{3}xx}{y})$ very nearly.

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hence $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} = -\frac{1}{2\cdot5\cdot9}q^4 + \frac{5}{4\cdot7\cdot27}q^6$ &c: and, supposing q not greater than 1, and rejecting the series, c will be $= y\sqrt{1 + \frac{1}{3}qq} = \sqrt{yy + \frac{4}{3}xx}$ nearly. Which is rule 3.

Again, from the 1st series, $\frac{c}{y} - 1 - \frac{1}{6}qq = -\frac{1}{2\cdot4\cdot5}q^4 + \frac{1}{4\cdot4\cdot7}q^6$ &c; or $\frac{4}{9} \times (\frac{c}{y} - 1 - \frac{1}{6}qq) = -\frac{1}{2\cdot5\cdot9}q^4 + \frac{3}{4\cdot7\cdot27}q^6$ &c; and from above, $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} = -\frac{1}{2\cdot5\cdot9}q^4 + \frac{5}{4\cdot7\cdot27}q^6$ &c; hence, by subtracting, $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} - \frac{4}{9}(\frac{c}{y} - 1 - \frac{1}{6}qq) = \frac{1}{2\cdot7\cdot27}q^6$ &c: and, consequently, the remaining series being very small, we shall obtain $c = \frac{1}{5}y \times (9\sqrt{1 + \frac{1}{3}qq} - 4(1 + \frac{1}{6}qq) =$ $\frac{1}{5} \times (9\sqrt{yy + \frac{4}{3}xx} - 4 \times \frac{yy + \frac{2}{3}xx}{y})$ very nearly. Which is rule 4.*

And in this manner we may proceed to any degree of accuracy required.

EXAMPLE.

Taking ſtill the ſame example; we ſhall have $4 \times \frac{yy + \frac{2}{3}xx}{y} = 4 \times \frac{36 + \frac{8}{3}}{6} = 25\frac{7}{9}$; but by the laſt rule $c = 6{\cdot}4291$; hence $9 \times 6{\cdot}4291 = 57{\cdot}8619$; and $\frac{57{\cdot}8619 - 25\frac{7}{9}}{5} = 6{\cdot}4168 = c$, the double of which is $12{\cdot}8336$.

Note. It muſt be obſerved, that, as theſe two approximations are derived from the 2d rule, they muſt be uſed only in thoſe caſes in which that rule might be applied, viz. thoſe in which the abſciſs does not exceed half the ordinate.

PROBLEM V.

To find the Area included by the Curve of a Parabola and Any Right Line, called the Baſe of the Segment or Area.

Take $\frac{2}{3}$ of its circumſcribed parallelogram for the area.*

Note. The baſe of the circumſcribed parallelogram, is the ſame with the baſe of the ſegment; their

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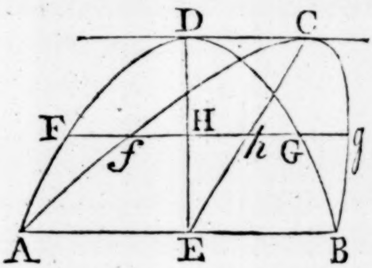
* DEMONSTRATION.

Put $y =$ an ordinate to any diameter, $x =$ its abſciſs, and $p =$ the parameter of that diameter; then the equation to the curve will be $px = yy$, and putting $s =$ the ſine of the angle made by the abſciſs and ordinate, the fluxion of the area $\dot{a}$ will be $= sy\dot{x}$ $= \frac{2sy^2\dot{y}}{p}$; hence the area a is $= \frac{2sy^3}{3p} = \frac{2}{3}sxy$; that is, $\frac{2}{3}$ of the parallelogram having the ſame baſe and altitude.

Corol.

their altitudes are likewiſe the ſame; and therefore the parabolic area will be equal to $\frac{2}{3}$ of its altitude multiplied by its baſe; that is $= \frac{2}{3}ab$; putting a to denote its altitude, and b its baſe.—It may farther be obſerved, that if the baſe be perpendicular to the diameter of the figure, then the altitude a will be the ſame with the abſciſs of the figure; otherwiſe, the altitude is equal to the abſciſs drawn into the natural ſine of the angle made by the abſciſs and ordinate, or baſe, the radius being 1. And this is to be obſerved in every other figure.

EX-



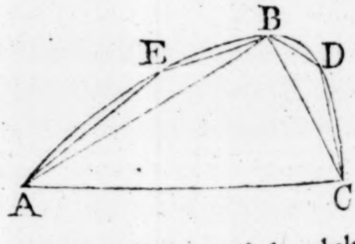
Corol. 1. Hence it is evident that all parabolas ADB, ACB, of the ſame baſe or equal baſe AB, and of equal altitudes, or between the ſame parallels AB, DC, are equal to one another.

Corol. 2. Any common ſections FG, fg, of the parabolas ADB, ACB, are equal to each other.

For CE : Ch :: DE : DH :: $AB^2 : fg^2$ :: $AB^2 : FG^2$; but AB = AB, therefore $fg = FG$.

Corol. 3. And hence, alſo, the ſegments FDG, fCg, are equal to each other.——For they are of equal baſes and altitudes.

Corol. 4. Moreover, the fruſtums AFGB, AfgB, of equal ends and altitudes, are equal to one another.



Corol. 5. Let ABC be a triangle, having the ſame baſe and altitude with the parabolic ſegment AEBDC; then, becauſe the triangle is half the circumſcribed parallelogram, it will be $\frac{3}{4}$ of the parabolic ſegment; and conſequently the ſegments AEB, BDC, together, will be $\frac{1}{4}$ of the whole portion AEBDC, or $\frac{1}{3}$ of the triangle ABC.

Again,

EXAMPLE I.

Required the area of a parabola, the abscis being 2, and the ordinate, perpendicular to the abscis, 6.

Here the altitude is 2, and the base or double ordinate is 12; therefore $\frac{2}{3} \times 2 \times 12 = 16$ is the area.

EXAMPLE II.

If the base of a parabolic segment be 12, and its abscis 2 make an angle of 30 degrees with it; what will be the area?

The sine of 30° being half radius, the altitude will be $2 \times \frac{1}{2} = 1$; and hence $\frac{2}{3} \times 12 = 8$ is the area.

PRO-

Again, it will appear that, if in the two segments AEB, BDC, be inscribed triangles AEB, BDC, of the same bases and altitudes with them, then these last triangles will be $\frac{3}{4}$ of their circumscribed segments, and, consequently, $\frac{1}{4}$ of the triangle ABC. And if, in like manner, in the last made segments be inscribed the greatest triangles, they will be $\frac{1}{4}$ of the triangles immediately preceding them, or $\frac{1}{4 \cdot 4} = \frac{1}{16}$ of the first triangle. And so on continually. Consequently all the inscribed triangles, taken together, will be expressed by the series $b \times (1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4}$ &c); where b denotes the first triangle; and which, when the series is infinitely continued, will denote the area of the parabolic segment.

Corol. 6. Hence the infinite series $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}$ &c, will be $= 1\frac{1}{3}$, as we also know from other principles; and consequently the sum of any finite number of terms of the series $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}$ &c, is less than $\frac{1}{3}$; and the sum of the infinite number of them $= \frac{1}{3}$.

PROBLEM VI.

To find the Area of a Frustum, or Zone of a Parabola, included by two Parallel Right Lines and the Intercepted Curves of the Parabola.

To one of the parallel ends, add the quotient arising from the division of the square of the other by the sum of the said ends; multiply the sum by the altitude of the frustum, or distance of the ends; and $\frac{2}{3}$ of the product will be the area.

Or divide the difference of the cubes of the diameters, by the difference of their squares; and multiply the quotient by $\frac{2}{3}$ of the altitude.

That is, $(\text{D} + \frac{dd}{\text{D}+d}) \times \frac{2}{3}a$, or $(d + \frac{\text{DD}}{\text{D}+d}) \times \frac{2}{3}a$, or $\frac{\text{D}^3 - d^3}{\text{D}^2 - d^2} \times \frac{2}{3}a =$ the area; D, d being the two ends, and a the altitude.*

EX-

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* DEMONSTRATION.

By the property of the parabola, $\text{DD} - dd : a :: \begin{cases} \text{DD} : \frac{a\text{DD}}{\text{DD}-dd} \\ dd : \frac{add}{\text{DD}-dd} \end{cases}$ = the altitudes of two complete segments whose bases are the ends D, d of the frustum; and consequently their difference, or the frustum, is $\frac{2}{3}a \times \frac{\text{D}^3 \sim d^3}{\text{D}^2 \sim d^2} = \frac{2}{3}a \times \frac{\text{DD} + \text{D}d + dd}{\text{D} + d} = \frac{2}{3}a \times (\text{D} + \frac{dd}{\text{D}+d}) = \frac{2}{3}a \times (d + \frac{\text{DD}}{\text{D}+d})$. *Q. E. D.*

Corollary. Hence a parabolic frustum is equal to a parabola of the same altitude, and whose base is equal to one end of the frustum, increased by a 3d proportional to the sum of the ends and the other. Therefore, in the one end DC, produced, of a parabolic

EXAMPLE I.

If the two parallel ends of the frustum of a parabola be 10 and 6, and the part of the abscis, perpendicular to, and connecting the middles of, those ends, be 4; what will be the area?

Here the abscis being perpendicular to the ends, will be the altitude of the figure, and therefore

$$(10 + \frac{6 \times 6}{10 + 6}) \times \frac{4 \times 2}{3} = \frac{196}{6} = 32\tfrac{2}{3} = \text{the area.}$$

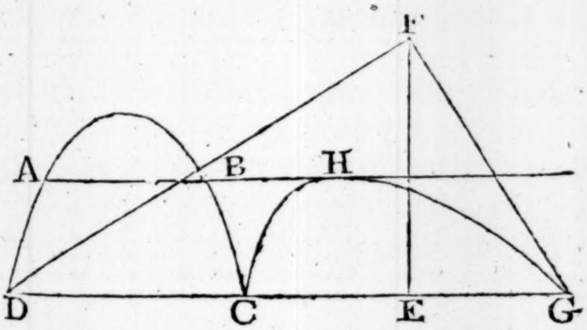
$$\text{Or } (6 + \frac{10 \times 10}{10 + 6}) \times \frac{4 \times 2}{3} = \frac{196}{6} = 32\tfrac{2}{3} = \text{the area.}$$

EXAMPLE II.

If the part of the abscis, connecting the middles of the two ends, make with them an angle of $48°\ 35\frac{5}{12}'$, required the area; the other dimensions being as in the first example.

The sine of $48°\ 35\frac{5}{12}'$ being $\cdot 75$ or $\frac{3}{4}$ very nearly, hence $4 \times \frac{3}{4} = 3 =$ the altitude $= \frac{3}{4}$ of that in the

Bb last



parabolic frustum ABCD, take CE = the other end AB; raise EF perpendicular to EC and = CD; draw DF, and perpendicular to it FG meeting CE produced in G: Then if upon the base CG be described any parabola CHG touching the line ABH, it will be equal to the frustum ABCD.

$$\text{For } CG = CE + EG = AB + EG = AB + \frac{EF^2}{DE} = AB + \frac{DC^2}{DC + AB}.$$

last example, and therefore the area here must be $\frac{3}{4}$ of that above; that is, $32\frac{2}{3} \times \frac{3}{4} = \frac{98}{4} = 24\frac{1}{2} =$ the area required.

PROBLEM VII.

To find the Area included by the Focal Distance, the Line drawn from the Focus to the Curve, and the Contained Arc of the Parabola.

RULE I.

Upon the axe, or focal distance AB, produced if necessary, having demitted the ordinate or perpendicular CD, cutting off the abscis AD; then

To the focal distance AB add $\frac{1}{3}$ of the abscis AD, multiply the sum by the ordinate CD; and half the product will be the area of the part ACB.

That is, $(AB + \frac{1}{3}AD) \times \frac{1}{2}DC =$ the area.*

Note. The focal distance AB, is $\frac{1}{4}$ of the parameter.

EXAMPLE.

If the abscis be 2, and the ordinate 6; required the area of ACB.

By case 1 prob. 2, we have $2 : 6 :: 6 : 18$ the parameter; and hence $\frac{18}{4} = 4\frac{1}{2} =$ AB.

Then $(4\frac{1}{2} + \frac{2}{3}) \times 3 = 5\frac{1}{6} \times 3 = 15\frac{1}{2}$ the area required.

RULE

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* DEMONSTRATION.

Putting $AD = x$, $DC = y$, and $AB = a$; $ACB = ACD \pm BCD$ will be $= \frac{2}{3}xy \pm \frac{1}{2}y\ (\mp x \pm a) = \frac{2}{3}xy - \frac{1}{2}xy + \frac{1}{2}ay = \frac{1}{2}y\ (\frac{1}{3}x + a) = \frac{1}{2}DC \times (AB + \frac{1}{3}AD)$. *Q. E. D.*

Corollary. The area cut off by AC is $= \frac{1}{3}AD \times \frac{1}{2}DC$.

RULE II.

Subtract the focal distance, or distance between the focus and the beginning of the arc, from the distance between the focus and the end of the arc, and multiply the remainder by the said focal distance; then multiply the root of the product by the sum of the distance between the end of the arc and focus, and double the focal distance; and $\frac{1}{3}$ of this product will be the area.

That is, $\frac{1}{3}\sqrt{(\text{CB} - \text{BA})\,\text{AB}} \times (2\text{AB} + \text{BC})$ is the area ACB.*

EXAMPLE.

Taking here the same example as before; we have $\text{AB} = 4\frac{1}{2}$, and $\text{BC} = \text{AD} + \text{AB} = 2 + 4\frac{1}{2} = 6\frac{1}{2}$; and hence $\frac{1}{3}\sqrt{4\frac{1}{2} \times 2} \times \overline{9 + 6\frac{1}{2}} = 15\frac{1}{2} =$ the area as before.

PROBLEM VIII.

To find the Curve Surface of a Paraboloid.

RULE I.

To the square of the ordinate, or semi-diameter of the base, add 4 times that of the axe; and the square root of the sum will be the tangent to the curve at the base, and intercepted by the axe produced; let this tangent be called t, viz. $t = \sqrt{yy + 4xx}$; x being the axe, and y the ordinate.

B b 2 Then,

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* DEMONSTRATION.

By the nature of the parabola, $z = \text{CB}$ is $= \text{AB} + \text{AD} = a + x$, or $x = z - a$, also $\frac{1}{2}y = \sqrt{ax} = \sqrt{(z-a)\,a}$; which values of x and y, written for them in the last rule, give $\frac{1}{3}\sqrt{(z-a)\,a} \times (2a + z) =$ the area ACB. *Q. E. D.*

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Then, by problem 6, find the area of the frustum of a parabola whose parallel ends are t and y, and its altitude $= y$.

And this area multiplied by 3·14159, will be the curve surface of the paraboloid ABC required.*

EX-

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* DEMONSTRATION.

Calling the ordinate y, the abscis x, the curve z, and the parameter p; also $a = 3{\cdot}14159$; the equation of the generating parabola will be $px = yy$; hence $p\dot{x} = 2y\dot{y}$, and the fluxion of the surface $\dot{s} = 2ay\dot{z} = 2ay\sqrt{\dot{y}^2 + \dot{x}^2} = \frac{4ay\dot{y}}{p}\sqrt{\frac{1}{4}pp + yy}$; and, by taking the correct fluents, the surface s will be $= \frac{4a}{3p} \times (\frac{1}{4}pp + yy)^{\frac{3}{2}} - \frac{1}{6}app = \frac{2}{3}ay \times \frac{(yy + 4xx)^{\frac{3}{2}} - y^3}{4xx} =$ $\frac{2}{3}ay \times \frac{(yy + 4xx)^{\frac{3}{2}} - y^3}{(yy + 4xx)^{\frac{2}{2}} - y^2} = \frac{2}{3}ay \times \frac{t^3 - y^3}{t^2 - y^2} = \frac{2}{3}ay \times \frac{tt + ty + yy}{t + y}$ $= \frac{2}{3}ay \times (t + \frac{yy}{t+y}) = \frac{2}{3}ay \times (y + \frac{tt}{t+y})$; putting $t = \sqrt{yy + 4xx}$ $=$ BE, the tangent to the curve at the extremity of the ordinate, and intercepted by the axe produced. *Q. E. D.*

Corol. 1. In CB produced take BG $=$ the tangent BE; erect GH perpendicular to GB and equal to BD; join DH, and perpendicular to it HI, meeting BG produced in I; take BK $= \frac{1}{3}$BI; and upon KC describe a circle meeting BL perpendicular to BC in L. And the circle whose radius is BL, will be equal to the curve surface of the paraboloid BAC.

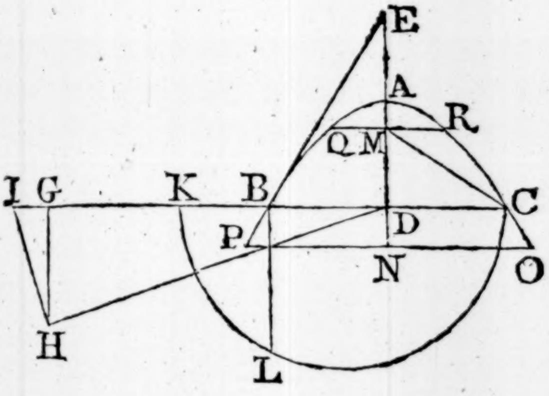
For, by the construction, $BL^2 = KB \times BC = BC \times \frac{1}{3}BI =$ $\frac{2}{3}BD \times (BG + GI) = \frac{2}{3}BD \times (BE + \frac{GH^2}{GD}) = \frac{2}{3}BD \times (BE + \frac{BD^2}{BE + BD})$ $= \frac{2}{3}y \times (t + \frac{yy}{t + y})$.

Corol.

EXAMPLE.

Required the curve ſurface of a paraboloid, whoſe axe is 20, and the diameter of its baſe 60.

Here $\sqrt{30^2 + 40^2} = 50$ is the tangent. Then, by prob. 5, $(50 + \frac{30 \times 30}{50+30}) \times \frac{2}{3} \times 30 \times 3{\cdot}14159 = \frac{4900}{80} \times 20 \times 3{\cdot}14159 = 1225 \times 3{\cdot}14159 = 3848{\cdot}451$ = the ſurface required.



Corol. 2. The curve ſurface of the paraboloid is to the area of its circular baſe, as IK is to BD.

For the baſe is $= a \times BD^2$, and the curve ſurface $= a \times \frac{2}{3} IB \times BD = a \times IK \times BD$.

Corol. 3. If M be the focus of the generating parabola, and there be taken $AN = MC = AD + AM$, and the double ordinates PNO, QMR be drawn; the ſurface generated by BAC, will be to the area PQRO, as $a = 3{\cdot}14159$, to 1, or as the circumference of a circle, to its diameter.

For the ſurface is $= \frac{4a}{3p} \times (\frac{1}{4}pp + yy)^{\frac{3}{2}} - \frac{1}{6}app = \frac{4a}{3p} \times (\frac{1}{4}pp + px)^{\frac{3}{2}} - \frac{1}{6}app = \frac{4}{3}a \times (\frac{1}{4}p + x)\sqrt{\frac{1}{4}pp + px} - \frac{1}{6}app = \frac{4}{3}a \times MC\sqrt{MC \times p} - \frac{1}{6}app = \frac{4}{3}a \times AN \times NO - \frac{4}{3}a \times AM \times MR = a \times (PAO - QAR) = a \times PQRO$.

RULE II.

Divide the difference between the cube of the tangent and the cube of the ordinate, by 4 times the ſquare of the axe; multiply the quotient by $\frac{2}{3}$ of the ordinate, and the product again by 3·14159 for the ſurface.

That is, $\frac{t^3 - y^3}{4xx} \times \frac{2}{3}cy =$ the ſurface; where $t = \sqrt{yy + 4xx}$ is the tangent, x the abſciſs or axe, y the ordinate or ſemi-diameter of the baſe, and $c = 3{\cdot}14159$.*

EXAMPLE.

Taking the ſame example as before, in which $x = 20$, $y = 30$, and therefore $t = \sqrt{40^2 + 30^2} = 50$; we ſhall have $\frac{50^3 - 30^3}{40 \times 40} \times 30 \times \frac{2}{3}c = \frac{980}{16} \times 20c = 1225 \times 3{\cdot}14159 = 3848{\cdot}451$ the ſurface, as before.

PROBLEM IX.

To find the Curve Surface of the Fruſtum of a Paraboloid, having given its Altitude and the Diameters of its Ends.

Divide the difference of the ſquares of the ſemi-diameters of the ends, by their diſtance, or by the altitude of the fruſtum; or divide the difference of the ſquares of the whole diameters, by 4 times the altitude; and the quotient will be the parameter of the axe of the generating parabola.

Find

* This rule appears in the inveſtigation of the laſt.

Find the two sums arising from the addition of the square of the parameter and the square of each diameter; multiply each sum by its root, and divide the difference of the products by the parameter; multiply the quotient by 3·14159, and $\frac{1}{6}$ of the product will be the surface of the frustum.

That is, $\frac{(pp + \text{DD})^{\frac{3}{2}} - (pp + dd)^{\frac{3}{2}}}{p} \times \frac{1}{6}c =$ the surface; D and d being the diameters of the ends, $p =$ the parameter $= \frac{\text{DD} - dd}{4a}$, $a =$ the altitude, and $c = 3·14159$.*

EXAMPLE.

Required the curve surface of the frustum of a paraboloid, the altitude being $25\frac{79}{80}$, and the diameters of its ends 48 and 15.

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* DEMONSTRATION.

In the investigation of rule 1 of the last problem, it appears that $\frac{4c}{3p} \times (\frac{1}{4}pp + yy)^{\frac{3}{2}} - \frac{1}{6}cpp$ or $\frac{c}{6p} \times (pp + \text{DD})^{\frac{3}{2}} - \frac{1}{6}cpp$ is the surface of the segment whose base diameter is D; and $\frac{c}{6p} \times (pp + dd)^{\frac{3}{2}} - \frac{1}{6}cpp =$ that whose diameter is d; and, by taking the difference, $\frac{1}{6}c \times \frac{(pp + \text{DD})^{\frac{3}{2}} - (pp + dd)^{\frac{3}{2}}}{p}$ will express the surface of the frustum, the diameters of whose ends are D, d.

And that the parameter is $= \frac{\text{DD} - dd}{4a}$, appears thus: Since $p\text{X} = \text{YY}$, and $px = yy$, therefore $p\text{X} - px = \text{YY} - yy$, and $p = \frac{\text{YY} - yy}{\text{X} - x} = \frac{\text{DD} - dd}{4a}$.

Here $D = 48$, $d = 15$, and $a = 25\frac{79}{80}$.

Therefore $\frac{DD - dd}{4a} = \frac{48^2 - 15^2}{4 \times 25\frac{79}{80}} = \frac{2079}{103\frac{19}{20}} = 20 = p$.

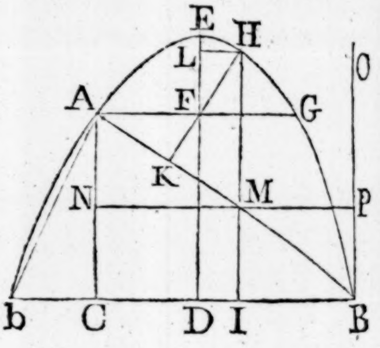
Then $\sqrt{pp + DD} = \sqrt{20^2 + 48^2} = \sqrt{2704} = 52$.

And $\sqrt{pp + dd} = \sqrt{20^2 + 15^2} = \sqrt{625} = 25$.

Hence $\frac{2704 \times 52 - 625 \times 25}{20 \times 6} \times c = \frac{41661}{40} \times 3{\cdot}14159$ $= 4166{\cdot}1 \times {\cdot}785398 = 3272{\cdot}0472885$ the surface required.

PROBLEM X.

If a Paraboloid be Cut by a Plane, Oblique to its Axe; it is required to find the Axes of the Elliptic Section.



It is evident that AB is the transverse axe. And if bB be a double ordinate to the axe, meeting the diameter AC in C, then BC will be the other or conjugate axe of the section. As is proved in corollary 7 to prop. 1 sect. 4.

EXAMPLE.

If a paraboloid, whose axe is $45\frac{5}{7}$, and its base or greatest double ordinate 32, be cut by a plane passing through the extremity of the base, and through the opposite side of the figure at the height of 20 above the base; what are the axes of the section?

Draw AF perpendicular to, and meeting the axe DE in F.——Then $EF = ED - DF = 45\frac{5}{7} - 20 = 25\frac{5}{7}$; and, by the nature of the parabola, $ED : EF :: DB^2 : AF^2 = 144$; and hence $AF = DC =$

$= 12$: therefore CB $=$ CD $+$ DB $= 12 + 16 = 28$ $=$ the conjugate axe. And AB $= \sqrt{AC^2 + CB^2} = \sqrt{20^2 + 28^2} = 4\sqrt{5^2 + 7^2} = 4\sqrt{74} = 34 \cdot 4093010682$ the transverse axe required.

EXAMPLE II.

Let the same paraboloid be cut through the extremity of the base, by a plane cutting the figure again on the same side of the axe, and at the same distance of 20 above the base; to find the axes of the section.

Here bc $=$ bD $-$ CD $= 16 - 12 = 4$ is the conjugate axe. And bA $= \sqrt{AC^2 + bc^2} = \sqrt{20^2 + 4^2} = 4\sqrt{5^2 + 1^2} = 4\sqrt{26} = 20 \cdot 396078$ is the transverse.

PROBLEM XI.

To find the Solidity of any Segment of a Paraboloid, whose Base is either Perpendicular or Oblique to the Axe.

RULE I.

For Both Right and Oblique Segments.

Multiply the base by half the altitude, and the product will be the content.*

Note.

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* DEMONSTRATION.

Let b denote the base, a the altitude, x any variable abscis, or part of the diameter drawn through the middle of the base, and s the sine of the angle formed by this diameter and its ordinate, to the radius 1.

Then as $a : sx :: b : \frac{bsx}{a}$ the section corresponding to the abscis x, by prop. 1 sect. 4. Therefore $\frac{sx}{a} \times bs\dot{x}$ is the fluxion

of

Note. When the base is perpendicular to the axe, it is a circle, and the altitude is equal to the whole length of the axe, contained between the base and the vertex.—But if the base be oblique to the axe, it

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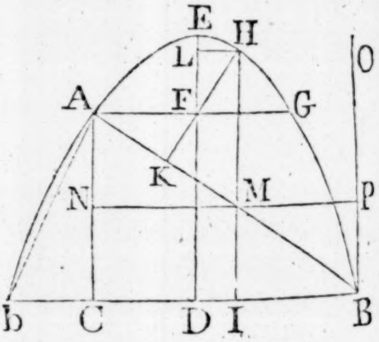
of the solid; and, by taking the fluent, the solid itself will be $\frac{bssxx}{2a}$; which, when $sx = a$, becomes $\frac{1}{2}ab$ for the whole solid, whose base is b and altitude a. *Q. E. D.*

Corol. 1. A paraboloid is equal to half a prism of the same base and altitude. Also a prism, a semi-spheroid or semi-sphere, a paraboloid, and a pyramid, all of equal bases and altitudes, are to one another as the numbers 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, or as 6, 4, 3, 2; and are, therefore, in a discontinued geometric proportion, whose ratio is that of 3 to 2. Which will appear by comparing the above value of the paraboloid with those of the other solids.

When the base of the paraboloid is perpendicular to the axe, or when the paraboloid is right, the semi-spheroid, or semi-sphere, will also be right; and the prism and pyramid will be the right or common cylinder and cone, the common base being a circle. But if the paraboloid be oblique, by having its base oblique to the axe, the common base of all the solids will be an ellipse; the semi-spheroid will also be oblique, by having its base oblique to the axe; and the prism and pyramid may be either right or oblique.



Corol. 2. If upon the diameter NP = EC = the conjugate axe of the section AB, be described a circle; the oblique paraboloid, or segment AHB, will be equal to the right paraboloid whose base is that circle, and its altitude MH. For, the diameter of the circle being the conjugate axe of the base of the oblique paraboloid, the circle will be to that base, as the conjugate is to the transverse axe, that is, as CB to AB; but HK is to HM in the same proportion of CB to AB; consequently HK × elliptic base AB will be = HM × circular base NP or EC.

It

it will be an ellipse, and the altitude will be the perpendicular demitted to the base, from the vertex of the diameter drawn through the middle of the base; or it will be equal to the product arising from the multiplication

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It is also evident, that all segments, right or oblique, having the same vertex H, and the transverse axes of whose bases pass through the same point M, and terminate in AC, OP, parallel to and equi-distant from HM, will be equal to one another.

Corol. 3. If B, b denote the two bases, or parallel ends, of the frustum of a paraboloid, either right or oblique; and d the distance of the ends, or the altitude of the frustum; then $\frac{1}{2}d \times (B + b)$ will be the solidity of the frustum.

For, by this problem, $\frac{1}{2}AB$ is the segment whose base is B and altitude A; and $\frac{1}{2}ab$ is that whose base is b and altitude a: therefore the frustum, or the difference of the segments, is $\frac{1}{2}AB - \frac{1}{2}ab$. But, by the nature of the paraboloid, $B-b : d$ or $A-a :: B : A = \frac{Bd}{B-b}$, and $B-b : d :: b : a = \frac{bd}{B-b}$; which values of A and a being substituted for them, we obtain $\frac{1}{2}AB - \frac{1}{2}ab = \frac{dB^2 - db^2}{2B - 2b} = \frac{1}{2}d \times (B + b)$.

Corol. 4. Let AGBb be a right frustum, HI the diameter to the double ordinate AB, and HK perpendicular to AB, that is, HK the altitude of the oblique segment AHB. Then the value of the oblique segment AHB is $\frac{1}{2}n \times AB \times BC \times HK$, putting $n = \cdot 785398$.

But, by the property of the figure, $DB^2 - DC^2 : DB^2 - DI^2 ::$ $AC : HI = \frac{DB^2 - DI^2}{DB^2 - DC^2} \times AC$; hence $HM = HI - IM = HI - \frac{1}{2}AC = \frac{DB + DC}{DB - DC} \times \frac{1}{4}CA = \frac{BC \times CA}{8DI}$; and, by similar triangles, $AB : CB :: HM : HK = \frac{HM \times CB}{AB} = \frac{AC \times CB^2}{AB \times 8DI}$; therefore the oblique segment AHB, or $\frac{1}{2}n \times AB \times CB \times HK$, will be $\frac{CB^3}{8DI} \times \frac{1}{2}AC \times n = \frac{CB^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n$.

Or,

multiplication of the part of the ſaid diameter, intercepted by the vertex and the baſe, by the ſine of the angle of inclination to the baſe.

EXAMPLE I.

Required the ſolidity of a right paraboloid whoſe axe is 30, and diameter of its baſe 40.

By the rule, $40^2 \times 15 \times \cdot 785398 = 18849\cdot 5559215$ is the ſolidity required.

EXAMPLE II.

If from the above right paraboloid be cut a part bAB, by a plane paſſing through the extremity of the baſe, and the oppoſite ſide at the height AC of $22\frac{1}{2}$ above the baſe; it is required to find the content of the oblique ſegment AHB.

Here $EF = ED - AC = 30 - 22\frac{1}{2} = 7\frac{1}{2}$; then, by the property of the parabola, $\sqrt{ED} : \sqrt{EF} :: DB : AF = DB\sqrt{\frac{EF}{ED}} = 20\sqrt{\frac{7\frac{1}{2}}{30}} = 20\sqrt{\frac{1}{4}} = 20 \times \frac{1}{2} = 10$; hence $CB = BD + DC = BD + AF = 20 + 10 = 30$ the conjugate diam. of the baſe, by the laſt problem; and AB

Or, by ſubſtituting the value of AC, viz. $\frac{DB^2 - DC^2}{DB^2} \times DE$ $= \frac{CB \times 2DI}{DB^2} \times DE$, inſtead of it, the ſame ſegment AHB will be $\frac{CB^4}{8DB^2} \times DE \times n = \frac{CB^4}{bB^2} \times DE \times \frac{1}{2}n$.

Corol. 5. If from $DE \times bB^2 \times \frac{1}{2}n$, the value of the right ſegment bEB, be taken the above value of the oblique ſegment AHB, there will remain $\frac{bB^4 - CB^4}{bB^2} \times DE \times \frac{1}{2}n$, or $\frac{bB^4 - CB^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n$ for the value of the greater ungula bAB.

Corol.

$AB = \sqrt{AC^2 + CB^2} = \sqrt{22{\cdot}5^2 + 30^2} = 7{\cdot}5\sqrt{3^2+4^2}$ $= 7{\cdot}5 \times 5 = 37{\cdot}5$ the tranſverſe. Now H being the vertex of the diameter HMI drawn through M the middle of AB, and HK perpendicular to AB, the triangles HKM, MIB, ACB, will be ſimilar; but $MB = \frac{1}{2}AB$, and therefore $MI = \frac{1}{2}AC = 11\frac{1}{4}$; but $LH = DI = \frac{1}{2}DB - \frac{1}{2}DC = 10 - 5 = 5$; then, by the property of the parabola, $DB^2 : DB^2 - LH^2 ::$ $DE : HI = 28\frac{1}{8}$; hence $HM = HI - IM = 28\frac{1}{8} - 11\frac{1}{4}$ $= 16\frac{7}{8}$; and, by ſimilar triangles, $AB : CB :: HM :$ $HK = 13\frac{1}{2}$ the altitude.

Then $CB \times AB \times \frac{1}{2}HK \times {\cdot}785398 = 30 \times 37{\cdot}5$ $\times 6{\cdot}75 \times {\cdot}785398 = 7593{\cdot}75 \times {\cdot}785398 =$ $5964{\cdot}1173033$ the ſegment AHB required.

RULE II.

For the Oblique Segment.

Divide the fourth power of CB by the ſquare of bB; multiply the quotient by $\frac{1}{2}DE$; then the product multiplied by ·785398 will be the content of the oblique ſegment AHB.

That is, $\frac{CB^4}{bB^2} \times \frac{1}{2}DE \times n = AHB$. By corollary 4.

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Corol. 6. And if from the ſaid value of the oblique ſegment AHB, be taken that of the right ſegment AEG, viz. $\frac{AG^4}{bB^2} \times DE \times \frac{1}{2}n$ or $\frac{AG^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n$, there will remain $\frac{CB^4 - AG^4}{bB^2} \times DE \times \frac{1}{2}n$ or $\frac{CB^4 - AG^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n$ for that of the leſs ungula AGB.

EXAMPLE.

Taking here the last example, we have $\frac{CB^4}{bB^2} \times \frac{1}{2}DE \times n = \frac{30^4}{40^2} \times 15 \times \cdot 785398 = \frac{30^2 \times 9 \times 15}{16} \times \cdot 785398 = 759375 \times \cdot 785398 = 5964 \cdot 1173033$, the same as before.

PROBLEM XII.

To find the Content of the Frustum of a Paraboloid, having its Ends Parallel to each other, and either Right or Oblique to the Axe.

Multiply the sum of the two ends by their distance, or the altitude of the frustum; and half the product will be the content.

As is proved in corollary 3 to the last problem.

Note. The same things may be observed here as in the note to the last problem.

EXAMPLE.

The length of a cask, composed of two equal frustums of a paraboloid, is 45 inches; required the content in ale and wine gallons, supposing the bung diameter to be 40, and the head diameter 20 inches.

Here $(40^2 + 20^2) \times \cdot 785398 \times 22\frac{1}{2} = 45000 \times \cdot 785398 = 35342 \cdot 917352885$ is the content in inches.

Then $\frac{35342 \cdot 91735}{282} = 125 \cdot 329494$ ale gallons.

And $\frac{35342 \cdot 91735}{231} = 152 \cdot 99964$ wine gallons.

PROBLEM XIII.

To find the Solidity of the Parabolic Ungula bAB, *or* AGB, *made by a Plane passing through the Opposite Extremities of the Ends of the Frustum* AGBb.

Divide the difference of the 4th powers of the diameter of the base, and half the sum of the diameters of the ends of the frustum, by the difference of the squares of the said diameters; multiply the quotient by ·785398, and the product multiplied by half the altitude will produce the content.

That is, $\frac{bB^4 - CB^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n =$ the ungula bAB,

And $\frac{CB^4 - AG^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n =$ the ungula BAG.

As is proved in corol. 4 and 5 of prob. 10.

EXAMPLE I.

If a vessel, in form of the frustum of a paraboloid, and open at the narrow end, be so placed, that the liquor in it may just cover the bottom, whose diameter is 40 inches, and rise to the height of $22\frac{1}{2}$ inches towards the top of it; required the quantity of liquor contained in the vessel, supposing the diameter of the vessel at the upper edge of the liquor to be 20 inches.

Here $bB = 40$, $AC = 22\frac{1}{2}$, and $AG = 20$; consequently $BC = 20 + 10 = 30$.

Hence $\frac{40^4 - 30^4}{40^2 - 20^2} \times 22\frac{1}{2} \times \frac{\cdot 785398}{2} = \frac{17500}{96} \times 90 \times$ $\cdot 785398 = 12885\cdot 438618$ cubic inches; which being divided by 282 or by 231, will give 45·693044 ale gallons, or 55·781119 wine gallons.

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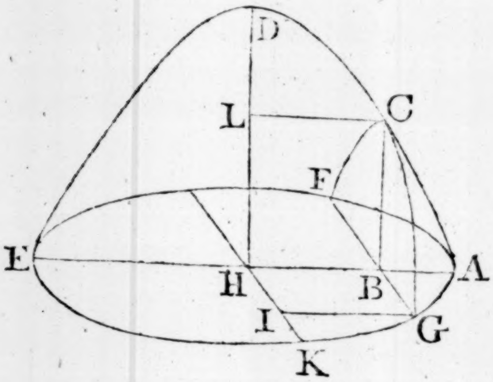
EXAMPLE II.

If the vessel have its dimensions inverted, that is, if AGB be the part filled, AB being the surface of the liquor, the bottom diameter AG being 20, the diameter bB at the top of the liquor 40, and the altitude AC $22\frac{1}{2}$ inches; required the quantity of liquor.

Here $\frac{CB^4 - AG^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n = \frac{30^4 - 20^4}{40^2 - 20^2} \times 22\frac{1}{2} \times \frac{\cdot785398}{2} = \frac{6500}{96} \times 90 \times \cdot785398 = 4786\cdot020058$ cubic inches $= 16\cdot971702$ ale gallons $= 20\cdot718701$ wine gallons.

PROBLEM XIV.

To find the Solidity of the Slice ABC *cut off a Paraboloid* ADE *by a Plane* FCG *Parallel to the Axe* DH.



Multiply the base of the slice by the square of the diameter of the base of the whole paraboloid, and divide the product by the square of the chord of the base of the slice; from the quotient subtract $\frac{1}{3}$ of the product of the said chord, and its distance from the center of the base of the paraboloid; and the difference multiplied by half the altitude of the slice will be the solidity.

That

That is, $(\frac{FAG \times EA^2}{FG^2} - \frac{FG \times HB}{3}) \times \frac{1}{2}BC$ is the slice ABC.*

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* DEMONSTRATION.

By corollary 4 prop. 1 sect. 4 the section FCG is a parabola, having the same parameter with the generating parabola EDA, and therefore $\frac{2}{3}FG \times CB \times \dot{BA}$ is the fluxion of the solid: but, putting $a = DH$, $r = HA$, $x = AB$, and $y = BG$, by the property of the parabola, it will be $rr : yy :: a : CB = \frac{ayy}{rr}$; and, by the property of the circle, $r - x = \sqrt{rr - yy}$; hence $\dot{BA} = \dot{x} = \frac{y\dot{y}}{\sqrt{rr - yy}}$; then, by substituting these values, the fluxion of the solid will become $\frac{4a}{3rr} \times \frac{y^4\dot{y}}{\sqrt{rr - yy}}$; whose fluent is $a \times$ area BAG $- \frac{ay^3\sqrt{rr - yy}}{3rr} =$ the solid required.

But, putting A for the altitude BC, it will be $yy : rr :: A : a = \frac{Arr}{yy}$; which being substituted for it, in the above found value of the solid, gives $A \times (\frac{rr}{yy} \times BAG - \frac{1}{3}y\sqrt{rr - yy}) = \frac{1}{2}BC \times (\frac{EA^2}{FG^2} \times FAG - \frac{1}{3}FG \times HB)$ for the slice ACB. *Q.E.D.*

Corol. 1. The slice ABC is equal to a paraboloid of the same altitude, and whose base is equal to the difference between $\frac{2}{3}$ of the rectangular area HBGI, and a fourth proportional to the areas BG^2, HA^2, FAGF.

Corol. 2. From the demonstration it appears that the slice is also equal to $a \times (BAG - \frac{y^3\sqrt{rr - yy}}{3rr})$.

Corol. 3. When B coincides with H, A will be $= a$, and $r = y$; and therefore the rule becomes $a \times HAK$ for half the paraboloid.

Corol.

EXAMPLE I.

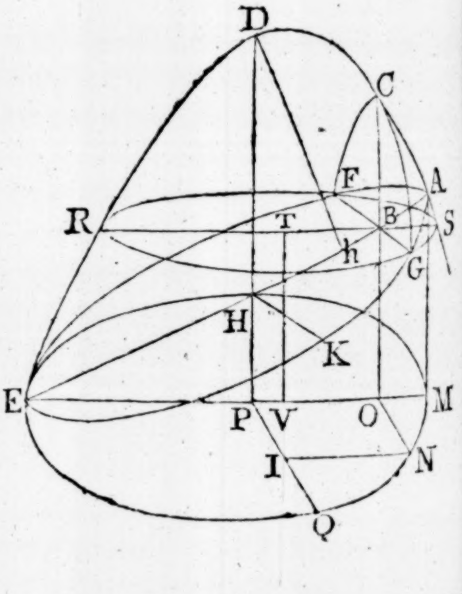
If a cask, composed of two equal paraboloidal frustums, having its length = 40, its bung diameter 32, and its head diameter 24 inches, when laid with its axe parallel to the horizon, want so much of being full, that there are $3\frac{1}{5}$ or 3·2 inches of the bung diameter left dry; it is required to find how many ale gallons will be necessary to fill the cask.

Here 3·2 is the versed sine of the base of the slice, and $3{\cdot}2 \div 32 = {\cdot}1 =$ the tabular versed sine, to which, in the table, corresponds the area ·04087528. And hence $\cdot 04087528 \times 32^2 = 41{\cdot}85628672 =$ the base of the slice.

But,

Corol. 4. From $a \times$ HAK = half the paraboloid, subtract $a \times$ BAG $- \dfrac{ay^3\sqrt{rr-yy}}{3rr} =$ the slice ABC, and there will remain $a \times$ HEGK $+ \dfrac{ay^3\sqrt{rr-yy}}{3rr} = a \times$ HBGK $+ \frac{1}{3}yy\sqrt{\text{DL} \times \text{LH}}$ for the value of the complemental slice DHBC; LC being drawn parallel to HA.

Corol. 5. After the same manner may be found the slice parallel to the axe of an oblique segment ADE. Thus, since $\frac{2}{3}$FG × CB × AB × S.∠ABC is the fluxion of the slice ABC; by putting $a =$ DH the diameter of the double ordinate AE, $t =$ HA the semi-transverse axe of the section AFEGA, $c =$ HK $= \frac{1}{2}$EM the semi-conjugate, $x =$ AB, and $y =$ BG; we shall have $cc : yy :: a : \dfrac{ayy}{cc} =$ BC; but cc :



yy

But, by the property of the circle, $2\sqrt{(32-3\cdot2)\times3\cdot2} = 2\sqrt{28\cdot8\times3\cdot2} = 19\cdot2$ is the chord of the base of the slice; and $16-3\cdot2 = 12\cdot8$ is its distance from the center or middle of the bung diameter. Also, by the nature of the parabola, $16^2-12^2 : 9\cdot6^2 :: 20 : 16\frac{16}{35}$ the altitude of the slice, or half the length of the empty part.

Then $\left(\frac{41\cdot85628672\times32^2}{19\cdot2\times19\cdot2} - \frac{19\cdot2\times12\cdot8}{3}\right)\times\frac{24^2}{35} = \left(\frac{41\cdot85628672\times5^2}{3^2} - 6\cdot4\times12\cdot8\right)\times\frac{24^2}{35} = \frac{8^2}{7}\times(41\cdot85628672\times5 - \cdot4\times19\cdot2^2) = 565\cdot2611072$ cubic inches $= 2\cdot004472$ ale gallons, the content of the empty part required.

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$yy :: tt : tt-(t-x)^2$; hence $t-x = \frac{t\sqrt{cc-yy}}{c}$, and $\dot{x} = \dot{\text{AB}} = \frac{ty\dot{y}}{c\sqrt{cc-yy}}$; also s. $\angle$ABC is $= \frac{c}{t}$; consequently the above fluxion will become $= \frac{4}{3}y\times\frac{ayy}{cc}\times\frac{ty\dot{y}}{c\sqrt{cc-yy}}\times\frac{c}{t} = \frac{4ay^4\dot{y}}{3cc\sqrt{cc-yy}}$; the fluent of which, viz. $\frac{ac}{t}\times$ elliptic area BAG $- \frac{ay^3\sqrt{cc-yy}}{3cc}$, will be the value of the slice BCA.

Or, if Db be drawn perpendicular to AE; since, by similar triangles, it is $t : c :: a : \frac{ac}{t} =$ Db, the same slice BCA will be expressed by D$b\times$ BAG $-$

$\frac{ay^3\sqrt{cc-yy}}{3cc} = \text{D}b\times\text{BAG} - \frac{\text{DH}\times\text{FG}^3\sqrt{\text{EM}^2-\text{FG}^2}}{12\text{EM}^2}$.

And if CB be produced to meet EM in O; and ON be perpendicular to EM, and meet the circle ENM in N; then will the circle ENM be to the ellipse EGA, as well as the segment NMO to the segment GAB, as EM to EA, or as c to t; and therefore the

EXAMPLE II.

If the ſame caſk be placed in the ſame manner, and $6\frac{2}{5}$ inches of the bung diameter be dry; required how many ale gallons the caſk wants of being full.

Here the dry inches being more than $16 - 12$ or 4, the difference of the bung and head ſemi-diameters, it ſhews that $6\frac{2}{5} - 4 = 2\frac{2}{5}$ inches of the head diameter alſo are dry; and therefore the empty part of the caſk will be double the difference of two ſlices, the verſed ſines of whoſe baſes are $6\frac{2}{5}$ and $2\frac{2}{5}$; the whole diameters being 32 and 24.

Now the tabular verſed ſines will be $6{\cdot}4 \div 32 = {\cdot}2$, and $2{\cdot}4 \div 24 = {\cdot}1$; whoſe tabular areas are ${\cdot}11182380$ and ${\cdot}04087528$; hence $32^2 \times {\cdot}1118238$ =

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the value of the ſlice BCA will alſo be $a \times \text{MNO} - \frac{ay^3\sqrt{cc - yy}}{3cc} = \text{DH} \times \text{MNO} - \frac{\text{DH} \times \text{FG}^3\sqrt{\text{EM}^2 - \text{FG}^2}}{12\text{EM}^2}$. Where the circular area MNO, is the projection of the elliptic area AEG, by lines parallel to the diameter DH, upon a plane perpendicular to the ſame.

Moreover, ſince $yy : cc :: \text{BC} = \text{A} : a = \text{DH} = \frac{\text{A}cc}{yy}$, by ſubſtituting this value of a inſtead of it, the ſolidity of the ſame ſlice will become

$$\frac{\text{A}c^3}{tyy} \times \text{AGE} - \tfrac{1}{3}\text{A}y\sqrt{c^2 - y^2} = \left(\frac{\text{PM}^2}{\text{ON}^2} \times \text{MNO} - \tfrac{1}{3}\text{PO} \times \text{ON}\right) \times \text{BC}.$$

Corol. 6. Hence the ſlice FGAC, is equal to a paraboloid whoſe altitude is BC, and its baſe equal to the difference between $\frac{2}{3}$ of the rectangular area PONI, and the fourth proportional to the areas ON^2, PM^2, 2OMN.

Corol. 7. When B coincides with H, the rule becomes barely $\text{DH} \times \text{PMQ} = \text{D}b \times \text{AHK}$, for the value of DAH or DEH or half the oblique ſegment.

$= 114\cdot5075712 =$ the base of the greater slice, and $24^2 \times \cdot04087528 = 23\cdot54416128 =$ that of the less.

Again, $16 - 6\cdot4 = 9\cdot6$ is the distance of the chord of the base of the greater slice from the middle of the bung diameter, and $12 - 2\cdot4 = 9\cdot6$ is the distance of that of the less from the middle of the head diameter; and hence the former chord will be $2\sqrt{16^2 - 9\cdot6^2} = 2\sqrt{25\cdot6 \times 6\cdot4} = 25\cdot6$, and the latter $= 2\sqrt{12^2 - 9\cdot6^2} = 2\sqrt{21\cdot6 \times 2\cdot4} = 14\cdot4$.

Moreover, by the nature of the parabola, as $16^2 - 12^2$ or $28 \times 4 : 20 :: \begin{cases} 12\cdot8^2 : 29\frac{9}{35} \\ 7\cdot2^2 : 9\frac{9}{35} \end{cases}$ the altitudes of the slices.

 Consequently,

Corol. 8. By taking the slice from the semi-segment, we obtain $\text{DH} \times \text{PONQ} + \text{DH} \times \frac{y^3\sqrt{cc - yy}}{3cc} = \text{DH} \times \text{PONQ} + \text{DH} \times \frac{\text{ON}^3 \times \text{PO}}{3\text{PM}^2}$ for the value of the complemental slice DCBH.

Corol. 9. And by adding this last to the semi-segment, there results $\text{DH} \times \text{EONQ} + \text{DH} \times \frac{\text{PO} \times \text{ON}^3}{3\text{PM}^2} = \frac{\text{BC} \times \text{PM}^2}{\text{ON}^2} \times \text{EONQ} + \text{BC} \times \frac{1}{3}\text{PO} \times \text{ON}$, for the value of the slice EDCBE greater than the semi-segment.

Corol. 10. If RGSFR be a circle perpendicular to BC; and if from its center T be demitted the perpendicular TV; then, by taking the oblique slice ABC from the right slice SBC, we shall obtain

$$\text{BC} \times \frac{\text{RS}^2 \times \text{BSG} - \text{EM}^2 \times \text{OMN} + \text{FG}^3 \times \frac{1}{3}\text{PV}}{\text{FG}^2}$$

for the value of the ungula, or cuneus, AFGSA, contained between the two sections RFSGR, EFAGE. And the same will hold, though both the sections be oblique.

Corol. 11. And, in the same manner,

$$\text{BC} \times \frac{\text{EM}^2 \times \text{EONQ} - \text{RS}^2 \times \text{BRG} + \text{FG}^3 \times \frac{1}{3}\text{PV}}{\text{FG}^2}$$

will be the value of the opposite ungula or cuneus EGFRE.

Consequently, $\left(\frac{114\cdot5075712 \times 32^2}{25\cdot6^2} - \frac{25\cdot6 \times 9\cdot6}{3}\right) \times \frac{32^2}{35}$ $= \left(\frac{114\cdot5075712 \times 5^2}{4^2} - 25\cdot6 \times 3\cdot2\right) \times \frac{32^2}{35} = \frac{8^2}{7} \times$ $(114\cdot5075712 \times 5 - \cdot4 \times 25\cdot6^2) = 2837\cdot886683$ is double the greater slice.

And $\left(\frac{23\cdot54416128 \times 24^2}{14\cdot4^2} - \frac{14\cdot4 \times 9\cdot6}{3}\right) \times \frac{18^2}{35} =$ $\left(\frac{23\cdot54416128 \times 5^2}{3^2} - \frac{14\cdot4 \times 9\cdot6}{3}\right) \times \frac{18^2}{35} = \frac{6^2}{7} \times$ $(23\cdot54416128 \times 5 - \cdot4 \times 14\cdot4^2) = 178\cdot852147$ is double the less slice.

Therefore $2837\cdot886683 - 178\cdot852147 =$ $2659\cdot034536$ inches $= 9\cdot4292$ ale gallons is the quantity required.

PROBLEM XV.

To find the Solidity of a Parabolic Spindle.

Multiply the area of the greatest circle, or middle section, by the length, and $\frac{8}{15}$ of the product will be the content.*

That

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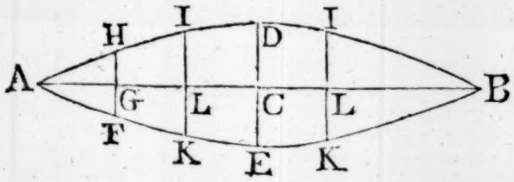
* DEMONSTRATION.

Putting $a =$ DC, $b =$ CA, $c = 3\cdot14159$, $x =$ AG, and $y =$ GH; we shall have, by the property of the parabola, as AC^2 : AG × GB :: CD : GH $= a \times \frac{2bx - xx}{bb} = \frac{2bx - xx}{p} = y$, putting $p = \frac{bb}{a}$ the parameter of DC; and hence the fluxion of the solid $cyy\dot{x}$, will be $\frac{c\dot{x}}{pp} \times (2bx - xx)^2$; the fluent of which, $cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{5}xx}{pp}$, will be a general expression for the

segment

That is, $\frac{8}{15}n \times DE^2 \times AB$ = the whole solid ADBEA; AB being the length or axe, DE the greatest diameter, or double the abscis of the generating parabola ADB, and $n = \cdot785398$.



EXAMPLE.

Required the solidity of a parabolic spindle, whose length is 80, and greatest diameter 32.

By the rule, $\cdot785398 \times 32^2 \times 80 \times \frac{8}{15} = \cdot785398 \times 43690\frac{2}{3} = 34314\cdot5693576$ is the content required.

PROBLEM XVI.

To find the Content of the Segment AFH *of a Parabolic Spindle.*

To $\frac{4}{3}$ of the square of the length of the semi-spindle, add $\frac{1}{5}$ of the square of the altitude of the segment;

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segment AFH; which, when $x = b$, gives $\dfrac{8cb^5}{15pp} = \frac{8}{15}caab$ for the value of the semi-spindle ADE. *Q. E. D.*

Corol. 1. A parabolic semi-spindle, is to a cylinder of the same base and altitude, as 8 to 15; and to a paraboloid of the same base and altitude, as 8 to $7\frac{1}{2}$, or as 16 to 15.

Corol. 2. It appears from the demonstration, that the segment AFH is expressed by $cx^3 \times \dfrac{\frac{4}{3}bb - bx + \frac{1}{5}xx}{pp} =$

$$\frac{\frac{4}{3}AC^2 - AC \times AG + \frac{1}{5}AG^2}{AC^4} \times c \times AG^3 \times DC^2.$$

Corol.

ſegment; from the ſum take the product of the ſaid length and altitude; and call the difference P.

Multiply the cube of the altitude of the ſegment by the ſquare of the greateſt diameter of the ſpindle; divide the product by the fourth power of the length of the ſemi-ſpindle; and call the quotient Q.

Then the product of P and Q multiplied by ·785398, will give the content of the ſegment.

That is, $\frac{\frac{4}{3}AC^2 - AC \times AG + \frac{1}{5}AG^2}{AC^4} \times AG^3 \times DE^2 \times$ ·785398, is the ſegment AFH, by corollary 2 of the laſt problem.

EXAMPLE.

If the diameter of the baſe of the ſegment of a parabolic ſpindle be 24, and its altitude 20; what will be the content, ſuppoſing the length of the whole ſpindle to be 80?

Here

Corol. 3. If from the value of the ſemi-ſpindle ADE, be taken that of the ſegment AFH, there will remain $\frac{8cb^5}{15pp} - cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{5}xx}{pp} = \frac{cz}{pp}$ or $\frac{ca^2z}{b^4} \times (b^4 - \frac{2}{3}b^2z^2 + \frac{1}{5}z^4)$ for the value of the fruſtum HDEF; putting z inſtead of $b - x$ or GC.

And if inſtead of z, in the two laſt terms of this expreſſion, be ſubſtituted its value $b\sqrt{\frac{a-y}{a}}$, the value of the ſaid fruſtum will be denoted by

$$cz \times \frac{8aa + 4ay + 3yy}{15} = c \times GC \times \frac{8DE^2 + 4DE \times FH + 3FH^2}{60},$$

Corol. 4. If to, or from, the fruſtum

$$HDEF = cA \times \frac{8D^2 + 4Dd + 3d^2}{60},$$

be

Here $AC = 40$, $AG = 20$, and $GH = 12$; then $GC = 40 - 20 = 20$; and, by the parabola, $AC^2 - GC^2 : AC^2 :: GH : CD = 16$; the double of which gives $DE = 32$.

Then $\left(\frac{4 \times 40^2}{3} - 40 \times 20 + \frac{20^2}{5}\right) \times \frac{20^3 \times 32^2}{40^4} \times \cdot785398 = \left(\frac{4}{3} - \frac{1}{3} + \frac{1}{20}\right) \times 5 \times 32^2 \times \cdot785398 = 4522\frac{2}{3} \times \cdot785398 = 3552\cdot094093659$ is the content required.

PROBLEM XVII.

To find the Content of the Frustum of a Parabolic Spindle, one of the Ends of the Frustum passing through the Center of the Spindle.

Add into one sum, 8 times the square of the diameter of the greater end, 3 times the square of the diameter of the less end, and 4 times the product of the

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be added, or subtracted, the frust. $DIKE = ca \times \frac{8D^2 + 4D\delta + 3\delta^2}{60}$, there will result $c \times \frac{8D^2(A \pm a) + dA(4D + 3d) \pm \delta a(4D + 3\delta)}{60}$ for the value of the frustum HIKF, neither of whose ends pass through the center of the spindle; the upper or under signs being used, according as the said center is within or without the frustum, and in which D represents the diameter through the center of the spindle, d the diameter of the less end of the frustum, and δ that of the greater; also A the distance of d from the center of the spindle, and a that of δ from the same. Or in a cask of this form, standing upon one end, D will be the bung diameter, d the head diameter, 2A the length of the cask, $A \pm a = w$ the wet inches of 2A, and δ the diameter at the surface of the liquor, the value of which diameter is as determined in the following corollary.

Corol. 5. Since, by the property of the parabola, $A^2 : a^2 :: D - d : D - \delta$, the value of δ will be $\frac{DA^2 - a^2(D - d)}{AA}$.

the diameters; multiply the ſum by the length; and the product multiplied by $\cdot 05236$, or $\frac{1}{15}$ of $\cdot 785398$, will be the content.

That is, $(3\text{DE}^2 + 4\text{DE} \times \text{FH} + 3\text{FH}^2) \times \text{CG} \times \cdot 05236$ is the fruſtum DEFH, by corollary 3 to prob. 15.

EXAMPLE.

Required the content of a caſk in the form of the middle zone of a parabolic ſpindle, the bung diameter being 32 inches, the head diameter 24, and length 40 inches.

The caſk is evidently two equal fruſtums, whoſe greater diameters are 32, and leaſt 24, and their lengths each 20; therefore

$$(8 \times 32^2 + 4 \times 32 \times 24 + 3 \times 24^2) \times 40 \times \cdot 05236$$
$$= 8^3 \times (8 \times 4^2 + 3 \times 4^2 + 3 \times 3^2) \times 5 \times \cdot 05236$$
$$= (11 \times 16 + 27) \times 8^3 \times 5 \times \cdot 05236 = 519680$$
$$\times \cdot 05236 = 27210\cdot\overset{44480}{\cancel{38117}}$$ inches $= 96\cdot 4907$ ale gallons, the content required.

PROBLEM XVIII.

To find the Content of the Fruſtum of a Parabolic Spindle, neither of whoſe Ends paſs through the Center of the Spindle.

Multiply the diameter of each end, by its diſtance from the diameter in the middle of the ſpindle; and multiply each product by the ſum of 3 times the ſaid diameter of the end, and 4 times the ſaid middle diameter; to or from the product belonging to the leſs diameter, add or ſubtract that which belongs to the greater, according as the center of the ſpindle is within or without the fruſtum; to the ſum or difference, add 8 times the product ariſing from the multiplication of the length of the fruſtum by the ſquare

square of the diameter in the middle of the spindle; and this sum multiplied by ·05236, or $\frac{1}{15}$ of ·785398, will produce the content of the frustum.

That is, $[8DE^2 \times GL + (4DE + 3FH)FH \times GC \pm (4DE + 3IK)IK \times LC] \times \cdot05236$ is the frustum HIKF; using the upper or under sign, according as GL is greater or less than GC; as is proved in corollary 4 to prob. 15.

Note. The value of the greater diameter IK is $\frac{DE \times GC^2 - LC^2 \times (DE - FH)}{GC^2}$; as in corollary 5 to the same problem.

EXAMPLE I.

If the liquor in the cask, in the example to the last problem, when standing upon its end, with its axe perpendicular to the horizon, rise to the height of 30 inches; what quantity of liquor will there be?

Here, by the note, $IK = \frac{32 \times 20^2 - 10^2 \times (32 - 24)}{20 \times 20} = 30$.

Then, by the rule, $[8 \times 32^2 \times 30 + (4 \times 32 + 3 \times 24) 24 \times 20 + (4 \times 32 + 3 \times 30) 30 \times 10] \times \cdot05236 = [8 \times 16^2 + (128 + 72)4 + (64 + 45)5] \times 120 \times \cdot05236 = 407160 \times \cdot05236 = 21318\cdot84775$ cubic inches $= 75\cdot59875$ ale gallons, the quantity required.

EXAMPLE II.

If in the same cask, placed as before, the liquor rise but to the height of 10 inches; how much will be in it?

Here all the dimensions are the same quantities as in the last example, excepting GL, which here is only 10, instead of 30; and the cask is less than half full.

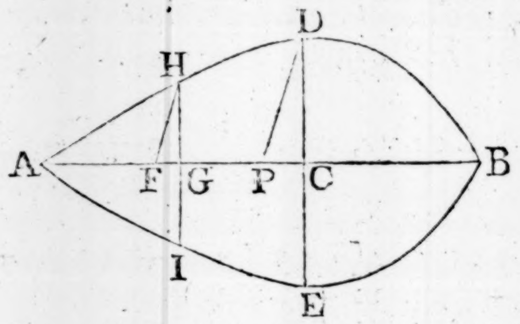
Wherefore,

Wherefore, by the rule, $[8 \times 32^2 \times 10 + (4 \times 32 + 3 \times 24) 24 \times 20 - (4 \times 32 + 3 \times 30) 30 \times 10] \times {\cdot}05236 = [8 \times 16^2 + (128 + 72) 12 - (64 + 45) 15] \times 40 \times {\cdot}05236 = 112520 \times {\cdot}05236 = 5891{\cdot}533423$ cubic inches $= 20{\cdot}89196$ ale gallons, the quantity required.

PROBLEM XIX.

To find the Content of the Universal Parabolic Spindle, or Solid generated by the Revolution of a Parabolic Segment about its Base, being an Ordinate to Any Diameter.

Multiply the area of the greatest section by the length, and $\frac{8}{15}$ of the product will be the content; as in the common spindle.



That is, $DE^2 \times AB \times \frac{8}{15}n =$ the solid ADBEA, generated from the revolution of the parabolic segment ADB about its base AB, which is a double ordinate to the diameter DP, and DCE being perpendicular to AB.*

EX-

* DEMONSTRATION.

Let GH be parallel to CD, and HF parallel to DP; and put $a =$ DP, $\frac{1}{2}b =$ AP $= \frac{1}{2}$AB, $\frac{1}{2}c =$ CD $= \frac{1}{2}$DE, $z =$ AF, and $p = 4n = 3{\cdot}14159$. Then, by the property of the parabola, $AP^2 : AF \times FB :: PD : FH ::$ by sim. triangles, CD : GH = CD

EXAMPLE.

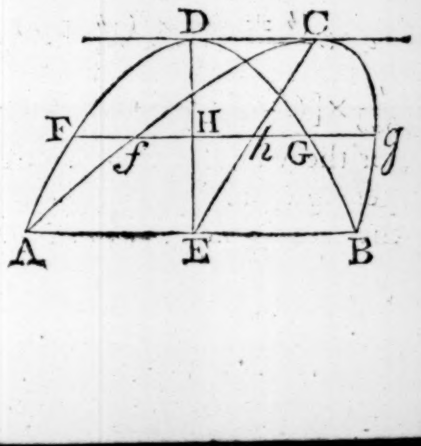
If an oblique parabola, whose base is inclined to its diameter in an angle of 30 degrees, be turned about its base; required the value of the solid generated by its revolution; the base of the parabola, or length of the spindle, being 80, and the diameter from the vertex to the base 32.

Here AB $= 80$, DP $= 32$, and the angle DPC $= 30°$; then, as radius $= 1 : \frac{1}{2} =$ sine $\angle$P or $30°$ :: PD $= 32$: DC $= 16$ the radius of the greatest circle of the spindle.

$\frac{CD \times AF \times FB}{AP^2} = 2c \times \frac{bz - zz}{bb}$ :: PC $= \sqrt{PD^2 - DC^2} = d$: FG $= \frac{GH \times PC}{CD} = 4d \times \frac{bz - zz}{bb}$; hence AG $=$ AF $+$ FG $= z \times \frac{bb + 4bd - 4dz}{bb}$; and the fluxion $p \times GH^2 \times AG$ of the solid will be $p\dot{z} \times \frac{bb + 4bd - 8dz}{bb} \times \frac{4cc}{b^4} \times (bz - zz)^2 = \frac{4pc^2z^2\dot{z}}{b^6} \times [b^3 \times (b + 4d) - 2b^2z \times (b + 8d) + bz^2 \times (b + 20d) - 8dz^3]$; the fluent of which, viz. $\frac{4pc^2z^3}{b^6} \times [\frac{1}{3}b^3 \times (b + 4d) - \frac{1}{2}b^2z \times (b + 8d) + \frac{1}{5}bz^2 \times (b + 20d) - \frac{4}{3}dz^3]$ will be a general expression for the value of the segment; which, when z becomes $= b$, will be $\frac{2}{15}pccb = \frac{8}{15}nbcc$ for the content of the whole solid ADBEA. *Q. E. D.*

Corollary. If any parabolas ADE, ACE, having the same base AB and between the same parallels AE, CD, be turned about their common base; they will generate equal spindles.

For the lengths and greatest diameters will be the same in all of them.

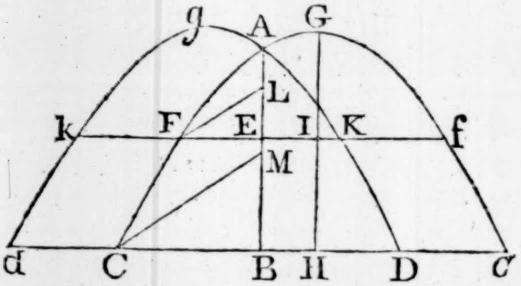


Therefore $32^2 \times 80 \times \frac{8}{15} \times \cdot 785398 = \cdot 785398 \times 43690\frac{2}{3} = 34314\cdot 5693576$, as in the example to prob. 15.

PROBLEM XX.

To find the Value of an Universal Paraboloid, or a Body Generated by the Revolution of a Parabola about Any Diameter.

INVESTIGATION.



Suppose AB to be the diameter about which the parabola CAc revolves, CAD a vertical section of the solid generated by AC, and dgAGc that of the solid generated by AGc; also let GIH be perpendicular, and FEIf parallel to Cc. Then, by the property of the parabola, $\text{CH}^2 = bb : \text{CH}^2 - \text{FI}^2 = \text{CH}^2 - (\text{FE} \mp \text{EI})^2 = bb - (y \mp d)^2 :: \text{GH} = a : \text{EB} = x :: -2\dot{y} \times (y \mp d) : \dot{x} = -2a\dot{y} \times \frac{y \mp d}{bb}$; hence the fluxion of the solid $2cyy\dot{x}$ will become $-2acy^2\dot{y} \times \frac{y \mp d}{bb}$; and the correct fluent gives $ac \times \frac{(b \pm d)^3 \times (3b \mp d) - y^3 \times (3y \mp 4d)}{6bb} =$

$$\frac{ac}{6bb} \times [(b \pm d)^3 \times (3b \mp d) - (b\sqrt{\tfrac{a-x}{a}} \pm d)^3 \times (3b\sqrt{\tfrac{a-x}{a}} \mp d)]$$

for the value of the solid required; in which the upper signs respect the solid $dkfc$ generated by cf, and the under ones the solid DKFC generated by CF.

 Corol.

Corol. 1. When E arrives at A, then $y = 0$, and the expression becomes $ac \times (b \pm d)^3 \times \frac{3b \mp d}{6bb} = c \times \text{GH} \times \text{CB}^3 \times \frac{\text{CB} + 2\text{CH}}{6\text{CH}} =$, by substituting for GH its value $\frac{\text{CH}^2 \times \text{BA}}{\text{CB} \times \text{B}c}$, $c \times \text{CB}^2 \times \text{AB} \times \frac{\text{B}c + 2\text{CH}}{6\text{B}c}$ for the solid CAD.

And if CM be an ordinate to the diameter AB; we shall have $\text{B}c : \text{BC} :: \text{AB} : \text{AM} = \frac{\text{AB} \times \text{BC}}{\text{B}c}$; hence $\text{AB} \times \frac{\text{B}c + 2\text{CH}}{6\text{B}c}$ will be $= \frac{2\text{AB} + \text{AM}}{6}$, and consequently the above value of the solid CAD becomes $c \times \text{CB}^2 \times \frac{2\text{AB} + \text{AM}}{6}$.

Corol. 2. When B coincides with H, d will be $= 0$, and then the rule becomes $ac \times \frac{b^4 - y^4}{2bb} = c \times \frac{1}{2}\text{HI} \times (bb + yy)$ for the frustum CFfc generated about HI, as in problem 12. Or $c \times \text{CH}^2 \times \frac{1}{2}\text{GH}$ for the whole paraboloid, as in prob. 11.

THE INVESTIGATION OTHERWISE.

Supposing FL to be an ordinate to the diameter AB, whose parameter is p; and putting m and n for the sine and cosine of the angle L, to the radius 1; we shall have $\text{FL} = \sqrt{p \times \text{AL}} = \sqrt{px}$, where x is now $= \text{AL}$; then $\text{FE} = m\sqrt{px}$, and $\text{LE} = n\sqrt{px}$; hence $\text{AE} = \text{AL} + \text{LE} = x + n\sqrt{px}$. Then the fluxion of the solid $c \times \text{FE}^2 \times \text{AE}$ is $cm^2px \times (\dot{x} + \frac{1}{2}n\dot{x}\sqrt{\frac{p}{x}})$; whose fluent gives $cm^2px \times (\frac{1}{2}x + \frac{1}{3}n\sqrt{px}) = \frac{1}{6}cyy \times (3x + 2n\sqrt{px}) = \frac{1}{6}c \times \text{FE}^2 \times (2\text{AE} + \text{AL})$ for the value of the solid AFK, generated by the revolution of AF. And $\frac{1}{6}c \times \text{CB}^2 \times (2\text{AB} + \text{AM})$ for the solid CAD, as above.

Scholium.

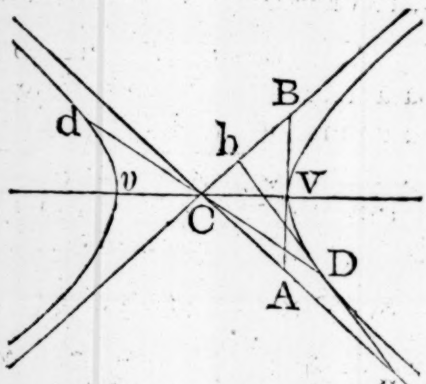
Scholium. It is evident that the ſurface of this ſolid might be found by the method by which we determined that of an elliptic ſpindle; and the ſurface of the parabolic ſpindle might eaſily be determined alſo, if there appeared any occaſion for it; but not by the ſame method.

SECTION VII.

OF HYPERBOLIC LINES, AREAS, SURFACES, AND SOLIDITIES.

DEFINITIONS.

TO what hath been ſaid of the hyperbola among the conic ſections in general, may be added the following obſervations.



If AVB touch the hyperbola in the vertex V, and be equal to the conjugate axe, viz. AV and VB each equal to half the conjugate axe; and if from the center

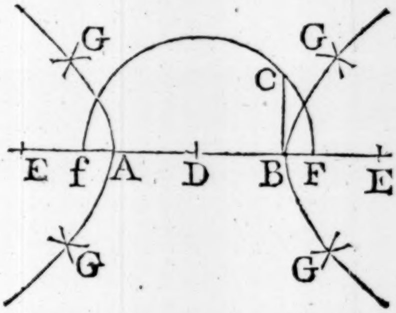
center C, through the extremities A, B, right lines CA, CB, be drawn; those lines will be what are called asymptotes to the hyperbola; that is, they are lines which continually approach to the curve.

And, like as the conjugate axe is a tangent to the curve at the vertex of the transverse, and bounded by the asymptotes; so, if to the extremity D, of any diameter *d* D, be drawn a tangent *a* D *b*, and terminated by the asymptotes, then *ab* will be the conjugate to *d* D.

When the asymptotes form a right angle at the center, the hyperbola is said to be right angled; as also equilateral, because its axes are then equal to each other.

PROBLEM I.

To Construct an Hyperbola, having given the Transverse and Conjugate Axes AB *and* 2BC.



The semi-conjugate BC being erected perpendicular to AB, and D being the middle of AB, or the center of the hyperbola; with the center D and radius DC, describe an arc meeting AB, both ways produced, in F and *f*; which will be the two foci.

Then, assuming several points E in the transverse produced, with the radii AE, BE, and centers *f*, F, describe arcs intersecting in the several points G; through all which points draw the hyperbolic curve.

PROBLEM II.

In an Hyperbola, to find any Two Conjugate Diameters, an Ordinate to one of them, and its Abſciſs, one from another; viz. having any three of them given, to find the Fourth.

CASE I.

To find the Ordinate.

As any diameter :
Is to its conjugate : :
So is the mean propor. between the abſciſſes :
To the ordinate.*

That is, as $d : c :: \sqrt{(d+x)x} : y = \frac{c}{d}\sqrt{(d+x)x}$; where d denotes the diameter, c its conjugate, y an ordinate to the diameter, and x its abſciſs, or its diſtance from the vertex of the diameter, meaſured upon it.

Note. In the hyperbola, the leſs abſciſs added to the diameter, gives the greater abſciſs.

EXAMPLE.

If the diameter be 24, its conjugate 21, and the leſs abſciſs 8; what is the ordinate?

By the rule, $\frac{c\sqrt{(d+x)x}}{d} = \frac{21\sqrt{32 \times 8}}{24} = \frac{7 \times 16}{8} =$ 14 the ordinate required.

CASE

* The values of the ſeveral quantities, in the four caſes of this problem, are eaſily found from the general property of the curve, viz. $dd : cc :: (d+x)x : yy$. The demonſtration of which, together with that of the preceding problem, belongs to conic ſections.

CASE II.

To find the Abſciſſes.

As the conjugate :
Is to the diameter : :
So is the root of the ſum of the ſquares of the ordinate and ſemi-conjugate :
To the diſtance between the foot of the ordinate and the center.

Then, to and from this diſtance, add and ſubtract the ſemi-diameter, and the ſum and difference will give the two abſciſſes.

That is, $\frac{d\sqrt{\frac{1}{4}cc+yy}}{c} \pm \frac{1}{2}d = x$ the greater or leſs abſciſs, according as the upper or under ſign is uſed.

EXAMPLE.

The diameter and its conjugate being 24 and 21, required the two abſciſſes to the ordinate 14.

Here $\frac{d\sqrt{\frac{1}{4}cc+yy}}{c} \pm \frac{1}{2}d = \frac{24\sqrt{\frac{1}{4}\times 21^2+14^2}}{21} \pm 12$ $= \frac{4\sqrt{21^2+28^2}}{7} \pm 12 = 4\sqrt{3^2+4^2} \pm 12 = 20$ $\pm 12 = 32$ and 8, the two abſciſſes required.

CASE III.

To find the Conjugate.

As the mean propor. between the abſciſſes :
Is to the ordinate : :
So is the diameter :
To its conjugate.

That is, $\sqrt{(d+x)x} : y :: d : \frac{dy}{\sqrt{(d+x)x}} = c.$

EXAMPLE.

What is the conjugate to the diameter 24, the leſs abſciſs being 8, and its ordinate 14?

Here $\frac{dy}{\sqrt{(d+x)x}} = \frac{24 \times 14}{\sqrt{32 \times 8}} = \frac{24 \times 14}{16} = 21$, the conjugate.

CASE IV.

To find the Diameter.

To or from the root of the ſum of the ſquares of the ordinate and ſemi-conjugate, add or ſubtract the ſemi-conjugate, according as the leſs or greater abſciſs is uſed: Then, as the ſquare of the ordinate, is to the product of the abſciſs and conjugate; ſo is the ſum or difference, above found, to the diameter.

That is, $cx \times \frac{\sqrt{\frac{1}{4}cc + yy} \pm \frac{1}{2}c}{yy} = d.$

EXAMPLE.

The leſs abſciſs being 8, and its ordinate 14, required the diameter, ſuppoſing the conjugate to be 21.

Here

$$cx \times \frac{\sqrt{\frac{1}{4}cc + yy} + \frac{1}{2}c}{yy} = 21 \times 8 \times \frac{\sqrt{\frac{1}{4} \times 21^2 + 14^2} + 10\frac{1}{2}}{14 \times 14}$$

$$= 3 \times \frac{\sqrt{21^2 + 28^2} + 21}{7} = 3 \times (\sqrt{3^2 + 4^2} + 3) =$$

$3 \times (5 + 3) = 24$, the diameter required.

PRO-

PROBLEM III.

Having given any Two Abscisses X, *x*, *and their Ordinates* Y, *y*; *to find the Diameter d to which they belong.*

Multiply each abscis by the square of the ordinate belonging to the other; multiply also the square of each abscis by the square of the other's ordinate; then divide the difference of the latter products by the difference of the former; and the quotient will be the diameter to which the ordinates belong.*

$$\text{That is, } \frac{\mathrm{XX}yy \backsim xx\mathrm{YY}}{x\mathrm{YY} \backsim \mathrm{X}yy} = d.$$

EXAMPLE.

If two abscisses be 1 and 8, and their two ordinates $4\frac{3}{8}$ and 14; what is the diameter to which they belong?

$$\text{Here } \frac{\mathrm{XX}yy - xx\mathrm{YY}}{x\mathrm{YY} - \mathrm{X}yy} = \frac{8^2 \times 4\frac{3}{8} \times 4\frac{3}{8} - 1^2 \times 14^2}{1 \times 14^2 - 8 \times 4\frac{3}{8} \times 4\frac{3}{8}} =$$

$$\frac{35 \times 35 - 14 \times 14}{14 \times 14 - 35 \times 4\frac{3}{8}} = \frac{5 \times 5 - 2 \times 2}{2 \times 2 - 5 \times \frac{5}{8}} = \frac{21 \times 8}{7} = 24, \text{ the}$$

diameter required.

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* DEMONSTRATION.

By the nature of the hyperbola, $d\mathrm{X} + \mathrm{XX} : dx + xx :: \mathrm{YY} : yy$, or $d\mathrm{X}yy + \mathrm{XX}yy = dx\mathrm{YY} + xx\mathrm{YY}$; hence $d = \frac{\mathrm{XX}yy - xx\mathrm{YY}}{x\mathrm{YY} - \mathrm{X}yy}$. *Q. E. D.*

PROBLEM IV.

Given any Three Equidistant Ordinates a, b, c, with their Common Distance, or Common Difference d of their Abscisses; to find the Diameter D *to which they belong, together with its Conjugate* C.

* From the sum of the squares of the extreme ordinates, take double the square of the middle one; and call the difference A.

From the square of the difference between A and double the said square of the middle ordinate, take 4 times the square of the product of the extreme ordinates; and call the square root of the remainder B.

Then, as A is to B, so is the given common distance, to the diameter to which the ordinates belong.

And

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* DEMONSTRATION.

If x be the abscis to the least ordinate a, by the nature of the hyperbola we shall have these three equations,

$$aa = \frac{CC}{DD} \times (D + x) \times x \qquad = \frac{CC}{DD} \times (Dx + xx),$$

$$bb = \frac{CC}{DD} \times (D + x + d) \times (x + d) = \frac{CC}{DD} \times (Dx + xx + 2dx + Dd + dd),$$

$$cc = \frac{CC}{DD} \times (D + x + 2d) \times (x + 2d) = \frac{CC}{DD} \times (Dx + xx + 4dx + 2Dd + 4dd).$$

And by subtracting the double of the second from the sum of the first and third, we obtain this equation,

$$aa - 2bb + cc = \frac{CC}{DD} \times 2dd; \text{ and hence } \frac{CC}{DD} = \frac{aa - 2bb + cc}{2dd}.$$

Again, taking the first equation from the second, leaves $bb - aa = \frac{CC}{DD} \times (2dx + Dd + dd)$; hence $x = \frac{DD}{CC} \times \frac{bb-aa}{2d} - \frac{D+d}{2} =$

$d \times$

And as the common distance of the ordinates, is to the root of $\frac{1}{2}$A, so is the diameter D, to its conjugate C. That is,

$$D = \frac{d\sqrt{(-aa+4bb-cc)^2-4aacc}}{aa-2bb+cc}, \text{ and } C = \frac{D\sqrt{\frac{1}{2}A}}{d}.$$

EXAMPLE.

Required the diameter, and its conjugate, to which belong the three ordinates $4\frac{3}{8}$, 14, and $\frac{21}{8}\sqrt{65}$, their common distance being 7.

Here the diam. $D = \frac{d\sqrt{(4bb-aa-cc)^2-4aacc}}{aa-2bb+cc} =$

$$\frac{7\sqrt{(28^2-\frac{35^2}{8^2}-\frac{65\times21^2}{8^2})^2-\frac{4\times65\times35^2\times21^2}{8^4}}}{\frac{35^2}{8^2}-2\times14^2+\frac{65\times21^2}{8^2}} =$$

$$\frac{7\sqrt{(32^2-5^2-65\times3^2)-65\times10^2\times3^2}}{5^2-2\times16^2+65\times3^2} = \frac{7\times336}{98} = 24.$$

D d 4 And

$$d\times\frac{bb-aa}{aa-2bb+cc} - \frac{D+d}{2} = d\times\frac{2bb-\frac{3}{2}aa-\frac{1}{2}cc}{aa-2bb+cc} - \frac{1}{2}D.$$

Consequently, $aa = \frac{CC}{DD}\times(D+x)\times x = \frac{aa-2bb+cc}{2dd}\times$ $[d^2\times(\frac{2bb-\frac{3}{2}aa-\frac{1}{2}cc}{aa-2bb+cc})^2-\frac{1}{4}D^2]$; and hence

$$\frac{d\sqrt{(-aa+4bb-cc)^2-4aacc}}{aa-2bb+cc} = \frac{dB}{A} = \text{the diameter D.}$$

Also from the equation $\frac{CC}{DD} = \frac{aa-2bb+cc}{2dd}$, is deduced

$$\frac{D}{d}\sqrt{\frac{aa-2bb+cc}{2}} = \frac{D\sqrt{\frac{1}{2}A}}{d} = \text{the conjugate C.} \quad \textit{Q. E. D.}$$

Corollary. Hence the sum of the squares of the extreme ordinates, is greater than double the square of the middle one.

And

$$\frac{24}{7}\sqrt{\frac{35^2}{2\times 8^2}-14^2+\frac{65\times 21^2}{2\times 8^2}}=8\sqrt{\frac{5^2}{2}-16^2+\frac{65\times 3^2}{2}}$$

$$=\tfrac{3}{2}\sqrt{50-32^2+9\times 130}=\tfrac{3}{2}\sqrt{196}=\tfrac{3}{2}\times 14=21,$$

the conjugate c.

PROBLEM V.

To find the Length of an Arc of an Hyperbola, the Arc beginning at the Vertex.

RULE I.*

Put $a =$ the semi-transverse axe, $c =$ the semi-conjugate, $q = \frac{cc + aa}{c^4}$, $y =$ an ordinate to the axe drawn from the end of the arc required, A = the hyperbolic logarithm of $\frac{y + \sqrt{cc + yy}}{c} = 2{\cdot}302585093$ $\times$ common log. of $\frac{y + \sqrt{cc + yy}}{c}$, $\text{B} = \frac{y\sqrt{cc + yy} - cc\text{A}}{2}$, $\text{C} = \frac{y^3\sqrt{cc + yy} - 3cc\text{B}}{4}$, $\text{D} = \frac{y^5\sqrt{cc + yy} - 5cc\text{C}}{6}$, $\text{E} = \frac{y^7\sqrt{cc + yy} - 7cc\text{D}}{8}$, &c.

Then

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* DEMONSTRATION.

Putting $x =$ the abscis of the ordinate y, we shall obtain, from the nature of the curve, $x = \frac{a\sqrt{cc + yy}}{c} - a$, and therefore $\dot{x} = \frac{ay\dot{y}}{c\sqrt{cc + yy}}$; hence $\dot{z} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{y}\sqrt{\frac{cc + \frac{cc + aa}{cc}yy}{cc + yy}} = \frac{c\dot{y}}{\sqrt{cc + yy}}\sqrt{1 + \frac{cc + aa}{c^4}yy} = \frac{c\dot{y}}{\sqrt{cc + yy}}\sqrt{1 + qyy} = \frac{c\dot{y}}{\sqrt{cc + yy}}$ $\times\ (1 + \frac{q}{2}y^2 - \frac{q^2}{2.4}y^4 + \frac{3q^3}{2.4.6}y^6 - \frac{3.5q^4}{2.4.6.8}y^8$ &c).

But

Then will the length of the arc be expressed by

$$c \times \left(A + \frac{q}{2}B - \frac{q^2}{2.4}C + \frac{3q^3}{2.4.6}D - \frac{3.5q^4}{2.4.6.8}E \text{ \&c}\right).$$

EXAMPLE.

Required the length of the curve corresponding to the ordinate 10, the transverse and conjugate axes being 80 and 60.

Here $a = 40$, $c = 30$, and $y = 10$; hence $q = \frac{cc+aa}{c^4} = \frac{2500}{810000} = \frac{1}{324}$; $\sqrt{cc+yy} = 10\sqrt{10} = 31{\cdot}62277662$; and $\frac{y+\sqrt{cc+yy}}{c} = 1{\cdot}387425887$; whose hyperbolic logarithm is $\cdot 3274501 = A$;

$$\text{also } \frac{y\sqrt{cc+yy} - ccA}{2} = 10{\cdot}76133 = B.$$

$$\frac{y^3\sqrt{cc+yy} - 3ccB}{4} = 641{\cdot}796405 = C.$$

$$\frac{y^5\sqrt{cc+yy} - 5ccC}{6} = 45698{\cdot}97\tfrac{1}{3} = D.$$

$$\text{and } \frac{y^7\sqrt{cc+yy} - 7ccD}{8} = 3540529{\cdot}3125 = E.$$

Then

But the fluent of $\frac{\dot{y}}{\sqrt{cc+yy}}$ is the hyp. log. of $\frac{y+\sqrt{cc+yy}}{c} = A$;

that of $\frac{y^2\dot{y}}{\sqrt{cc+yy}}$ is $\frac{y\sqrt{cc+yy} - ccA}{2} = B$;

that of $\frac{y^4\dot{y}}{\sqrt{cc+yy}}$ is $\frac{y^3\sqrt{cc+yy} - 3ccB}{4} = C$;

that of $\frac{y^6\dot{y}}{\sqrt{cc+yy}}$ is $\frac{y^5\sqrt{cc+yy} - 5ccC}{6} = D$, &c.

Theref. $z = c \times \left(A + \frac{q}{2}B - \frac{q^2}{2.4}C + \frac{3q^3}{2.4.6}D - \frac{3.5q^4}{2.4.6.8}E \text{ \&c}\right).$

$$\text{Then} + \quad A = \cdot 327450, - \quad \frac{q^2}{2.4}C = \cdot 000764$$
$$+ \quad \frac{q}{2}B = \cdot 016607, - \frac{3.5q^4}{2.4.6.8}E = \cdot 000012$$
$$+ \frac{3q^3}{2.4.6}D = \cdot 000084 \qquad\qquad - \cdot 000776$$

$$+ \cdot 344141$$
$$- \cdot 000776$$

$\cdot 343365$ the sum of the series,
which multip. by $c =$ 30

produces $10 \cdot 30095$ the length of the arc.

RULE II.*

Using the same symbols as in the last rule, the arc will be expressed by the series $y \times (1 + \frac{a^2}{6c^4}y^2 - \frac{a^4 + 4a^2c^2}{40c^8}y^4 + \frac{a^6 + 4a^4c^2 + 8a^2c^4}{112c^{12}}y^6 - \frac{5a^8 + 24a^6c^2 + 48a^4c^4 + 64a^2c^6}{1152c^{16}}y^8$ &c).

or

---

* DEMONSTRATION.

For, by the demonst. of the last rule, $\dot{z}$ is $= \dot{y}\sqrt{\frac{cc + \frac{aa+cc}{cc}yy}{cc+yy}}$ =, by extracting the root of the numerator and denominator, and then dividing the one by the other, $\dot{y} \times$ $(1 + \frac{a^2}{2c^4}y^2 - \frac{a^4 + 4a^2c^2}{2.4c^8}y^4 + \frac{a^6 + 4a^4c^2 + 8a^2c^4}{4.4c^{12}}y^6 - \frac{5a^8 + 24a^6c^2 + 48a^4c^4 + 64a^2c^6}{4.4.8c^{16}}y^8$ &c). And, by taking the fluents, we obtain the series as in the rule.

or $y \times (1 + \frac{a^2y^2}{6c^4}\text{A} - \frac{a^2+4c^2}{c^4}.\frac{3y^2}{20}\text{B} + \frac{a^4+4a^2c^2+8c^4}{a^2+4c^2}.\frac{5y^2}{14c^4}\text{C}$

$- \frac{5a^6+24a^4c^2+48a^2c^4+64c^6}{a^4+4a^2c^2+8c^4}.\frac{7y^2}{72c^4}\text{D} \ \&c).$

Putting A, B, C, &c, for the 1st, 2d, 3d, &c, term.

EXAMPLE.

Taking the same example as to the last rule, in which $a = 40$, $c = 30$, and $y = 10$; we shall obtain

A =	1	C =	·003170
B =	0·032922	E =	59
D =	398	−	·003229
+	1·033320		
−	·003229		

the sum 1·030091 which multiplied
by $y = 10$
produces 10·30091 the arc, nearly as before.

RULE III.*

1. To 21 times the square of the conjugate, add 9 times the square of the transverse; and to the same 21 times

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* DEMONSTRATION.

First $x = \frac{a\sqrt{c^2+y^2}}{c} - a$, using the same symbols as in the last rules, $= \frac{ay^2}{2c^2} - \frac{ay^4}{2.4c^4} + \frac{3ay^6}{2.4.6c^6} - \frac{3.5ay^8}{2.4.6.8c^8}$ &c.

Then proceed as in rule 5 for the elliptic arc.

Thus $y \times \frac{\text{A}+(\text{B}+1)x}{\text{A}+\text{B}x} = y \times \frac{\text{A}+(\text{B}+1)\times(\frac{ay^2}{2c^2} - \frac{ay^4}{8c^4}\ \&c)}{\text{A}+\text{B}\times(\frac{ay^2}{2c^2} - \frac{ay^4}{8c^4}\ \&c)}$

$= y$

21 times the ſquare of the conjugate, add 19 times the ſquare of the tranſverſe; and multiply each ſum by the abſciſs.

2. To each of theſe two products add 15 times the product of the tranſverſe and ſquare of the conjugate.

3. Then as the leſs ſum is to the greater, ſo is the double ordinate to the length of the curve, nearly.

That is, $\frac{15c^2t + (19t^2 + 21c^2)x}{15c^2t + (9t^2 + 21c^2)x} \times y =$ the length of the curve nearly; where t is the tranſverſe axe, c the conjugate, x the abſciſs, and y the ordinate correſponding to the arc.

EXAMPLE.

Taking, again, the ſame example, we have $t =$ 80, $c = 60$, $y = 10$, and $x = \frac{a\sqrt{c^2 + y^2}}{c} - a$ $= a \times \frac{\sqrt{c^2 + y^2} - c}{c} = 40 \times \frac{\sqrt{1000} - 30}{30} = 4 \times \frac{10\sqrt{10} - 30}{3}$ $= 2\cdot16370216$.

Then

$= y \times (1 + \frac{ay^2}{2Ac^2} - \frac{Aa + 2Ba^2}{8A^2c^4}y^4$ &c): which put $=$ $y \times (1 + \frac{a^2y^2}{6c^4} - \frac{a^4 + 4a^2c^2}{40c^8}y^4$ &c) the value of the arc in the laſt rule. Then, by equating the correſponding terms, we have $\frac{a}{2Ac^2} = \frac{a^2}{6c^4}$,

or $A = \frac{3c^2}{a} = \frac{3}{2}p$; and $\frac{Aa + 2Ba^2}{8A^2c^4} = \frac{a^4 + 4a^2c^2}{40c^8}$,

hence $B = \frac{9a^2 + 21c^2}{10aa} = \frac{9a + \frac{21cc}{a}}{10a} = \frac{9t + 21p}{10t}$, putting p for the parameter, and t for the whole tranſverſe axe.

Conſequently

Then $y \times \frac{15c^2t + (19t^2 + 21c^2)x}{15c^2t + (9t^2 + 21c^2)x}$

$= 10 \times \frac{15 \times 60^2 \times 80 + (19 \times 80^2 + 21 \times 60^2)2\cdot16}{15 \times 60^2 \times 80 + (9 \times 80^2 + 21 \times 60^2)2\cdot16}$

$= 10 \times \frac{11866\cdot70516488}{11520\cdot51281928} = 10\cdot3005$, the length of the arc required, very near.

PROBLEM VI.

To find the Area of a Segment cut off an Hyperbola by a Double Ordinate.

RULE I.

From the product of the ordinate and its diſtance from the center, ſubtract the product ariſing from the multiplication of the product of the ſemi-axes, by the hyperbolic logarithm of the quotient reſulting from the diviſion of the ſum of the products of the ſemi-tranſverſe multiplied by the ordinate, and the ſemi-conjugate multiplied by the ſaid diſtance of the ordinate from the center, by the product of the ſemi-

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Conſequently $y \times \frac{A + (B + 1)x}{A + Bx}$ is $= y \times \frac{\frac{3}{2}p + \frac{19t + 21p}{10t}x}{\frac{3}{2}p + \frac{9t + 21p}{10t}x} =$

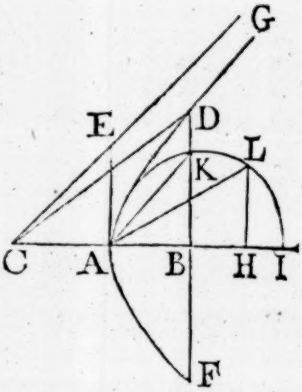
$y \times \frac{15p + \frac{19t + 21p}{t}x}{15p + \frac{9t + 21p}{t}x} = \frac{15c^2t + (19t^2 + 21c^2)x}{15c^2t + (9t^2 + 21c^2)x} \times y.$

Q. E. D.

Corollary. A right line may be found nearly equal to an hyperbolic arc, by proceeding in the ſame manner as in the corollary to rule 5 for the arc of the ellipſe, taking here $PG = \frac{1}{4}AP + \frac{19AK + 21AP}{10AK} \times AD.$

ſemi-axes; and the remainder will be the area cut off by the double ordinate.*

That is, CB × BD — CA × AE × hyp. logarithm of $\frac{CA \times BD + CB \times AE}{CA \times AE} = vy - ac \times$ hyp. log. of $\frac{ay + cv}{ac}$ = the area ADF; putting a = the ſemi-tranſverſe axe CA, c = AE the ſemi-conjugate, y = BD the ordinate, and v = CB its diſtance from the center.



Note

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* DEMONSTRATION.

The fluxion of the area ABD is $y\dot{v} = \frac{c\dot{v}\sqrt{vv - aa}}{a}$, by the nature of the curve; the fluent of which is $\frac{cv\sqrt{vv - aa}}{2a} - \frac{ac}{2} \times$ h. l. of $\frac{v + \sqrt{vv - aa}}{a}$ = the area AED; the double of which will be $\frac{cv\sqrt{vv - aa}}{a} - ac \times$ h. l. of $\frac{v + \sqrt{vv - aa}}{a} = vy - ac \times$ h. l. of $(\frac{v}{a} + \frac{y}{c}) = vy - ac \times$ h. l. of $\frac{ay + cv}{ac}$ the whole ſegment ADF. *Q. E. D.*

Corol. 1. The ſame ſeg. ADF or $\frac{cv\sqrt{vv - aa}}{a} - ac \times$ h. l. of $\frac{v + \sqrt{vv - aa}}{a}$, is alſo $= \frac{ay\sqrt{cc + yy}}{c} - ac \times$ h. l. of $\frac{y + \sqrt{cc + yy}}{c}$
$= ac \times (q\sqrt{1 + qq} -$ h. l. of $(q + \sqrt{1 + qq})$
$= ac \times (Q\sqrt{QQ - 1} -$ h. l. of $Q + \sqrt{QQ - 1})$;
putting $q = \frac{y}{c}$, and $Q = \frac{v}{a}$.

Note 1. The same rules will serve for any other two conjugate diameters, and their absciss and ordinate; multiplying the result by the sine of the angle formed by the absciss and ordinate, to the radius 1.

Note 2. The hyperbolic logarithm, is equal to the common logarithm multiplied by 2·302585093.

EXAMPLE.

Required the area of an hyperbola, whose base is 24, and altitude 10; the transverse axe being 30.

Here

Corol. 2. If from the triangle CBD $= \frac{1}{2}vy$, be taken the semi-segment AED $= \frac{1}{2}vy - \frac{1}{2}ac \times$ h. l. of $\frac{ay + cv}{ac}$, there will remain $\frac{1}{2}ac \times$ h. l. of $\frac{ay + cv}{ac}$ for the value of the hyperbolic sector CAD. When y or v are supposed infinite; then the sector becomes the space CAG between the asymptote and the curve infinitely produced, and the expression for it becomes infinite.

Corol. 3. When the hyperbola is equilateral, then a is $= c$, and the area ADF $= vy - aa \times$ h. l. of $\frac{v + y}{a}$.

Corol. 4. When A and C coincide, the hyperbola becomes a triangle; and because CA and AE are then each $= 0$, the rule will become barely CB $\times$ BD or AB $\times$ ED for the area.

Corol. 5. Similar segments of any hyperbolas, are to each other as the rectangles of their axes. Where by similar segments, are meant those whose ordinates or abscisses, are as their like axes or diameters. For, by corollary 1, the segment being expressed by $ac \times (q\sqrt{1 + qq}$ — hyp. log. of $q + \sqrt{1 + qq})$ or $ac \times$ $Q\sqrt{QQ - 1}$ — hyp. log. of $Q + \sqrt{QQ - 1}$; and $q = \frac{y}{c}$, or $Q = \frac{v}{a}$, being the same in all similar segments, those segments will be as ac.——And hence, similar segments of equilateral hyperbolas, are as the squares of their axes, or of their like diameters.

4 Hence

Here $a = 15$, $y = 12$, and $v = 25$.

Hence $c = \frac{ay}{\sqrt{vv - aa}} = \frac{15 \times 12}{\sqrt{40 \times 10}} = \frac{180}{20} = 9$.

Then $\frac{ay + cv}{ac} = \frac{15 \times 12 + 9 \times 25}{15 \times 9} = \frac{4 + 5}{3} = 3$;

whose

Hence also, hyperbolas of the same transverse axe and abscis, are to one another as their conjugate axes; but if their bases or ordinates and conjugate axes be the same, they will be as their transverse axes. For when a and v are the same, the quantity $\frac{cv\sqrt{vv - aa}}{a} - ac \times$ h. l. of $\frac{v + \sqrt{vv - aa}}{a}$ is as c; and when c and y are the same, then $\frac{ay\sqrt{cc + yy}}{c} - ac \times$ h. l. of $\frac{y + \sqrt{cc + yy}}{c}$ is as a.

And consequently, having the quadrature of any one hyperbola, from it we may find that of any other hyperbola; viz. knowing the area answering to every abscis in any one hyperbola, we can find the area answering to any abscis in any other hyperbola. For we can find a similar abscis in the squared hyperbola, with the corresponding area. Then, as the rectangle of the axes of the squared hyperbola, is to the rectangle of those of the hyperbola proposed, so is the area of the former, to that of the latter. ——Thus, let t, c be the transverse and conjugate axes of any proposed hyperbola, and x its abscis; also T, C the axes of the squared hyperbola: Then, $t : \text{T} :: x : \frac{\text{T}x}{t} =$ the similar abscis in the squared hyperbola, whose corresponding area call A: Then, as $\text{TC} : tc :: \text{A} : \frac{tc\text{A}}{\text{TC}} =$ the area of the hyperbola proposed.——Now, it will be most convenient to have the squared hyperbola an equilateral one, whose axes are each an unit. For then we need only divide the abscis of the hyperbola proposed, by its transverse axe, and multiply the product of its axes by the area answering to this quotient in the squared, or equilateral, hyperbola. And this is similar to finding the segments of circles, and ellipses, from those of a circle whose segments are already calculated, and arranged in tables.

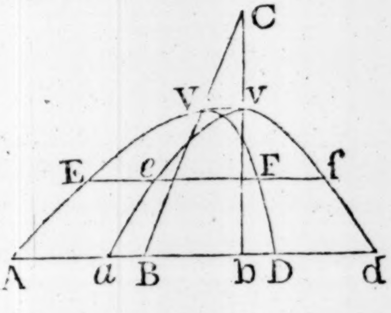
Corol. 6. If a and c be any other two conjugate diameters, and s the sine of the angle formed by a and its ordinates, to the radius 1;

whose hyperbolic log. is $1\cdot0986122288\frac{2}{3}$; which being multiplied by $ac = 15 \times 9$, produces $148\cdot312658 97$; and this being taken from $vy = 25 \times 12 = 300$,

leaves the area required $= 151\cdot68734103$.

1; it is very evident that $\frac{scv\sqrt{vv - aa}}{a} - sac \times$ h. l. of $\frac{v + \sqrt{vv - aa}}{a}$ will denote the oblique segment cut off by a double ordinate to the diameter a. And if, in this expression, for sa and sv be substituted respectively A and V, the said oblique hyperbola will be expressed by $\frac{cV\sqrt{VV - AA}}{A} - Ac \times$ h. l. of $\frac{V + \sqrt{VV - AA}}{A}$; which also expresses a right hyperbola, whose transverse and conjugate axes are A, c, and the distance of its base from the center V.

And hence is evident the following construction, which SCHOOTEN first demonstrated, in a very operose manner, in his *Exercitationes Mathematicæ*, *Lib.* 4.



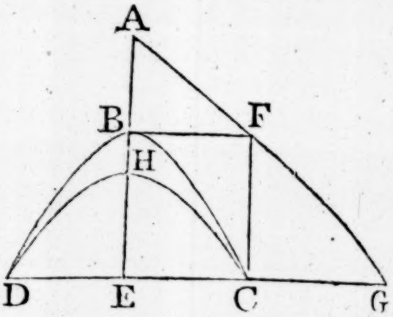
Viz. Let AD be a double ordinate to the semi-diameter CV, of the oblique hyperbola AVD. Draw Vv parallel and Cvb perpendicular to AD; and with the semi-transverse Cv and semi-conjugate = the semi-conjugate of CV, describe the right hyperbola avd. And the right hyperbola avd will be = AVD the oblique one proposed; also AD = ad; and every other pair of corresponding ordinates EF, ef, equal to each other.

Corol. 7. Hence it appears that all hyperbolas, having the same center, and equal bases, and between the same parallels Ad, Vv, infinitely produced, are equal to each other; as are also their corresponding sections EF, ef, parallel to the bases; and likewise the intercepted frustums AEFD, $aefd$.

RULE II.*

Putting x = the abscifs AB, and the reft of the letters as in the firft rule, alfo $z = \frac{x}{2a + x}$. Then

$2xy$



Corol. 8. Let AB be the femi-tranfverfe axe of the hyperbola DBC; draw CF parallel, and BF perpendicular to AB; and draw AFG meeting DC produced in G; then upon the bafe CD of the hyperbola defcribe a parabola DHC, having its parameter equal to the conjugate axe of the hyperbola. And (EG − CH) × AB will be equal to the hyperbolic area EBC.

For the area EBC $= A = \frac{1}{2}vy - \frac{1}{2}ac \times$ h. l. of $\frac{ay + cv}{ac}$,

and $a \times$ the curve CH $= aC = \frac{1}{2}vy + \frac{1}{2}ac \times$ h. l. of $\frac{ay + cv}{ac}$,

by addition, $A + aC = vy$, and therefore $A = vy - aC =$

$a \times (\frac{vy}{a} - C) = (\frac{AE \times EC}{AB} - CH) \times AB = (EG - CH) \times AB$.

So that the quadrature of the hyperbola depends upon the rectification of the parabola.

Corol. 9. By fubtracting the feg. $vy - ac \times$ h. l. of $\frac{ay + cv}{ac}$

from the fegment $VY - ac \times$ h. l. of $\frac{aY + cV}{ac}$,

there will refult $VY - vy - ac \times$ h. l. of $\frac{aY + cV}{ay + cv}$

for the fruftum included by two parallel double ordinates $2Y$, $2y$; their diftances from the center being V, v.

* DEMONSTRATION.

The fluxion of the area is

$\dot{y}x = \frac{cx\sqrt{2ax + xx}}{a}$. Then inftead of $\frac{x}{2a + x}$ write z, in order that the fluent quantity z may be lefs than 1, and by that means

$2xy \times (\frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 - \frac{1}{5.7.9}z^3 - \frac{1}{7.9.11}z^4$ &c) or

$2xy \times (\frac{1}{3} - \frac{1}{5}Az - \frac{1}{7}Bz - \frac{3}{9}Cz - \frac{5}{11}Dz$ &c)

---

means the series be made to converge by its powers, and the said fluxion

$\dot{y}x = \frac{c\dot{x}\sqrt{2ax + xx}}{a}$ will be $= \frac{4acz^{\frac{1}{2}}\dot{z}}{\overline{1-z}|^3} = 2acz^{\frac{1}{2}}\dot{z} \times$

$(1.2 + 2.3z + 3.4z^2 + 4.5z^3$ &c);

and the fluent is $4acz^{\frac{3}{2}} \times (\frac{1.2}{3} + \frac{2.3}{5}z + \frac{3.4}{7}z^2 + \frac{4.5}{9}z^3$ &c) = the semi-segment ABD.

Now although this series converge by the powers of z, yet the coefficients actually diverge: in order, then, to make the coefficients converge, let the series be multiplied by some convenient quantity less than 1, as $1 - z$, and the quantity $4acz^{\frac{3}{2}}$ divided by the same, and the result will be

$\frac{8acz^{\frac{3}{2}}}{1-z} \times \frac{1.1}{1.3} + \frac{2.2}{3.5}z + \frac{3.3}{5.7}z^2 + \frac{4.4}{7.9}z^3$ &c) for the said area ABD. And, to make the series converge still quicker, let the same operation be performed upon this series, and there will result

$\frac{8acz^{\frac{3}{2}}}{\overline{1-z}|^2} \times (\frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 - \frac{1}{5.7.9}z^3 - \frac{1}{7.9.11}z^4$ &c) for the value of the said area.

But $\frac{8acz^{\frac{3}{2}}}{\overline{1-z}|^2} = \frac{2cx\sqrt{2ax + xx}}{a} = 2xy$; therefore

$2xy \times (\frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 - \frac{1}{5.7.9}z^3$ &c) = the area ABD. *Q. E. D.*

Corol. 1. An hyperbola is always less than a parabola of the same base and altitude; for $2xy \times (\frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2$ &c)

is

will be the area of the ſemi-ſegment ABD (Fig. to rule 1); putting A, B, C, &c, for the firſt, ſecond, third, &c, term.

EXAMPLE.

Taking here the laſt example, in which $x = 10$, $y = 12$, and $2a = 30$;

$$z \text{ will be } = \frac{x}{2a + x} = \frac{10}{30 + 10} = \frac{10}{40} = \frac{1}{4}.$$

Then

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is always leſs than $2xy \times \frac{1}{3}$ or $\frac{2}{3}xy$, the parabola. But as the altitude is diminiſhed, the hyperbola approaches nearer and nearer to the meaſure of the parabola; till at laſt they vaniſh in a ratio of equality.

Corol. 2. An hyperbola is evidently, for any finite altitude, always greater than a triangle of the ſame baſe and altitude. But as the altitude is increaſed, the triangle approaches nearer and nearer to the value of the hyperbola, till at length, when the altitude is infinite, they become accurately equal to each other. Which I demonſtrate, by the method of increments, after Mr. *Emerſon*'s notation, thus:

When the altitude x is infinite, then $z = \frac{x}{2a + x} = \frac{x}{x} = 1$, and the general ſeries expreſſing the value of the area will become $2xy \times (\frac{1}{3} - \frac{1}{1\cdot3\cdot5} - \frac{1}{3\cdot5\cdot7} - \frac{1}{5\cdot7\cdot9}$ &c)—Now to find the ſum s of any number n of the negative terms; ſince it is evident that the nth term is $\frac{1}{\overline{2n-1}\,.\,\overline{2n+1}\,.\,\overline{2n+3}}$; or, putting $v = 2n - 1$, then $\dot{v} = 2\dot{n} = 2$, and the nth term $= \frac{1}{v\,.\,\overline{v+2}\,.\,\overline{v+3}} = \frac{1}{v\,v_1\,v_2}$; the $n + 1$th term, or $\dot{s}$, will be $\frac{1}{v_1\,v_2\,v_3}$; and the integral $s = A - \frac{1}{2\dot{v}\,v\,v_1\,v_2} = A - \frac{1}{4v_1\,v_2}$: But when n is $= 1$, then s ought to be $\frac{1}{v\,v_1\,v_2} =$, in this caſe, $A - \frac{1}{4v_1\,v_2}$; hence

A =

Then $A = \frac{1}{3} = \cdot 33$ &c.

$$
\begin{array}{rlr}
B = & \frac{1}{5}Az = & \cdot 01666667 \\
C = & \frac{1}{7}Bz = & 51524 \\
D = & \frac{3}{9}Cz = & 4960 \\
E = & \frac{5}{11}Dz = & 5639 \\
F = & \frac{7}{13}Ez = & 76 \\
G = & \frac{9}{15}Fz = & 11 \\
H = & \frac{11}{17}Gz = & 2
\end{array}
$$

the sum of these negative terms is $\cdot 017318039$
which being taken from the first term $\cdot 333333333$
leaves $\cdot 316015294$
and this multiplied by $2xy = 240$

produces $75\cdot 84367056$
$=$ the semi-segment,
the double of which is $151\cdot 68734112$
$=$ the area required.

 RULE

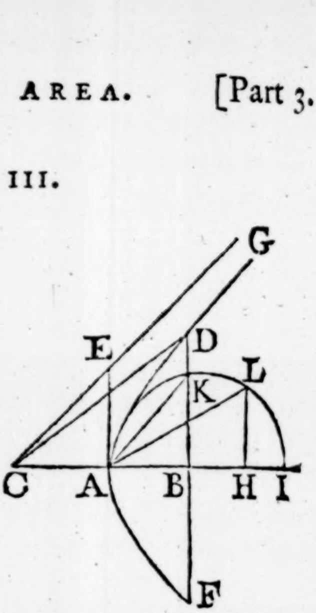
$A = \frac{1}{4\overset{1}{v}\overset{2}{v}} + \frac{1}{\overset{1}{v}vv} = \frac{v+4}{4vvv} = \frac{1+4}{4\cdot 1\cdot 3\cdot 5} = \frac{5}{60} = \frac{1}{12}$; and consequently the correct integral is

$I = \frac{1}{12} - \frac{1}{4vv} = \frac{1}{12} - \frac{1}{4.\overline{2n+1}.\overline{2n+3}}$; which, when n is infinite, becomes accurately $= \frac{1}{12}$; that is, the sum of the infinite series $\frac{1}{1\cdot 3\cdot 5} + \frac{1}{3\cdot 5\cdot 7}$ &c is $= \frac{1}{12}$; which being taken from $\frac{1}{3}$, leaves $\frac{1}{3} - \frac{1}{12} = \frac{1}{4}$; and hence $2xy \times \frac{1}{4} = \frac{1}{2}xy =$ the area of the hyperbola when x is infinite; and which is therefore equal to a triangle of the same base and altitude.

Corol. 3. Hence the space included between the arc of an hyperbola, infinitely produced, and its chord, is of a finite magnitude, although it be of an infinite length. For this space is the difference between the hyperbola and triangle, which, as above, is nothing, that is, nothing in comparison with an infinite space.

Corol.

RULE III.



Take BH $= \frac{3AB^2}{8AC} =$ to three-fourths of a third proportional to the transverse axe 2AC, and the abscifs AB; upon the diameter AI, = the parameter of the axe, describe a circle meeting BD in K, and HL, parallel to BD, in L; and draw AK, AL.

Then will the fegment DAF be nearly equal to $\frac{4AL + AD}{15} \times 4AB$, or $= \frac{4\sqrt{tx + \frac{3}{4}xx} + \sqrt{tx}}{15} \times \frac{4cx}{t}$; putting

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Corol. 4. By cor. 3 rule 2 for the fegment of a circle, the area of a circle whofe diameter is 1, is

$n = 2 \times (\frac{1}{3} + \frac{1}{1\cdot3\cdot5} - \frac{1}{3\cdot5\cdot7} + \&c)$, and by cor. 2 to this prob.

$\frac{1}{2} = 2 \times (\frac{1}{3} - \frac{1}{1\cdot3\cdot5} - \frac{1}{3\cdot5\cdot7} - \&c)$; from hence are deduced the following equations:

I. $n = \frac{5}{6} - 4 \times (\frac{1}{3\cdot5\cdot7} + \frac{1}{7\cdot9\cdot11} + \frac{1}{11\cdot13\cdot15} + \frac{1}{15\cdot17\cdot19}\ \&c)$

II. $n = \frac{1}{2} + 4 \times (\frac{1}{1\cdot3\cdot5} + \frac{1}{5\cdot7\cdot9} + \frac{1}{9\cdot11\cdot13} + \frac{1}{13\cdot15\cdot17}\ \&c)$

III. $n = \frac{2}{3} + 12 \times (\frac{1}{1\cdot3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9\cdot11} + \frac{1}{9\cdot11\cdot13\cdot15}\ \&c)$

IV. $\frac{1}{96} = \frac{1}{1\cdot3\cdot5\cdot7} + \frac{2}{5\cdot7\cdot9\cdot11} + \frac{3}{9\cdot11\cdot13\cdot15}\ \&c)$

putting t = the tranſverſe 2AC, c = the conjugate, and x = AB the abſciſs.*

EXAMPLE.

Taking the ſame example, in which $t = 30$, $c = 18$, and $x = 10$; we ſhall have $\frac{\sqrt{tx} + 4\sqrt{tx + \frac{3}{4}x^2}}{15} \times \frac{4cx}{t} = \frac{\sqrt{300} + 4\sqrt{375}}{450} \times 720 = \frac{10\sqrt{3} + 20\sqrt{15}}{5} \times 8 = (\sqrt{3} + 2\sqrt{15}) \times 16 = 151{\cdot}64828$ = the area nearly.

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* DEMONSTRATION.

By proceeding as in rule 2 for the circular ſegment, we obtain $\frac{cx\sqrt{tx}}{t} \times (\frac{2}{3} + \frac{x}{5t} - \frac{x^2}{28t^2}$ &c) for the true value of the ſemi-ſegment ADB. Then, after the method uſed in demonſtrating rule 6 for the circular ſegment, ſuppoſe ADB to be $\frac{cmx\sqrt{tx + \frac{3}{4}xx}}{t} + \frac{cnx\sqrt{tx}}{t} = \frac{cx\sqrt{tx}}{t} \times (m + n + \frac{3mx}{8t}$ &c): hence, comparing the like terms, $m + n = \frac{2}{3}$, and $\frac{3m}{8} = \frac{1}{5}$; and therefore $m = \frac{8}{15}$, and $n = \frac{2}{3} - \frac{8}{15} = \frac{2}{15}$. Which values being ſubſtituted for them, we have ADB =

$$\frac{2cx}{t} \times \frac{4\sqrt{tx + \frac{3}{4}xx} + \sqrt{tx}}{15} = 2x \times \frac{4\sqrt{\frac{c^2x}{t} + \frac{3c^2x^2}{4tt}} + \sqrt{\frac{c^2x}{t}}}{15}$$

$$= \frac{2}{15}\text{AB} \times [4\sqrt{(\text{AB} + \frac{3\text{AB}^2}{8\text{AC}}) \times \text{AI}} + \sqrt{\text{AB} \times \text{AI}}] = 2\text{AB} \times \frac{4\sqrt{\text{AH} \times \text{AI}} + \sqrt{\text{AB} \times \text{AI}}}{15} = 2\text{AB} \times \frac{4\text{AL} + \text{AD}}{15}.$$ *Q. E. D.*

And this rule was firſt given by Sir *I. Newton*; but without demonſtration.

RULE IV.

If BH be taken $= \frac{5xx}{7t} = \frac{5AB^2}{14AC} =$ to five-ſevenths of the ſaid third proportional to the axe 2AC, and the abſciſs AB; and the lines drawn as in the laſt rule.

Then $\frac{21AL + 4AK}{75} \times 4AB$, or $\frac{21\sqrt{tx + \frac{5}{7}xx} + 4\sqrt{tx}}{75} \times \frac{4cx}{t}$ will expreſs the area nearer than by the laſt rule.*

EXAMPLE.

Taking, ſtill, the ſame example, we have $\frac{21\sqrt{tx + \frac{5}{7}xx} + 4\sqrt{tx}}{75} \times \frac{4cx}{t} = \frac{21\sqrt{371\frac{3}{7}} + 4\sqrt{300}}{75} \times \frac{720}{30} = \frac{30\sqrt{182} + 40\sqrt{3}}{75} \times 24 = \frac{3\sqrt{182} + 4\sqrt{3}}{5} \times 16 = 151{\cdot}68133$ the area nearly.

PRO-

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* DEMONSTRATION.

Taking BH $= \frac{Axx}{t}$, let, as before, the area be expreſſed by AB $\times$ ($m \times$ AL $+ n \times$ AK) $=$ AB $\times (m\sqrt{HA \times AI} + n\sqrt{BA \times AI})$ $= mx\sqrt{(x + \frac{Axx}{t}) \times \frac{cc}{t}} + nx\sqrt{x \times \frac{cc}{t}} = (m\sqrt{tx + Axx} + n\sqrt{tx})$ $\times \frac{cx}{t} = \frac{cx\sqrt{tx}}{t} \times (m + n + \frac{mAx}{2t} - \frac{mA^2x^2}{8tt}$ &c) : which being compared with the true ſeries $\frac{cx\sqrt{tx}}{t} \times (\frac{2}{3} + \frac{x}{5t} - \frac{x^2}{28t^2}$ &c), we

PROBLEM VII.

To find the Area of the Fruſtum of an Hyperbola, included between two Parallel Double Ordinates.

Multiply the ſemi-tranſverſe by the one ordinate, and its diſtance from the center by the ſemi-conjugate, and take the ſum of the products; do the ſame by the other ordinate and its diſtance from the center, taking the ſum of the products alſo; divide the greater ſum by the leſs, and multiply the hyperbolic logarithm of the quotient by the product of the ſemi-axes; then ſubtract this laſt product from the difference of the products ariſing from the multiplication of each ordinate by its own diſtance from the center, and the remainder will be the area required.

That is, $VY - vy - ac \times$ hyp. log. of $\frac{aY + cV}{ay + cv} =$ the area included by $2Y$ and $2y$, V and v being their reſpective diſtances from the center, and a and c the ſemi-axes. As is proved in cor. 8 to rule 1 to the laſt problem.

Or,

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we obtain theſe three equations, $m + n = \frac{2}{3}$, $\frac{mA}{2} = \frac{1}{5}$, and $\frac{mA^2}{8} = \frac{1}{28}$; hence $A = \frac{5}{7}$, $m = \frac{14}{25}$, and $n = \frac{8}{75}$.

Conſequently $BH = \frac{Axx}{t} = \frac{5xx}{7t}$, and the ſemi-ſegment $= 2AB \times \frac{21AL + 4AK}{75} = \frac{2cx}{t} \times \frac{21\sqrt{tx} + 4\sqrt{tx + \frac{5}{7}xx}}{75}$.

Q. E. D.

Which likewiſe is another rule given by Sir *I. Newton*, without demonſtration.

Or, if there be calculated the two ſegments whoſe baſes are 2Y and $2y$, their difference will be the area ſought.

EXAMPLE.

To find the area of the fruſtum whoſe altitude is 5; its leſs end being, as in the example to the laſt problem, 24, its diſtance from the center of the hyperbola 25, the tranſverſe axe 30, and the conjugate 18.

Here $a = 15$, $c = 9$, $v = 25$, $y = 12$, and $\text{V} = 30$; hence $\text{Y} = \frac{c\sqrt{\text{VV} - aa}}{a} = \frac{9\sqrt{30^2 - 15^2}}{15} = 9\sqrt{2^2 - 1^2}$ $= 9\sqrt{3} = 15{\cdot}58845727$.

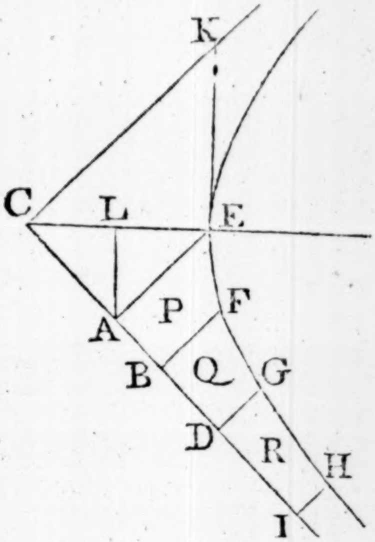
Then $\frac{a\text{Y} + c\text{V}}{ay + cv} = \frac{15 \times 9\sqrt{3} + 9 \times 30}{15 \times 12 + 9 \times 25} = \frac{9\sqrt{3} + 3 \times 6}{12 + 3 \times 5} = \frac{\sqrt{3} + 2}{3} = 1{\cdot}244016936$, whoſe hyp. log. is $\cdot 2183456$, which multiplied by $ac = 15 \times 9$, gives $29{\cdot}476656$.

And $\text{VY} - vy = 30 \times 9\sqrt{3} - 25 \times 12 = 30 \times (9\sqrt{3} - 10) = 30 \times 5{\cdot}58845727 = 167{\cdot}653718$.

Theref. $167{\cdot}653718 - 29{\cdot}476656 = 138{\cdot}177062$ is the area required.

PROBLEM VIII.

To find the Area included between two Ordinates BF, DG, *to an Asymptote* CD *of an Hyperbola, drawn parallel to the other Asymptote.*



From the vertex E of the curve, draw EA parallel to BF, or CK.

Then the rectangle of CA and AE multiplied by the hyp. log. of the quotient arising from the division of CD by CB, or from the division of BF by DG, will produce the area of BFGD.*

E X-

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* DEMONSTRATION.

Put CA $= a$, AE $= c$, AB $= x$, and BF $= y$; then, by the property of the curve, $ac =$ CB $\times$ BF $= (a + x) \times y$, and hence

$$y = \frac{ac}{a+x}, \text{ and } y\dot{x} = \frac{ac\dot{x}}{a+x}; \text{ whose fluent is}$$

$$ac \times \text{h. l. of } \frac{a+x}{a} = \text{CA} \times \text{AE} \times \text{h. l. of } \frac{\text{CB}}{\text{CA}} = \text{the area ABFE.}$$

In

EXAMPLE.

Required the area BDGF included by two ordinates BF, DG, whose lengths are 8 and 6; the semi-transverse CE being 15, and the semi-conjugate EK 9.

Here $CK = \sqrt{CE^2 + EK^2} = \sqrt{15^2 + 9^2} = 3\sqrt{5^2+3^2} = 3\sqrt{34}$.

And in the similar triangles CEK, CAE,

$$CK : CE :: CE : CA = \frac{75}{\sqrt{34}},$$

$$CK : CE :: EK : AE = \frac{45}{\sqrt{34}}.$$

Consequently $CA \times AE = \frac{75 \times 45}{34} = \frac{3375}{34}$.

But the hyp. log. of $\frac{BF}{DG} = \frac{8}{6} = \frac{4}{3}$ is ·287682.

Theref.

In like manner ADGE is $= CA \times AE \times$ h. l. of $\frac{CD}{CA}$.

And, taking the difference, we have $BDGF = CA \times AE \times$ h. l. of $\frac{CD}{CA}$ − h. l. of $\frac{CB}{CA} = CA \times AE \times$ h. l. of $\frac{CD}{CB}$; or by writing $\frac{CA \times AE}{BF}$ for CB, and $\frac{CA \times AE}{DG}$ for CD, the same area will be $= CA \times AE \times$ h. l. of $\frac{BF}{DG}$. *Q. E. D.*

Corol. 1. If CB, CD, CI, &c, be in geometrical progression; then $\frac{CD}{CB}$ will be $= \frac{CI}{CD}$, &c, and consequently the spaces P, Q, R, &c, equal to one another; or the spaces ABFE, ADGE, AIKE, &c, a series of arithmeticals, and are the logarithms of the geometricals $\frac{CB}{CA}, \frac{CD}{CA}, \frac{CI}{CA}$, &c, to the modulus CA; or the said spaces will form a scale of hyperbolic logarithms, whose absolute numbers are CB, CD, CI, &c, when CA is 1.

Corol.

Theref. $\cdot 287682 \times \frac{3375}{34} = 28\cdot 55667$ is the area required.

PROBLEM IX.

To find the Curve Surface of an Hyperboloid.

Let a and c be the semi-axes of the generating hyperbola, and v the distance of its base from the center. Also let $A = \frac{aa}{\sqrt{aa+cc}}$ be the semi-transverse of another hyperbola, whose semi-conjugate is c, the same with that of the former. Then find, by problem 7, the area of the frustum of this latter hyperbola, whose two ends are distant from the center by v and a; multiply this area by $3\cdot 14159$, and the product will be the surface required.

That

Corol. 2. The space ABFE is also equal to

$$ac \times \left(\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \text{ \&c}\right).$$

For the fluxion of the area

$$\dot{y}x = \frac{ac\dot{x}}{a+x} \text{ is } = ac\dot{x} \times \left(\frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} \text{ \&c}\right)$$

whose fluent is $ac \times \left(\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \text{ \&c}\right)$

$$= CA \times AE \times \left(\frac{AB}{CA} - \frac{AB^2}{2CA^2} + \frac{AB^3}{3CA^3} \text{ \&c}\right).$$

After the same manner

$$ADGE \text{ is } = CA \times AE \times \left(\frac{AD}{CA} - \frac{AD^2}{2CA^2} + \frac{AD^3}{3CA^3} \text{ \&c}\right).$$

And consequently, by taking the difference, we obtain EDGF

$$= CA \times AE \times \left(\frac{AD - AB}{CA} - \frac{AD^2 - AB^2}{2CA^2} + \frac{AD^3 - AB^3}{3CA^3} \text{ \&c}\right).$$

Corol. 3. Hence the hyperbolic logarithm of

$$\frac{a+x}{a} \text{ is } \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \text{ \&c.}$$

That is, $p \times (v\text{Y} - ay - \text{A}c \times \text{hyp. log. of } \frac{\text{AY}+cv}{\text{A}y+ac})$ $=$ the surface required; p being $= 3{\cdot}14159$, and Y, y, the ordinates, of the latter hyperbola, whose distances from the center are v, a.*

EX-

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* DEMONSTRATION.

Put here, $w =$ the ordinate to the abscifs v, and $z =$ the curve; then $a : c :: \sqrt{vv-aa} : w = \frac{c\sqrt{vv-aa}}{a}$, and $\dot{w} = \frac{cv\dot{v}}{a\sqrt{vv-aa}}$;

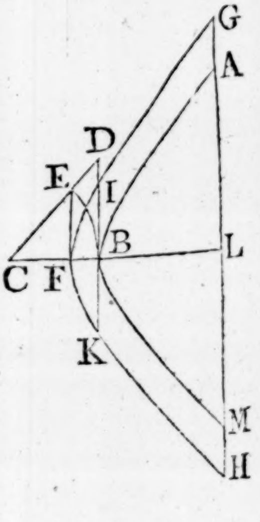
hence the fluxion of the surface will be $2pw\dot{z} = 2pw\sqrt{\dot{v}\dot{v}+\dot{w}\dot{w}}$

$$= \frac{2pc\sqrt{vv-aa}}{a}\sqrt{\dot{v}\dot{v}+\frac{ccvv\dot{v}\dot{v}}{aa(vv-aa)}} = \frac{2pc\dot{v}\sqrt{aa(vv-aa)+ccvv}}{aa}$$

$$= \frac{2pc\dot{v}\sqrt{aa+cc}}{aa}\sqrt{vv-\frac{aaaa}{aa+cc}} = \frac{2pc\dot{v}\sqrt{vv-\text{AA}}}{\text{A}};$$

which is evidently equal to p drawn into the fluxion of an hyperbolic segment, the semi-axes being A, c, and abscifs v. And, by taking the correct fluent, we obtain the expression in the rule above given; which is the difference of two segments whose bases are distant from the center by v and a; or the frustum whose two ends are distant from the center by the same quantities v and a. *Q. E. D.*

Corol. 1. Hence if ABM be the generating hyperbola, BC its semi-transverse, ED perpendicular to BC, and equal to the semi-conjugate.——Draw CD; make CE = CB, and on CB let fall the perpendicular EF; with the semi-transverse CF, and semi-conjugate of ABM, describe the hyperbola GFH.



Then, as the diameter of a circle is to its circumference, so is the frustum GIKH, included by the parallels BI, AM, produced, to the surface generated by AB.

For,

EXAMPLE.

Required the curve surface of an hyperboloid whose altitude is 10, the transverse and conjugate axes of the generating hyperbola being 30 and 18.

Here $a = 15$, $c = 9$, $A = \frac{aa}{\sqrt{aa+cc}} = \frac{15^2}{\sqrt{15^2+9^2}} = \frac{75}{\sqrt{34}}$,

$v = 25$, $Y = \frac{c\sqrt{vv - AA}}{A} = \frac{9\sqrt{25^2 - \frac{75^2}{34}}}{\frac{75}{\sqrt{34}}} = 15$,

and $Y = \frac{c\sqrt{aa - AA}}{A} = \frac{9\sqrt{15^2 - \frac{75^2}{34}}}{\frac{75}{\sqrt{34}}} = \frac{27}{5} = 5\frac{2}{5}$.

Hence $vY - ay = 25 \times 15 - 15 \times 5\frac{2}{5} = 19\frac{3}{5} \times 15 = 294$; $Ac = \frac{75 \times 9}{\sqrt{34}} = \frac{75 \times 9\sqrt{34}}{34} = 115{\cdot}7615451$; and

For, by similar triangles, $CD : CB :: CE : CF = A = \frac{CB^2}{CD} = \frac{CB^2}{\sqrt{CB^2 + BD^2}} = \frac{aa}{\sqrt{aa + cc}}$. And all the rest is evident.

Corol. 2. Putting $Z = \frac{FL}{2CF + FL}$, and $z = \frac{FB}{2CF + FB}$; and proceeding as in rule 2 for the hyperbolic segment, we shall obtain the surface generated by BA, equal to $3{\cdot}14159$ drawn into the difference between the series

$$2FL \times GH \times \left(\frac{1}{3} - \frac{1}{1 \cdot 3 \cdot 5}Z - \frac{1}{3 \cdot 5 \cdot 7}Z^2 - \frac{1}{5 \cdot 7 \cdot 9}Z^3 \ \&c\right)$$

and the series

$$2FB \times BK \times \left(\frac{1}{3} - \frac{1}{1 \cdot 3 \cdot 5}z - \frac{1}{3 \cdot 5 \cdot 7}z^2 - \frac{1}{5 \cdot 7 \cdot 9}z^3 \ \&c\right).$$

and

$$\frac{\text{AY}+cv}{\text{A}y+ca}=\frac{\frac{75\times 15}{\sqrt{34}}+9\times 25}{\frac{75\times 27}{5\sqrt{34}}+9\times 15}=\frac{5}{3}\times\frac{5+\sqrt{34}}{3+\sqrt{34}}=\frac{19+2\sqrt{34}}{15}$$

$= 2{\cdot}0441269$; whose hyperbolic logarithm is $\cdot 7149706$; which multiplied by $\text{A}c = 115{\cdot}7615451$ produces $82{\cdot}7661049$; which, being taken from $v\text{Y} - ay = 294$, leaves $211{\cdot}2338951$, which multiplied by $3{\cdot}14159$, produces $663{\cdot}610853$, the curve surface required.

PROBLEM X.

To find the Solidity of an Hyperboloid.

RULE I.*

As the sum of the transverse axe of the generating hyperbola, and the height of the solid, is to the sum of

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* DEMONSTRATION.

Let t be the transverse, and c the conjugate axe of the generating hyperbola, x its abscifs, or the altitude of the solid, y the ordinate, or radius of the base, and $p = 3{\cdot}14159$.

Then $yy = cc \times \frac{tx + xx}{tt}$, and the fluxion of the solid or $pyy\dot{x}$ is $pccx\dot{x} \times \frac{t + x}{tt}$, whose fluent is $pccxx \times \frac{\frac{1}{2}t + \frac{1}{3}x}{tt} = \frac{1}{2}pxyy \times \frac{t + \frac{2}{3}x}{t + x} = \frac{1}{2}$ base $\times$ altitude $\times \frac{t + \frac{2}{3}x}{t + x}$ the measure of the solid. *Q. E. D.*

Corol. 1. An hyperboloid is to a paraboloid of the same base and altitude, as $t + \frac{2}{3}x$ to $t + x$; and therefore the former is always less than the latter, by the quantity $\frac{1}{6}pxyy \times \frac{x}{t + x}$: which difference, when x is infinitely little, or nothing is nothing; and the hyperboloid approaches nearer and nearer to the

of the said transverse and $\frac{2}{3}$ of the height; so is half the cylinder of the same base and altitude, to the solidity of the hyperboloid.

That is, the solidity is $= \frac{1}{2}$ base $\times$ altitude $\times$

$$\frac{t + \frac{2}{3}a}{t + a} = \frac{1}{2}parr \times \frac{t + \frac{2}{3}a}{t + a};$$

putting a for the altitude, r the radius of the base, t the transverse axe, and $p = 3{\cdot}14159$.

EXAMPLE.

Required the content of an hyperboloid whose altitude is 10, the radius of its base 12, and the ordinate to the middle of the abscis or height $3\sqrt{7}$.

 Here,

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the paraboloid, as the common altitude is diminished; till at last they vanish in a ratio of equality. Also when the altitude is very little, the hyperboloid is equal to the paraboloid very nearly. But when x is infinite, the same difference $\frac{1}{6}pxyy \times \frac{x}{t+x}$ becomes barely $\frac{1}{6}pxyy$, and the value of the infinitely long hyperboloid is $\frac{1}{2}pxyy - \frac{1}{6}pxyy = \frac{1}{3}pxyy =$ a cone of the same base and altitude, to which measure the hyperboloid continually approaches; and when the altitude is very great, it is equal to the cone very nearly.

Corol. 2. When x is $= nt$, the general expression becomes $\frac{1}{2}$ base $\times$ altitude $\times \frac{1 + \frac{2}{3}n}{1 + n}$; where n may be any number, integral or fractional. When n is infinitely little, this expression becomes $\frac{1}{2}$ base $\times$ altitude or $=$ the paraboloid of the same base and altitude, as before.—If n be infinitely great, it will become $\frac{1}{2}$ base $\times$ altitude $\times \frac{2}{3}$, that is, $\frac{2}{3}$ of the paraboloid, or $=$ the cone of the same base and altitude, as above. When n is $= 1$, or $t = x$, the expression becomes $\frac{1}{2}$ base $\times$ altitude $\times \frac{5}{6}$, or $\frac{5}{6}$ of the paraboloid. Moreover, when n is between 1 and 0, the hyperboloid is between $\frac{5}{6}$ and $\frac{6}{6}$ of the paraboloid; and when

Here, by prob. 3, we ſhall have $\frac{10^2 \times 3^2 \times 7 \backsim 5^2 \times 12^2}{5 \times 12^2 \backsim 10 \times 3^2 \times 7} = \frac{70 - 40}{8 - 7} = \frac{30}{1} = 30$ the tranſverſe.

Then $\frac{1}{2} parr \times \frac{t + \frac{2}{3}a}{t + a} = p \times 10 \times 144 \times \frac{36\frac{2}{3}}{80} = p \times 1440 \times \frac{11}{24} = 3{\cdot}14159 \times 660 = 2073{\cdot}451151369$, the content required.

RULE II.

To the ſquare of the radius of the baſe, add the ſquare of the diameter in the middle between the baſe and top; multiply the ſum by the altitude, the product by $3{\cdot}14159$, and $\frac{1}{6}$ of the laſt product will be the content of the ſegment.

That is, $\frac{rr + dd}{6} \times ap =$ the content of the ſegment; putting r and d for the radius of the baſe and diameter in the middle, a for the altitude, and $p = 3{\cdot}14159$.

This is proved in cor. 1 to rule 2 for the next prob.

EX-

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when n is between 1 and infinite, the hyperboloid is between $\frac{5}{6}$ and $\frac{4}{6}$ of the paraboloid of the ſame baſe and altitude.

Corol. 3. If the generating hyperbola be equilateral; ſince t is then $= c$, yy will be $= tx + xx$, and hence $t = \frac{yy - xx}{x}$; which being ſubſtituted inſtead of it, the general expreſſion for the hyperboloid will become $px \times \frac{yy - \frac{1}{3}xx}{2}$, which is the ſame expreſſion as that for the ſpherical ſegment, differing only in the ſign of the latter term, it being $- \frac{1}{3}xx$ here and $+ \frac{1}{3}xx$ there.

Corol. 4. When t is $= 0$, the rule becomes $\frac{1}{3}$ baſe $\times$ altitude, as it ought; for when t is $= 0$, the hyperboloid becomes a cone.

EXAMPLE.

Taking here the last example, in which $r = 12$, $d = 6\sqrt{7}$, and $a = 10$;

We shall have $\frac{12^2 + 36 \times 7}{6} \times 10p = 11 \times 6 \times 10p = 660p$, the content the same as before.

PROBLEM XI.

To find the Content of the Frustum of an Hyperboloid.

RULE I.*

From the sum of the squares of the semi-diameters of the two ends, subtract $\frac{1}{3}$ of a fourth proportional to the square of the transverse, the square of the conjugate, and the square of the altitude of the frustum; multiply the remainder by the said altitude, and the product by 3·14159, and the half of the last product will be the content of the frustum.

That is, $(\text{DD} + dd - \frac{aacc}{3tt}) \times \frac{1}{2}pa =$ the content; putting D and d for the semi-diameters of the ends, a the

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* DEMONSTRATION.

Using here y, Y for the ordinates or semi-diameters of the ends, x for the altitude; and putting A for the distance of the less ordinate y from the vertex of the whole solid; since YY is $= \frac{(t + \text{A} + x) \times (\text{A} + x)}{tt} \times cc$, we shall have the fluxion of the solid $\dot{s} = p\text{YY}\dot{x} = pcc\dot{x} \times \frac{\text{A}t + \text{AA} + 2\text{A}x + tx + xx}{tt}$; and the fluents give $s = pccx \times \frac{\text{A}t + \text{AA} + \text{A}x + \frac{1}{2}tx + \frac{1}{3}xx}{tt}$; and this,

a the altitude, t the transverse, c the conjugate, and $p = 3\cdot14159$.

EXAMPLE.

If a cask, in the form of two frustums of an hyperboloid, have its bung diameter 32 inches, its head diameter 24, and the diameter in the middle between the bung and head $\frac{8}{5}\sqrt{310}$; required the content in ale gallons, the length being 40 inches.

By problem 4 we shall have

$$\frac{10\sqrt{\left(-12^2+\frac{64}{25}\times 310-16^2\right)^2-4\times 12^2\times 16^2}}{12^2-\frac{32}{25}\times 310+16^2} =$$

$$\frac{10\sqrt{\left(\frac{64\times 310}{25}-20^2\right)^2-2^2\times 12^2\times 16^2}}{20^2-\frac{32\times 310}{25}} = \frac{10\sqrt{(248-125)^2-120^2}}{125-124}$$

$$= 30$$

this, by substituting $\frac{yy}{cc}$ for $\frac{At+AA}{tt}$, and $\frac{YY}{cc}$ for $\frac{At+AA+2Ax+tx+xx}{tt}$, becomes $(YY+yy-\frac{ccxx}{3tt})\times\frac{1}{2}px$.

Q. E. D.

Corol. 1. When the generating hyperbola is equilateral, t is $= c$, and the rule becomes $(YY+yy-\frac{1}{3}xx)\times\frac{1}{2}px$; which is the same with the rule for the frustum of the sphere.

Corol. 2. When the axe is $= 0$, or the hyperboloid becomes a cone; since then $Y:y::A+x:A$, or $Y-y:y::x:A = \frac{xy}{Y-y}$, and $t:c::A:y$, we shall have $\frac{cc}{tt}=\frac{yy}{AA}=\frac{(Y-y)^2}{xx}$, and therefore $\frac{ccxx}{3tt}=\frac{(Y-y)^2}{3}$; which being substituted for it in the rule $(YY+yy-\frac{ccxx}{3tt})\times\frac{1}{2}px$, produces $(YY+Yy+yy)\times\frac{1}{3}px$ for the frustum of the cone, as it ought.

$= 30\sqrt{41^2 - 40^2} = 30\sqrt{81} = 270 = t$ the transverse axe.

And
$\frac{270}{10}\sqrt{\frac{12^2}{2} - \frac{16 \times 310}{25} + \frac{16^2}{2}} = 27 \times 4\sqrt{\frac{3^2}{2} - \frac{62}{5} + \frac{4^2}{2}}$
$= 108\sqrt{\frac{5^2}{2} - \frac{62}{5}} = 108\sqrt{\frac{125-124}{10}} = 108\sqrt{\frac{1}{10}} =$
the conjugate axe c.

Then
$(DD + dd - \frac{aacc}{3tt}) \times \frac{1}{2}ap \times 2 = (16^2 + 12^2 - \frac{20^2 \times 108^2}{30 \times 270^2})$
$\times 20p = (20^2 - \frac{20^2 \times 2^2}{30 \times 5^2}) \times 20p = \frac{30 \times 25 - 4}{3} \times 4^2 \times 2p =$
$\frac{746 \times 32}{3} \times p = 7957\frac{1}{3} \times 3{\cdot}14159 = 24998{\cdot}69994216$
inches $= 88{\cdot}64787213$ ale gallons.

* RULE II.

Add together the squares of the greatest and least semi-diameters and the square of the whole diameter in

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* DEMONSTRATION.

Put $x =$ the abscis, whose ordinate is δ, the axes being t and c; and, from the nature of the hyperbola, we shall have these three equations:

$tt\delta\delta = cc(t+x) \times x = cc(tx+xx)$,
$ttdd = cc(t+x-\frac{1}{2}a) \times (x-\frac{1}{2}a) = cc(tx+xx-ax-\frac{1}{2}at+\frac{1}{4}aa)$,
$ttDD = cc(t+x+\frac{1}{2}a) \times (x+\frac{1}{2}a) = cc(tx+xx+ax+\frac{1}{2}at+\frac{1}{4}aa)$;
from the sum of the two latter of which subtract double the former, and there will result $tt \times (DD - 2\delta\delta + dd) = \frac{1}{2}aacc$;
and hence $\frac{aacc}{3tt}$ will be $= \frac{2DD - 4\delta\delta + 2dd}{3}$. Which being substituted instead of it, in the last rule, will give
$(DD + 4\delta\delta + dd) \times \frac{1}{6}ap$ for the content of the frustum required. *Q. E. D.*

Corol.

in the middle, multiply the ſum by the altitude, and the product by 3·14159, and one-ſixth of the laſt product will be the content.

That is, $(DD + 4\delta\delta + dd) \times \frac{1}{6}ap =$ the content, putting D, δ, and d for the greateſt, middle, and leaſt ſemi-diameters, $a =$ the altitude, and $p = 3·14159$.

EXAMPLE.

Taking here the ſame example as before, we ſhall have $\dfrac{16^2 + \dfrac{64 \times 310}{25} + 12^2}{6} \times 40p = \dfrac{20^2 + \dfrac{64 \times 62}{5}}{6} \times 40p = \dfrac{5 \times 20^2 + 64 \times 62}{3} \times 4p = 7957\frac{1}{3}p$, the content the ſame as before.

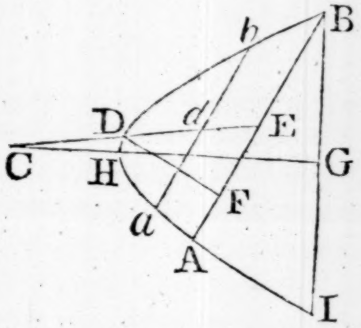
PRO-

Corol. 1. If d the leaſt diameter be ſuppoſed to become infinitely little, or nothing, the above rule will become $(DD + 4\delta\delta) \times \frac{1}{6}ap$ for the content of a ſegment.

Corol. 2. An hyperbolic fruſtum is equal to a cylinder of the ſame altitude, and whoſe diameter is $\sqrt{\dfrac{DD + 4\delta\delta + dd}{6}}$. And the ſegment is equal to the cylinder whoſe diameter is $\sqrt{\dfrac{DD + 4\delta\delta}{6}}$, the altitude being the ſame.

PROBLEM XII.

To find the Content of the Segment cut off an Hyperboloid by a Plane Oblique to the Axe of the Solid.



As the sum of DE and 2CD,
Is to the sum of $\frac{2}{3}$DE and 2CD,
So is $\frac{1}{2}$ the cylindroid of the same base and alt.
To the solidity.

That is, $\frac{2CD + \frac{2}{3}DE}{2CD + DE} \times \frac{1}{2}DF \times$ base is the content of ADB.—Where 2CD is the diameter to the double ordinate AB, or transverse diameter of the base, DE the continuation of the diameter to the middle of AB, and DF perpendicular to AB.*

* DEMONSTRATION.

Put 2CD $= a$, DE $= b$, EB $= c$, D$d = x$, $db = y$, B $=$ the elliptic base whose transverse axe is AB, and $s =$ the sine of the angle E to the radius 1.

By the property of the hyperbola, $ab + bb : ax + xx :: cc : yy = cc \times \frac{ax + xx}{ab + bb}$; and because parallel sections are similar, and similar figures are as the squares of their like diameters, we obtain $cc : yy ::$ B $:$ B $\times \frac{ax + xx}{ab + bb} =$ the section at ab.

Hence

PROBLEM XIII.

To find the Content of an Hyperbolic Spindle.

RULE I.

Proceed as in rule 1 for the elliptic ſpindle, and the content will be obtained.*

That

Hence the fluxion of the ſolid is $\text{B}s\dot{x} \times \frac{ax + xx}{ab + bb}$; whoſe fluent $\text{B}sxx \times \frac{\frac{1}{2}a + \frac{1}{3}x}{ab + bb} = \frac{\text{B}sxx}{2b} \times \frac{a + \frac{2}{3}x}{a + b}$ will be the content of $a\text{D}b$.

And when x or $\text{D}d$ becomes $= b$ or DE, the rule will be $\frac{1}{2}\text{B}bs \times \frac{a + \frac{2}{3}b}{a + b} = \frac{1}{2}\text{B} \times \text{DF} \times \frac{a + \frac{2}{3}b}{a + b}$ for the ſolid ADB.

Q. E. D.

Corol. 1. When E is a right angle, DE, DF and HG all coincide, as do AB and BI; and the rule becomes $\frac{1}{2}$ baſe $\times$ HG $\times$ $\frac{2\text{CH} + \frac{2}{3}\text{HG}}{2\text{CH} + \text{HG}}$ for the right ſegment, the ſame as in problem 10.

Corol. 2. If from $\frac{1}{2}$ baſe BI $\times$ HG $\times$ $\frac{2\text{CH} + \frac{2}{3}\text{HG}}{2\text{CH} + \text{HG}}$ the right ſegment IHB, be taken $\frac{1}{2}$ baſe B $\times$ DF $\times$ $\frac{2\text{CD} + \frac{2}{3}\text{DE}}{2\text{CD} + \text{DE}}$ the oblique ſegment ADB, there will remain the part BAI.

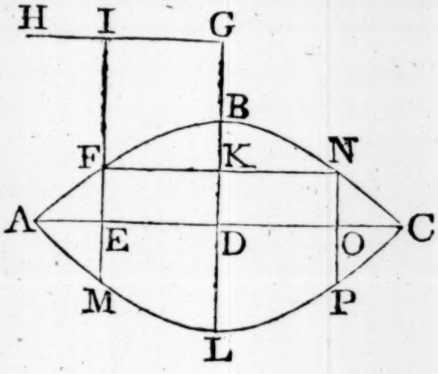
SCHOLIUM.

I might here proceed to find the ſlice parallel to the axe, and the other parts depending on it; but as there appears little occaſion for it, and it would run into a complex infinite ſeries, I have omitted it.

* GENERAL INVESTIGATION.

Put x = DE, and y = EF; then, by the property of the curve, $c : t :: \sqrt{cc + xx} : \text{FI} = \frac{t\sqrt{cc + xx}}{c}$; hence

$y =$



That is, $(2\text{CS} - \frac{t^2 \text{L}^3}{6cc}) \times p =$ the content of the ſpindle ABCLA; where C is $=$ GD the central diſtance, or the diſtance of the centers of the ſpindle and generating area ABC; S $=$ the generating area ABC, $t =$ BG the ſemi-tranſverſe, $c =$ GH the ſemi-conjugate, L $=$ AC the length of the ſpindle, and $p = 3{\cdot}14159$.

As in corollary 1.

Note.

$y =$ DG $-$ IF $= \text{C} - \frac{t\sqrt{cc + xx}}{c}$; and the fluxion of the ſolid will be $py y\dot{x} = p\dot{x} \times (\text{CC} - \frac{2\text{C}t\sqrt{cc + xx}}{c} + tt + \frac{ttxx}{cc}) =$ $p\dot{x} \times [tt - \text{CC} + \frac{ttxx}{cc} + (\text{C} - \frac{t\sqrt{cc + xx}}{c})\, 2\text{C}] = p\dot{x} \times$ $(tt - \text{CC} + \frac{ttxx}{cc} + 2\text{C}y) = 2p\text{C}y\dot{x} - \frac{ptt}{cc}\dot{x} \times (\frac{1}{4}\text{LL} - xx)$; whoſe fluent gives $2p\text{C} \times$ area EFBD $- \frac{pttx}{cc} \times (\frac{1}{4}\text{LL} - \frac{1}{3}xx) = 2p\text{C} \times$ area EFBD $- \frac{pttx}{cc} \times \frac{3\text{LL} - 4xx}{12}$ for the content of the fruſtum BFML. Or $2p\text{C} \times$ area EFBNO $- \frac{pttl}{cc} \times \frac{3\text{LL} - ll}{12} =$ the content of the middle zone FBNPLM; putting $l = 2x =$ EO its length.

Corol.

Note. If the generating hyperbola be equilateral, the theorem will be barely $(\frac{LL + DD}{D} \times S - \frac{1}{3}L^3) \times \frac{1}{2}p$ for the content of the ſpindle; putting $D = BL$ the greateſt diameter. As in corollary 3.

RULE

Corol. 1. When l becomes $= L$, the laſt theorem becomes $2pcS - \frac{pt^2L^3}{6cc}$ for the whole ſpindle ABCLA.

Corol. 2. If from the ſemi-ſpindle be ſubtracted the fruſtum, there will remain $2pc \times$ area AFE $- \frac{paatt}{cc} \times \frac{3L - 2a}{6}$ for the ſegment MAF; putting a for AE the altitude.

Corol. 3. When the generating hyperbola is equilateral, we ſhall have $LL = 4cc - 4tt = 4cc - (2c - D)^2 = 4cD - DD$; and hence $c = \frac{LL + DD}{4D}$; which being ſubſtituted in the foregoing rules, we obtain $\frac{LL + DD}{2D} \times pS - \frac{1}{6}pL^3$ for the whole ſpindle, $\frac{LL + DD}{2D} \times p \times$ area EFENO $- pl \times \frac{3LL - LL}{12}$ the middle zone FBNPLM, and $\frac{LL + DD}{2D} \times p \times$ area AFE $- paa \times \frac{3L - 2a}{6}$ for the ſegment MAF.

Corol. 4. Putting d for MF the leaſt diam. of the fruſt. or zone, $n = \frac{1}{4}p = .785398$, and the reſt of the quantities as above; ſince $LL = ll \times \frac{Dc - \frac{1}{4}DD}{Dc - dc - \frac{1}{4}DD + \frac{1}{4}dd}$, $\frac{tt}{cc} = \frac{Dc - dc - \frac{1}{4}DD + \frac{1}{4}dd}{\frac{1}{4}ll}$, and the area EFBNO $= \frac{1}{2}dl + s$, putting s for the area FBN; if theſe values be ſubſtituted in the general forms, we ſhall obtain $\frac{1}{3}nl \times (2DD + dd + (-D + d + \frac{3s}{l})$ for the value of the middle zone FENPLM,

and $\frac{1}{3}nL \times (2DD + (-D + \frac{3S}{L}) \times 8c)$ for that of the whole ſpindle.

Corol.

RULE II.

Divide 3 times the generating area, by the length of the spindle; from the quotient subtract the greatest diameter of the spindle; multiply the remainder

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Corol. 5. And if here, again, the generating hyperbola be supposed to be equilateral, since in that case ll is $= (2c - d)^2 - 4tt = (2c - d)^2 - (2c - D)^2 = -4cd + dd + 4cD - DD$, and hence $c = \frac{ll + DD - dd}{4D - 4d} = \frac{LL + DD}{4D}$; we shall then have $\frac{1}{3}nL \times [2DD + (-D + \frac{3s}{L}) \times \frac{2LL + 2DD}{D}]$ or $\frac{1}{6}pL \times (3s \times \frac{LL + DD}{LD} - LL)$ for the whole spindle, and

$\frac{1}{3}nl \times [2DD + dd + (-D + d + \frac{3s}{l}) \times \frac{2ll + 2DD - 2dd}{D - d}]$

or $\frac{1}{6}pl \times (\frac{3}{2}dd - ll + \frac{3s}{l} \times \frac{ll + DD - dd}{D - d})$ for the zone.

Hence it may be observed, that the content of an equilateral hyperbolic spindular cask, may be found from having only its length, with the bung and head diameters given.

Corol. 6. The fluent of

$p\dot{y}y\dot{x}$ or $p\dot{x} \times (cc + tt + \frac{ttxx}{cc} - \frac{2ct\sqrt{cc + xx}}{c})$ is also

$= px \times (cc + tt + \frac{ttxx}{3cc}) - \frac{ctx\sqrt{cc + xx}}{c} - ctc \times$ hyp. log. of $\frac{x + \sqrt{cc + xx}}{c}$, which is another expression for the content of the frustum FELM.

Corol. 7. Putting m for the diameter in the middle of the frustum or semi-spindle; then, by the nature of the hyperbola, we shall have $(c - \frac{1}{2}d)^2 - (c - \frac{1}{2}D)^2 : (c - \frac{1}{2}m)^2 - (c - \frac{1}{2}D)^2 :: 4 : 1$; hence $(c - \frac{1}{2}d)^2 = 4(c - \frac{1}{2}m)^2 - 3(c - \frac{1}{2}D)^2$, and $c = \frac{1}{4} \times \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d}$ in the frustum, or $c = \frac{1}{4} \times \frac{4m^2 - 3D^2}{4m - 3D}$

in

mainder by four times the central diſtance; and to the product add the ſquare of the greateſt diameter; then the ſum multiplied by the length, and the product by 3·14159, $\frac{1}{6}$ of the laſt product will be the content.

That is, $[\text{DD} + (\frac{3\text{S}}{\text{L}} - \text{D}) \times 4\text{C}]\, \frac{1}{6}p\text{L}$ = the content; the letters as before.——By corollary 4.

Note. If the generating hyperbola be ſuppoſed to be equilateral, 4 times the central diſtance will be $= \frac{\text{LL} + \text{DD}}{\text{D}}$, and then $(3\text{S} \times \frac{\text{LL} + \text{DD}}{\text{LD}} - \text{LL}) \times \frac{1}{6}p\text{L}$ is the content.——By corollary 5.

RULE III.

Divide the difference between 4 times the ſquare of the middle diameter and 3 times the ſquare of the greateſt, by the difference between 4 times the ſaid middle diameter and 3 times the greateſt, and $\frac{1}{4}$ of the quotient will be the central diſtance.

Then proceed as in the laſt rule.

That is, $[\text{DD} + (\frac{3\text{S}}{\text{L}} - \text{D}) \times \frac{4mm - 3\text{DD}}{4m - 3\text{D}}] \times \frac{1}{6}p\text{L}$ = the content.——By corollary 7.

Note.

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in the whole ſpindle; and hence the rules in corol. 4 will become $[\text{DD} + (\frac{3\text{S}}{\text{L}} - \text{D}) \times \frac{4m^2 - 3\text{D}^2}{4m - 3\text{D}}] \times \frac{1}{6}p\text{L}$ for the whole ſpindle; and $[\text{DD} + \frac{1}{2}dd + (\frac{3s}{l} + d - \text{D}) \times \frac{4m^2 - 3\text{D}^2 - d^2}{4m - 3\text{D} - d}] \times \frac{1}{6}pl$ for the zone.

Corol. 8. And if the generating hyp. be equilateral, then $\frac{1}{4}ll = (\text{C} - \frac{1}{2}d)^2 - (\text{C} - \frac{1}{2}\text{D})^2$ =, by writing for C its value found in the laſt

Note. When the generating hyperbola is equilateral, the content may be found without having the length L given, for then L is =

$$2\sqrt{\frac{(mm - \frac{3}{4}DD)^2 - [(D - m)^2 - \frac{1}{4}DD]^2}{(4m - 3D)^2}}, \text{ by corollary 8.}$$

Another rule might be drawn from corollary 6.

PROBLEM XIV.

To find the Content of a Zone or Double Frustum of an Hyperbolic Spindle.

RULE I.

Use here rule 1 for the zone of an elliptic spindle, and the content will be obtained.

That is, $2pc \times$ area EFBNO $- \frac{pttl}{cc} \times \frac{3LL - ll}{12} =$ the content of the zone FBNPLM; where $p =$ 3·14159, $C =$ GD the central distance, $t =$ BG the semi-transverse axe, $c =$ GH the semi-conjugate, $l =$ EO the length of the zone, and $L =$ AC the length of the whole spindle.

By the general investigation of the last problem.

Note.

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last corollary, $\frac{[(m-d)^2 - \frac{3}{4}(D-d)^2]^2 - [(D-m)^2 - \frac{1}{4}(D-d)^2]^2}{(4m - 3D - d)^2}$;

hence $2\sqrt{\frac{[(m-d)^2 - \frac{3}{4}(D-d)^2]^2 - [(D-m)^2 - \frac{1}{4}(D-d)^2]^2}{(4m - 3D - d)^2}} = l$ the length of the zone, and when d vanishes, we have

$2\sqrt{\frac{(m^2 - \frac{3}{4}D^2)^2 - [(D-m)^2 - \frac{1}{4}D^2]^2}{(4m - 3D)^2}} =$ L the length of the whole spindle. So that when the hyperbola is equilateral, the length of the spindle need not be given, if the diameters D and m be given.

Note. When the generating hyperbola is equilateral, the rule will be barely $p \times \frac{\text{LL} + \text{DD}}{2\text{D}} \times$ area EFBNO $- pl \times \frac{3\text{LL} - ll}{12}$.—By corollary 3 to the last problem. Putting $\text{D} = \text{BL}$ the greatest diameter.

RULE II.

Use the second rule for the zone of an elliptic spindle, only here add the first general product instead of subtracting it, and the content will be got.

That is, $[2\text{DD} + dd + 8c \times (\frac{3s}{l} + d - \text{D}) \times 8c] \times \frac{1}{12}pl = \text{FBNPLM}$. Where $d = \text{FM}$ the least diameter, $s =$ the area FBN, and the other letters as before.——By corollary 4 to the last problem.

Note. If the generating hyperbola be equilateral, 4 times the central distance will be $= \frac{ll + \text{DD} + dd}{\text{D} - d}$, and then the content of the zone will be

$$[\tfrac{3}{2}dd - ll + \tfrac{3s}{l} \times (\frac{ll + \text{DD} - dd}{\text{D} - d}) \times \tfrac{3s}{l}] \times \tfrac{1}{6}pl.$$

By corollary 5 to the last problem.

RULE III.

From 4 times the square of the diameter equidistant from the greatest and least, subtract the sum of the square of the least and 3 times the square of the greatest diameter; and from 4 times the said middle diameter, take the sum of the least and 3 times the greatest; then divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central distance.

That

That is, $\frac{1}{4} \times \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d} = c$, m being the middle diameter.—By corollary 7 to the laſt problem.

Then proceed as in the laſt rule.

Note. When the generating hyperb. is equilateral, l is $= 2\sqrt{\frac{[(m-d)^2 - \frac{3}{4}(D-d)^2]^2 - [(D-m)^2 - \frac{1}{4}(D-d)^2]^2}{(4m - 3D - d)^2}}$, by corollary 8 to the laſt problem; ſo that the content may then be found without having the length given.

Another rule might be drawn from corollary 6 to the laſt problem.

PROBLEM XV.

To find the Content of the Segment of an Hyperbolic Spindle.

Uſe here the rule for the ſegment of an elliptic ſpindle; only inſtead of what is there called the leſs axe, take here the tranſverſe, and inſtead of the greater, the conjugate axe; and the content will be obtained.——By corollary 2 to problem 13.

That is, $2pc \times$ area AFE $- \frac{paatt}{cc} \times \frac{3L - 2a}{6} =$ the ſegment MAF. Where a is $=$ AE its altitude, and the other letters as in the laſt problems.

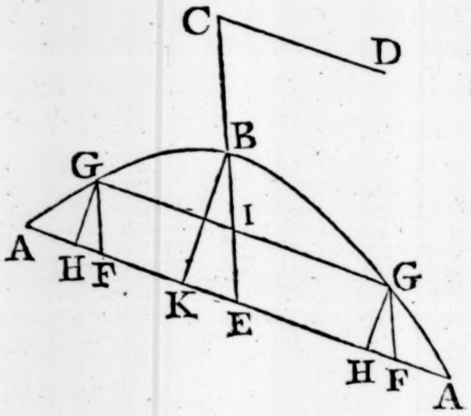
Other rules might be found by ſubtracting the fruſtum from the ſemi-ſpindle, according to the correſponding rules of the laſt two problems.

PROBLEM XVI.

To find the Content of an Universal Hyperbolic Spindle; that is, of a Solid Generated from the Revolution of an Hyperbola about an Ordinate to Any Diameter.

INVESTIGATION.

Put a = CB the semi-diameter to which belongs the double ordinate AA, about which the figure ABA revolves, c = CD its semi-conjugate, b = AE, C = CE, z = EF, y = GF parallel to CE, m = sine of the angle E or F, and n = its cosine, to the radius 1, also $p = 3{\cdot}14159$; and let GH be perpendicular to AE.



Then, by the nature of the hyperbola, $c : a :: \sqrt{cc+zz} : \frac{a\sqrt{cc+zz}}{c}$ = CI; hence y = C − CI = $\text{C} - \frac{a\sqrt{cc+zz}}{c}$: but GH is $= my$, and HF $= ny$; hence EH is $= z \pm ny$, and the fluxion of the solid is $p \times \text{GH}^2 \times \text{HK} = pmmyy \times (\dot{z} \pm n\dot{y}) = pmm \times (\pm ny^2\dot{y} + \text{C}^2\dot{z} - \frac{2\text{C}az\dot{}\sqrt{cc+zz}}{c} + a^2\dot{z} + \frac{a^2z^2\dot{z}}{cc})$;

whose

whose correct fluent is $pmm \times [\mp \frac{1}{3}n(\text{BE}^3 - y^3)$ $+ c^2z + a^2z + \frac{a^2z^3}{3c^2} - \frac{caz\sqrt{cc+zz}}{c} - cac \times$ hyp. log. of $\frac{z+\sqrt{cc+zz}}{c}$ = the solid generated by KBGH.

And when z becomes $= b$, the above fluent gives $pmm \times (c^2b + a^2b + \frac{a^2b^3}{3c^2} - \frac{cab\sqrt{cc+bb}}{c}$ $- cac \times$ hyp. log. of $\frac{b+\sqrt{cc+bb}}{c})$ for half the content of the whole solid ABA.

Corollary. From the general investigation above it appears that $pm^2 \times (\mp \frac{1}{3}n\,\text{BE}^3 + c^2b + a^2b + \frac{a^2b^3}{3cc} - \frac{cab\sqrt{cc+bb}}{c} - cac \times$ hyp. log. of $\frac{b+\sqrt{cc+bb}}{c})$ expresses the solid generated by ABK; the sign of the first term $\frac{1}{3}n\,\text{BE}^3$ being $-$ or $+$, according as K falls between A and E, or without them: and consequently half the diff. of the solids generated by the spaces ABK, KBA, is $\frac{1}{3}pnm^2 \times \text{BE}^3 = \frac{1}{3}p \times \text{BK}^2 \times \text{KE}$ = the cone generated by the triangle BKE. And hence the solids generated by the two equal areas ABE, EBA, are also equal; the value of each being the half content of the whole solid mentioned above in the investigation.

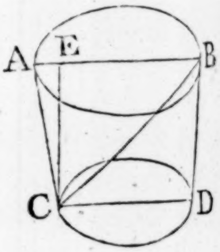
Scholium. The solution of this problem was never but once before attempted, viz. in a periodical performance; but the solution there given is erroneous, the fluxion and fluent of the general frustum being both falsely assigned.

SECTION VIII.

PROMISCUOUS QUESTIONS CONCERNING SOLIDS.

QUESTION I.

REQUIRED the content of a tub whose greater diameter AB is 60, diagonal BC 66, and the length of the stave AC 30 inches.



Here AB : BC + AC :: BC − AC : $\frac{36 \times 96}{60} = 6 \times 9 \cdot 6 = 57 \cdot 6$ = BE − EA = DC.

Hence AE $= \frac{1}{2}$AB $- \frac{1}{2}$CD $=$ $30 - 28 \cdot 8 = 1 \cdot 2$.

And EC $= \sqrt{AC^2 - AE^2} = \sqrt{30^2 - 1 \cdot 2^2} = 6\sqrt{5^2 - \cdot 2^2}$ $= 6\sqrt{25 - \cdot 04} = 6\sqrt{24 \cdot 96}$.

Then $(AB^2 + AB \times DC + DC^2) \times \frac{1}{3}CE \times \cdot 785398$ $= (60^2 + 60 \times 57 \cdot 6 + 57 \cdot 6^2) \times 2\sqrt{24 \cdot 96} \times \cdot 785398$ $= 10373 \cdot 76 \times 4\sqrt{\cdot 39} \times 3 \cdot 14159 = 81410 \cdot 112$ cubic inches $= 288 \cdot 688$ ale gallons, the content required.

QUESTION II.

Three persons having bought a sugar loaf, would divide it equally among them by sections parallel to the base; it is required to find the altitude of each person's share, supposing the loaf to be a cone whose height is 20 inches.

Similar

Similar cones being as the cubes of their altitudes, we shall have as $\sqrt[3]{3} : \sqrt[3]{1} :: 20 : 20\sqrt[3]{\frac{1}{3}} = 13{\cdot}8672247$ = the height of the upper part; and as $\sqrt[3]{3} : \sqrt[3]{2} :: 20 : 20\sqrt[3]{\frac{2}{3}} = 17{\cdot}4716107$ = that of the upper and middle part together; consequently $17{\cdot}4716107 - 13{\cdot}8672247 = 3{\cdot}604386$ is the height of the middle part, and $20 - 17{\cdot}4716107 = 2{\cdot}5283893$ is that of the lower part.

QUESTION III.

A silver cup, in form of the frustum of a cone, whose top diameter is 3 inches, its bottom diameter 4, and its altitude 6 inches, being filled with liquor, a person drank out of it till he could see the middle of the bottom; it is required to find how much he drank?

By problem 14 of section 1 we have $D = 4$, $d = 3$, $b = 6$, and $BD = \frac{1}{2}D = 2$; hence $D^3 - d^3 = 64 - 27 = 37$, P = the tabular area whose versed sine is $\frac{BD}{D}$ or $\frac{1}{2}$, $= \frac{1}{2}n = \frac{1}{2}$ of $\cdot78539816$; Q = the tabular area whose versed sine is $\frac{BD - D + d}{d}$ or $\frac{1}{3}$, $= \cdot22945528$; $\frac{BD}{BD - D + d} = 2$; and $\frac{\frac{1}{3}b}{D - d} = 2$: consequently $(37n - 32n + 54Q\sqrt{2}) \times 2 = (5n + 54Q\sqrt{2}) \times 2 = 7{\cdot}8539816 + 35{\cdot}0458624 = 42{\cdot}899844$ cubic inches $= \cdot152127$ ale gallons, or 1 gill and $\frac{1}{5}$ nearly, the quantity required.

QUESTION IV.

I have a right cone which cost me £5 13 7, at 10s a cubic foot, the diameter of its base being to its altitude as 5 to 8; and would have its convex surface

ſurface divided in the ſame ratio by a plane parallel to the baſe, the upper part to be the greater: required the ſlant height of each part.

Here $\frac{£5\ 13\ 7}{10s} = \frac{1363}{120}$ is the ſolidity of the cone in feet; and $\cdot785398 \times 5^2 \times \frac{8}{3} = 200 \times \cdot785398 \times \frac{1}{3}$ $= \frac{200n}{3}$ is that of a cone ſimilar to it, whoſe altitude is 8.

Now, the ſurfaces of ſimilar ſolids being as the ſquares of their like dimenſions, we have $\sqrt{5+8} : \sqrt{8} :: \sqrt{2\cdot5^2 + 8^2}$ the ſide of the ſaid ſimilar cone $\sqrt{\frac{562}{13}}$ the ſlant height of the upper part of this cone, when its ſurface is divided in the ratio propoſed; and conſequently $\sqrt{70\frac{1}{4}} - \sqrt{\frac{562}{13}} = \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}}$ is the ſlant height of the under part of it.

Then, becauſe ſimilar ſolids are as the cubes of their like dimenſions, we ſhall have $\sqrt[3]{\frac{200n}{3}} : \sqrt[3]{\frac{1363}{120}} ::$ $\sqrt{\frac{562}{13}} : \frac{1}{20}\sqrt[3]{\frac{1363}{n}} \times \sqrt{\frac{562}{13}} = 3\cdot9506486$ the ſlant height of the upper part required; and $\sqrt[3]{\frac{200n}{3}} :$ $\sqrt[3]{\frac{1363}{120}} :: \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}} : \frac{1}{20}\sqrt[3]{\frac{1363}{n}} \times \left(\sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}}\right)$ $= \frac{1}{20}\sqrt[3]{\frac{1363}{n}} \times \sqrt{\frac{562}{8}} - 3\cdot9506486 = 5\cdot0361098 -$ $3\cdot9506486 = 1\cdot0854612$ the ſlant height of the under part.

QUES-

QUESTION V.

There is a mill-hopper in the form of a square pyramid, whose solid content is $13\frac{1}{2}$ feet; but one foot is cut off its perpendicular altitude, to make a passage for the grain from the frustum or hopper to the mill-stone: The sides of its greater and less end are in proportion as $4\frac{1}{2}$ to 1. Required the content of the frustum, in corn measure, in which $268{\cdot}802\frac{1}{2}5$ cubic inches make a gallon, or $2150{\cdot}42$ a bushel.

Since similar solids are as the cubes of their like dimensions, we shall have $4{\cdot}5^3 : 1^3 :: 13\frac{1}{2} : \frac{13\frac{1}{2}}{4{\cdot}5^3} = \frac{3}{4{\cdot}5^2} = \frac{12}{81} = \frac{4}{27}$ the content of the part cut off.

Therefore $13\frac{1}{2} - \frac{4}{27} = \frac{27}{2} - \frac{4}{27} = \frac{27^2-8}{54} = \frac{729-8}{54} = \frac{721}{54}$ cubic feet $= \frac{721}{54} \times 1728$ or $\frac{721 \times 288}{9}$ or 721×32 or 23072 cubic inches, the content of the hopper. Which being divided by $2150{\cdot}42$, gives $10{\cdot}729067$ bushels for the content in corn measure.

QUESTION VI.

A piece of round tapering timber, whose top and bottom diameters are 40 and 50 inches, and its height 6 feet, is to be cut through the extremity of the less diameter, and parallel to the tapering direction; required the content of the two parts or hoofs into which it is cut.

By prob. 14 sect. 1, $D = 50$, $d = 40$, $h = 6$ feet $= 72$ inches; then $A = 50^2 \times$ tabular segment whose height is $\frac{50-40}{50}$ or $\frac{10}{50}$ or $\cdot 2 = 50^2 \times \cdot 1118238 = 279{\cdot}5595$.

 Hence

Hence $[\frac{A \times D}{D - d} - \frac{4}{3} d \sqrt{(D - d) d}] \times \frac{1}{3} h =$ $[\frac{279.5595 \times 50}{50 - 40} - \frac{160}{3} \sqrt{(50 - 40) 40}] \times 24 =$ $(279.5595 \times 5 - \frac{160 - 20}{3}) \times 24 = (279.5595 \times 3$ $- 160 \times 4) \times 40 = 198.6785 \times 40 = 7947.14$, the content of the hoof cut off.

But, by prob. 8 sect. 1, $\frac{50^3 - 40^3}{50 - 40} \times \frac{72}{3} \times .785398$ $= (5^3 - 4^3) \times 100 \times 24 \times .785398 = 166400$ $\times .785398 = 250690.2543893$, the content of the whole piece.

Consequently $130690.2543893 - 7947.14 =$ 122743.1143893 is the solidity of the complemental hoof.

QUESTION VII.

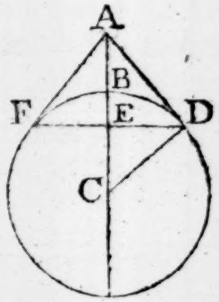
"Two Oxonians meeting at an inn, encountered with a tankard of negus; the one, being pot-valiant, gave it a black-eye, as it is called, that is, he drank till he could see the center of the bottom of the tankard; the other drank the rest: Now, if the liquor cost 1s 6d, and the tankard measure 4 inches diameter at the top and bottom, and 6 inches in depth, what must each person pay, proportionable to the liquor he drank?"

By prob. 4 sect. 1, we have $\frac{1}{6} ddh$ for the quantity left by the first person; and, by prob. 2 sect. 1, $nddh$ is the content of the whole tankard; d being the diameter, h the height, and $n = .785398$; that is, the whole is to the less share as n to $\frac{1}{6}$; consequently $n : \frac{1}{6} :: 18\text{d} : \frac{3}{n} = \frac{3}{.785398} =$ 3.8197185

3·8197185 pence, which is the sum the latter person must pay; and hence 18d − 3·8197185d = 14·1802815 pence is the sum the first drinker must pay.

QUESTION VIII.

How many acres of the earth's surface may be seen from the top of a steeple whose height is 400 feet; the earth being supposed a perfect sphere whose circumference is 25000 miles?



Having drawn from A, the top of the steeple, two lines AD, AF, to touch the earth, whose center is C, and the other lines as in the figure; FBD will be the surface required.

By similar triangles, CA : CD or CB :: CD or CB : CE; hence CA : CA − CB = AB :: CB : CB − CE =

$$\text{BE} = \frac{\text{CB} \times \text{BA}}{\text{AC}} = \frac{\text{CB} \times \text{BA}}{\text{CB} + \text{BA}} = \frac{\frac{12500}{3\cdot14159} \times \frac{400}{5280}}{\frac{12500}{3\cdot14159} + \frac{400}{5280}} =$$

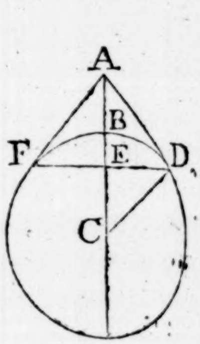
$$\frac{12500 \times 4}{1250 \times 528 + 3\cdot14159 \times 4} = \frac{50000}{660012\cdot5663706144} =$$

$\cdot07575613336416945$.

Then, by problem 19 section 1, we have $\cdot07575613336416945 \times 250000 \times 640$ (the acres in a square mile) $= 12120981\cdot338267112$, the number of acres required.

QUESTION IX.

How high above the surface of the earth must a person be raised, that he may see one third of its surface?



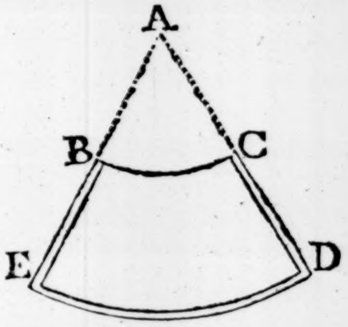
The surfaces of segments being, by prob. 19 sect. 1, as their altitudes, the altitude of the segment in the question must be one-third of the diameter, or two-thirds of the radius of the sphere; that is, $BE = \frac{2}{3}BC$; and consequently $CE = \frac{1}{3}CB$.

But, by similar triangles, as $CE : CD$ or $CB :: CD$ or $CB : CA$, and hence $CE : CB - CE = BE :: CB : CA - CB = AB = \frac{CB \times BE}{CE}$, which is a general expression for the height above the surface, and which when, as above, BE is $2EC$, becomes $2BC =$ the diameter of the earth, which is the height required.

QUESTION X.

If from a piece of tin AED, in the form of the sector of a circle, whose radius AE or AD is 30 inches, and the length of its arc ED 36 inches, be cut another sector ABC whose radius AB or AC is 20 inches; and if then the remaining frustum BCDE be rolled up so as to form the frustum of a cone; it is required to find its content, supposing one-eighth of an inch to be allowed off its slant height BE for the bottom, and the same allowance off the circumference, of both top and bottom, for what the sides BE, CD, fold over each other, in order to their being soldered together.

By



By similar figures, as AE : AB :: ED : BC $= \frac{36 \times 20}{30} = 24$. Then the circumferences of the top and bottom are $36 - \frac{1}{8} = 35\frac{7}{8} = 35{\cdot}875$, and $23{\cdot}875$; and the side of the vessel is $30 - 20 - \frac{1}{8} = 9\frac{7}{8} = 9{\cdot}875$.

Hence the diameters of the ends will be D = $\frac{35{\cdot}875}{3{\cdot}14159}$ and $d = \frac{23{\cdot}875}{3{\cdot}14159}$; half their difference is $\frac{6}{3{\cdot}14159}$; consequently the perpendicular altitude will be found, by the property of right-angled triangles, to be $\sqrt{9{\cdot}875^2 - \frac{6^2}{3{\cdot}14159^2}} = \sqrt{\frac{9{\cdot}875^2 \times 3{\cdot}14159^2 - 6^2}{3{\cdot}14159^2}} = \frac{30{\cdot}43749}{3{\cdot}14159}$.

Then, by problem 8 section 1, we shall have

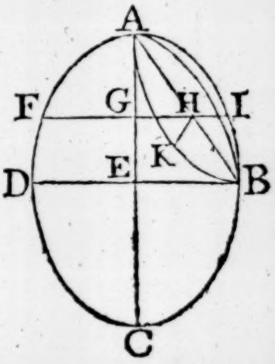
$$\frac{35{\cdot}875^2 + 35{\cdot}875 \times 23{\cdot}875 + 23{\cdot}875^2}{3{\cdot}14159^2} \times \frac{30{\cdot}43749}{3{\cdot}14159} \times \frac{3{\cdot}14159}{12}$$

$$= \frac{287^2 + 287 \times 191 + 191^2}{8^2 \times 3{\cdot}14159^2} \times \frac{30{\cdot}43749}{12} = \frac{170667}{768} \times \frac{30{\cdot}43749}{3{\cdot}14159^2}$$

$= 685{\cdot}3263$ cubic inches, the content required.

QUESTION XI.

It is required to find the area of the section of any spheroid, formed by a plane passing through the extremities of the two axes, the one axe being 80, and the other 60.



By prop. 1 sect. 4, the section AKB will be an ellipse, whose axes are AB and 2HK, H being the middle of AB.

But

But $AB = \sqrt{AE^2 + EB^2} = \sqrt{40^2 + 30^2} = 50$.

And, having drawn FGHI parallel to DB, by the nature of the ellipſe, $AE^2 : EB^2 :: AE^2 - GE^2 = AE^2 - \frac{1}{4}AE^2 = \frac{3}{4}AE^2 : GI^2 = \frac{3}{4}EB^2$; and hence, by the nature of the circle, $2\sqrt{FH \times HI} = 2\sqrt{GI^2 - GH^2} = 2\sqrt{\frac{3}{4}EB^2 - \frac{1}{4}EB^2} = EB\sqrt{2} =$ the other axe 2HK.

Conſequently $AB \times 2HK \times \cdot785398 = 50 \times 30\sqrt{2} \times \cdot785398 = 1666\cdot081101807$, the area of the ſection, when the ſpheroid is oblong. And $50 \times 40\sqrt{2} \times \cdot785398 = 2221\cdot441469076$, the area when it is oblate.

QUESTION XII.

There is a punch bowl in form of the ſegment of an oblong ſpheroid, whoſe axes are to each other in the proportion of 3 to 4, the depth of the bowl being one-fourth of the whole tranſverſe axe, and the diameter of its top 20 inches: it is required to determine what number of rounds a company of 10 men may drink out of it, when filled with liquor, uſing a conical glaſs, whoſe depth is 2 inches, and the diameter of its top an inch and a half.

The ſegment, whoſe altitude is 1, of the ſpheroid, whoſe axes are 3 and 4, is ſimilar to the propoſed one; and, by the nature of the ellipſe, $4 : 3 :: 2\sqrt{3 \times 1} : \frac{3\sqrt{3}}{2}$ the diameter of its top; but by prob. 14 ſect. 5, its content is $\frac{6-1}{3 \times 4^2} \times 3^2 \times 3\cdot14159 = \frac{15 \times \cdot785398}{4}$; and ſimilar ſolids are as the cubes of their like dimenſions; therefore $(\frac{3\sqrt{3}}{2})^3 : 20^3 ::$

15

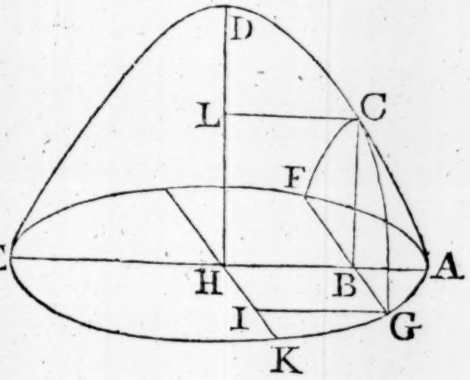
$\frac{15 \times \cdot 785398}{4} : \frac{40^3}{3^4 \times \sqrt{3}} \times \frac{15 \times \cdot 785398}{4} = \frac{80000\sqrt{3}}{81} \times$ $\cdot 785398$, the content of the bowl.

But $1 \cdot 5^2 \times 2 \times \frac{\cdot 785398}{3} = \frac{3}{2} \times \cdot 785398$, the content of the glass, and $15 \times \cdot 785398$ = each round.

Therefore $\frac{80000\sqrt{3} \times \cdot 785398}{81 \times 15 \times \cdot 785398} = \frac{16000\sqrt{3}}{243} =$ $114 \cdot 0444976$ is the number of rounds required.

QUESTION XIII.

Two persons would divide between them, by a plane perpendicular to the base, a hay-rick, in the form of a paraboloid, whose altitude is 40, and the diameter of its base 30 feet; it is required to find the difference between the solidities of the parts, supposing the altitude of the section to be 28 feet.



Here DH = 40, EA = 30, and BC = HL = 28.

Hence DL = DH − HL = 40 − 28 = 12; and, by prob. 3 sect. 6, as $\sqrt{DH} : \sqrt{DL} :: HA : LC = HB$ $= \frac{15\sqrt{12}}{\sqrt{40}} = \frac{3\sqrt{30}}{2}$; hence $\sqrt{HA^2 - HB^2} = \sqrt{15^2 - \frac{9 \times 30}{4}}$ $= \sqrt{\frac{315}{2}} = BG$, and $\frac{BA}{EA} = \frac{15 - \frac{3}{2}\sqrt{30}}{30} = \frac{10 - \sqrt{30}}{20} =$ $\cdot 2261387$ the tabular versed sine similar to AB; whose tabular area is $\cdot 13322474$; which taken from $\cdot 39269908$, the tabular semi-circle, leaves $\cdot 25947434$ the

the tabular area ſimilar to double the area HBGK; conſequently $\frac{1}{2} \times 30^2 \times \cdot 25947434 = 116\cdot 763453$ is the area HBGK.

Then, by cor. 4 to prob. 14 ſect. 6, we ſhall have $116\cdot 763453 \times 40 \times 2 + \frac{315\sqrt{28 \times 12}}{3} = 9341\cdot 07624 + 420\sqrt{21} = 9341\cdot 07624 + 1924\cdot 681794 = 11265\cdot 75803$, the difference required.

QUESTION XIV.

The curve ſurface of a paraboloid being $(109\sqrt{109} - 27) \times 2 \times 3\cdot 14159 = 6980\cdot 57746$, it is required to find its altitude, and the diameter of its baſe, ſuppoſing them in proportion as 5 to 6.

By prob. 7 ſect. 6, $\frac{(3^2 + 10^2)^{\frac{3}{2}} - 3^3}{3^2 + 10^2 - 3^2} \times 3 \times \frac{2}{3} \times 3\cdot 14159 = \frac{109\sqrt{109} - 27}{100} \times 2 \times 3\cdot 14159$ is the ſurface of a paraboloid whoſe altitude is 5 and baſe diameter 6, which is ſimilar to the propoſed paraboloid; but the ſurfaces of ſimilar bodies are as the ſquares of their lineal dimenſions, or the dimenſions are as the roots of the ſurfaces; therefore as $\sqrt{\frac{109\sqrt{109} - 27}{100} \times 2 \times 3\cdot 14159}$ is to $\sqrt{\frac{109\sqrt{109} - 27}{1} \times 2 \times 3\cdot 14159}$, or as 1 : 10 :: 5 : 50 the altitude, and :: 6 : 60, the baſe-diameter required.

QUESTION XV.

If a cubical foot of braſs were to be drawn into wire, of one-fortieth of an inch in diameter, it is required to determine the length of the ſaid wire, allowing no loſs in the metal.

Here $1 \times 12^3 = 1728$ is the ſolidity of the wire in inches; and $(\frac{1^2}{40^2}) \times \cdot 785398 = \frac{1}{1600} \times \cdot 785398$ the

the area of its end; consequently

$$\frac{1728}{\frac{1}{1600} \times \cdot 785398} = \frac{1728 \times 1600}{\cdot 785398} = 3520252\cdot 69329$$

inches $= 55\frac{5}{9}$ miles, is the length required.

QUESTION XVI.

A gentleman having a bowling green, 300 feet long and 200 feet broad, which he would raise one foot higher by means of the earth to be dug out of a ditch around it; it is required to find to what depth the ditch must be dug, its breadth being every where 8 feet.

Here the ditch is a kind of ring whose breadth is 8, and length $= 300 \times 2 + 200 \times 2 + 8 \times 4 = 516 \times 2 = 1032$, and consequently the area of its bottom $= 8 \times 1032 = 8256$; but $300 \times 200 \times 1 = 60000$ is its content; consequently $\frac{60000}{8256} = 7\frac{23}{86}$ feet is the depth required.

QUESTION XVII.

Required the weight of a bomb-shell, or hollow sphere of cast iron, whose outside diameter is 15, and the thickness of the metal $2\frac{1}{2}$ inches; the gravity of water being to that of cast iron, as 1 to 7, and a cubic inch of water weighing $\cdot 5787$ ounces avoirdupois.

Here $15^3 \times 3\cdot 14159 \times \frac{1}{6} = 15^3 \times \cdot 5236$ is the solidity of the whole sphere, supposing it all solid; and, since $15 - 2\frac{1}{2} \times 2 = 15 - 5 = 10$ is the diameter of the cavity, $10^3 \times \cdot 5236$ will be the content of the cavity; consequently their difference $(15^3 - 10^3) \times \cdot 5236 = (3^3 - 2^3) \times 5^3 \times \cdot 5236 = 19 \times 125 \times \cdot 5236$ is the solidity of the metal in inches.

But

But 1 : 7 :: ·5785 : 4·0509 ounces, the weight of a cubic inch of the metal.

Consequently $4·0509 \times 19 \times 125 \times ·5236 = 5037·4849152$ ounces $= 314·8428072$ pounds avoirdupois is the weight required.

QUESTION XVIII.

Of what diameter must the bore of a cannon be cast for a ball of 24 pound weight, so that the diameter of the bore may be one tenth of an inch more than that of the ball?

By the last, the weight of a cubic inch of cast iron is 4·0509 ounces, consequently as 4·0509 ounces : 1 inch :: 24 lb. or 24×16 oz. : $\frac{24 \times 16}{4·0509} = \frac{128}{1·3503}$ inches, the solidity of the ball; but the solidity is equal to the cube of the diam. multiplied by ·5236; therefore $\sqrt[3]{\frac{128}{1·3503 \times ·5236}} = 5·657098$ inches, the diameter of the ball; to which adding ·1 makes 5·757098 inches, the diameter required.

PART

PART IV.

Having given the menſuration of all figures which are uſually thought to be met with in real practice, viz. the figures formed by right or circular lines, or by the conic ſections; I ſhall, under this part, give a very brief treatiſe of other things relating to menſuration in general; ſuch as the quadrature and cubature of figures, from general equations expreſſing their nature or property; the method of equidiſtant ordinates; and the relation between the areas or ſolidities of figures, and their centers of gravity.

SECTION I.

OF THE TRUE QUADRATURE AND CUBATURE OF CURVES IN GENERAL.

PROPOSITION I.

If z be the abſciſs of any curve, and the ordinate y be equal to $z^{p-1} \times (\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n} + \&c.)^{q-1} \times (a + bz^n + cz^{2n} + dz^{3n} + ez^{4n}\ \&c)$; *and if* $\frac{p}{n}$ *be put equal to* r, $r + q = s$, $s + q = t$, $t + q = v$, $v + q = w$, *&c. I ſay the area will be equal to*

$$z^p \times (\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n} + \&c)^q$$

drawn into the ſeries

$+$

$$+ \frac{\frac{a}{n}}{\alpha r}$$

$$+ \frac{\frac{b}{n} - A\beta s}{\alpha(r+1)} z^{n}$$

$$+ \frac{\frac{c}{n} - A\gamma t - B\beta(s+1)}{\alpha(r+2)} z^{2n}$$

$$+ \frac{\frac{d}{n} - A\delta v - B\gamma(t+1) - C\beta(s+2)}{\alpha(r+3)} z^{3n}$$

$$+ \frac{\frac{e}{n} - A\varepsilon w - B\delta(v+1) - C\gamma(t+2) - D\beta(s+3)}{\alpha(r+4)} z^{4n}$$

$+$ &c.

Where A, B, C, &c, are the whole coefficients of the preceding terms, with their signs $+$ or $-$;

viz. $A = \frac{\frac{a}{n}}{\alpha r}$, $B = \frac{\frac{b}{n} - A\beta s}{\alpha(r+1)}$, &c.

DEMONSTRATION.

The fluxion of the area is $y\dot{z} = z^{p-1}\dot{z} \times (\alpha + \beta z^{n} + \gamma z^{2n}\ \&c)^{q-1} \times (a + bz^{n} + cz^{2n} + \&c)$.

Let the fluent of this expression, or the area required, be represented by $z^{p}(\alpha + \beta z^{n} + \gamma z^{2n} + \&c)^{q} \times (\mathrm{A} + \mathrm{B}z^{n} + \mathrm{C}z^{2n} + \&c)$; where A, B, C, &c, are not supposed to represent the same quantities as in the proposition, but other quantities yet to be determined.

Then

Then let the fluxion of this last assumed area or fluent be taken, and compared with the given fluxion, so shall we have equations for determining the values of the assumed letters A, B, C, &c.

Thus the fluxion of the assumed area being $z^{p-1}\dot{z} \times (\alpha + \beta z^n + \gamma z^{2n} + \&c)^q \times (Ap + (p+n)Bz^n + (p+2n)Cz^{2n} + \&c, + nqz^{p-1}\dot{z} \times (\alpha + \beta z^n + \gamma z^{2n} + \&c)^{q-1} \times (\beta z^n + 2\gamma z^{2n} + 3\delta z^{3n} + \&c) \times (A + Bz^n + Cz^{2n} + \&c)$.

Or, by proper multiplication, &c,

$z^{p-1}\dot{z} \times (\alpha + \beta z^n + \gamma z^{2n} + \&c)^{q-1}$ drawn into

$$\begin{aligned}
&p\alpha A + (p+qn)\beta A z^n + (p+2qn)\gamma A z^{2n} + (p+3qn)\delta A z^{3n} + \&c, \\
&\quad + (p+n)\alpha B z^n + (p+qn+n)\beta B z^{2n} + (p+2qn+n)\gamma B z^{3n} \\
&\qquad + (p+2n)\alpha C z^{2n} + (p+qn+2n)\beta C z^{3n} \\
&\qquad\qquad + (p+3n)\alpha D z^{3n} \\
&\qquad\qquad\quad \&c.
\end{aligned}$$

If the several terms of this series be compared with the corresponding terms of the series in the given fluxion, we shall obtain these equations, viz.

$$\begin{aligned}
a &= p\alpha A, \\
b &= (p+qn)\beta A + (p+n)\alpha B, \\
c &= (p+2qn)\gamma A + (p+qn+n)\beta B + (p+2n)\alpha C, \\
d &= (p+3qn)\delta A + (p+2qn+n)\gamma B + (p+qn+2n)\beta C \\
&\quad + (p+3n)\alpha D, \\
e &= p+4qn)\varepsilon A + (p+3qn+n)\delta B + (p+2qn+2n)\gamma C \\
&\quad + (p+qn+3n)\beta D + (p+4n)\alpha E, \\
\&c. &\quad \&c.
\end{aligned}$$

 From

From whence we obtain

$$\text{A} = \frac{a}{\alpha p},$$

$$\text{B} = \frac{b - (p + qn)\beta\text{A}}{\alpha(p + n)},$$

$$\text{C} = \frac{c - (p + 2qn)\gamma\text{A} - (p + qn + n)\beta\text{B}}{\alpha(p + 2n)},$$

$$\text{D} = \frac{d - (p + 3qn)\delta\text{A} - (p + 2qn + n)\gamma\text{B} - (p + qn + 2n)\beta\text{C}}{\alpha(p + 3n)},$$

$$\text{E} = \frac{e - (p + 4qn)\varepsilon\text{A} - (p + 3qn + n)\delta\text{B} - (p + 2qn + 2n)\gamma\text{C} - (p + qn + 3n)\beta\text{D}}{\alpha(p + 4n)},$$

&c. &c.

Where now the letters A, B, C, &c, in the terms on the right hand side of these equations, denote the preceding terms, as specified in the proposition.

And if $\frac{p}{n}$ be put $= r$, $r + q = s$, $s + q = t$, $t + q = v$, &c, these last equations will become

$$\text{A} = \frac{\frac{a}{n}}{\alpha r},$$

$$\text{B} = \frac{\frac{b}{n} - \text{A}\beta s}{\alpha(r + 1)},$$

$$\text{C} = \frac{\frac{c}{n} - \text{A}\gamma t - \text{B}\beta(s + 1)}{\alpha(r + 2)},$$

$$\text{D} = \frac{\frac{d}{n} - \text{A}\delta v - \text{B}\gamma(t + 1) - \text{C}\beta(s + 2)}{\alpha(r + 3)},$$

$$\text{E} = \frac{\frac{e}{n} - \text{A}\varepsilon w - \text{B}\delta(v + 1) - \text{C}\gamma(t + 2) - \text{D}b(s + 3)}{\alpha(r + 4)},$$

&c. &c.

Which

Which values of A, B, C, &c, being ſubſtituted for them in the aſſumed fluent or area, will give the area as in the propoſition. *Q. E. D.*

And much after the ſame manner we may find the area of the curve whoſe abſciſs is z, and ordinate $(\alpha+\beta z^n+\gamma z^{2n}+\&c)\times z^{p-1}\times(\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}+\&c)^{q-1}$ $\times(A+Bz^n+Cz^{2n}+Dz^{3n}+\&c)^{r-1}\times$ &c, whatever be the number of the ſeries.

When, after ſome of the firſt terms, the numerators of each of the following terms of the ſeries $\frac{\frac{a}{n}}{\alpha r}+\frac{\frac{b}{n}-A\beta s}{\alpha(r+1)}z^n+$ &c, become equal to nothing, the ſeries will break off, and terminate; and then the curve is ſaid to be quadrable. If otherwiſe, it is ſaid to be non-quadrable.——If r be either nothing, or a negative integer number, it is evident that the denominator of one of the terms of the above ſeries will become equal to nothing, and then that term will be infinite; and if this happen before the ſeries terminate, by means of the numerators becoming equal to nothing, the value of the area will come out infinite; in which caſe the ſeries is ſaid to fail.

The curve is denominated from the number of terms contained in the quantity

$(\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}+\&c)^q$. So, if it contain only one term α, it is a ſimple nomial; if it contain two terms $(\alpha+\beta z^n)^q$, it is a binomial; if three $(\alpha+\beta z^n+\gamma z^{2n})^q$, it is a trinomial; and ſo on.

 But

But in what follows I ſhall conſider none beyond the trinomial.

The area might be expreſſed by a deſcending ſeries, and from thence other caſes of the termination of the ſeries might be pointed out; but this I ſhall do in the particular forms, as in them it will be done with greater eaſe.

Nor ſhould it be wondered at that the area admits of two different values; for when an ordinate flows, the area on one ſide of it will increaſe as faſt as that on the other decreaſes; conſequently the fluxions of thoſe two areas will be equal to each other, that is, the fluxions of the areas are both expreſſed by the ſame quantity; and it is therefore but right that the fluxion ſhould admit of two different fluents, anſwering to the two areas on the oppoſite ſides of the ordinate. When the expreſſion comes out affirmative, it denotes the area lying on that ſide of the ordinate from which it is ſuppoſed to move; but when it comes out negative, it denotes the area on the other ſide of the ordinate. When it comes out infinite, it denotes the area lying along the abſciſs infinitely produced.

If all the terms of the ſeries $a + bz^n + cz^{3n} +$ &c, after the firſt, vaniſh, and by that means the expreſſion for the ordinate become only $az^{p-1} \times (\alpha + \beta z^n + \gamma z^{2n} + \&c)^{q-1}$, the curve or area is ſaid to be ſimple; but if there be more terms than one, it is ſaid to be compound.

In what follows I ſhall conſider chiefly thoſe caſes in which the curve is ſaid to be ſimple; not only becauſe thoſe are the caſes that commonly happen

happen in practice, but becauſe every compound caſe may be reſolved into as many ſimple ones, as there are terms in the ſeries $a + bz^n + cz^{2n} + \&c$); and then, by finding the area for every ſimple caſe, the aggregate of thoſe will be the area for the whole compound one.

In the quantity expreſſing any area, write that particular value of the abſciſs which it is ſuppoſed to have where the area commences, or when it is equal to nothing, and the value of the area reſulting from that ſubſtitution, will be equal to nothing, if the firſt area be rightly aſſigned, and then it needs no correction; but if the area, by this ſubſtitution, come out of ſome value, then by juſt ſo much will the firſt area differ from the truth, and it muſt be corrected by ſubtracting the ſaid value from it.

It may alſo be obſerved, that when the ordinate is oblique to the abſciſs, the area, as found by the ſeries, muſt be drawn into the ſine of the angle of inclination of the ordinate to the abſciſs.

EXAMPLE.

If an ordinate y be $= \dfrac{3a - bzz}{zz\sqrt{az - bz^3 + cz^4}}$.

To have this in the ſame form with the general ſeries, we muſt expreſs it thus

$$y = z^{-\frac{3}{2}-1} \times (3a - bzz) \times (a - bz^2 + cz^3)^{\frac{1}{2}-1};$$

or it may be expreſſed thus

$$y = z^{-1-1} \times (-b + 3az^{-2}) \times (c - bz^{-1} + az^{-3})^{\frac{1}{2}-1}.$$

Now by comparing the firſt of theſe forms with the general expreſſion of the area, we obtain

$a =$

$a = 3a$, $b = 0$, $c = -b$; $\alpha = \alpha$, $\beta = 0$, $\gamma = -b$, $\delta = c$; $p = -\frac{3}{2}$, $n = 1$, $q = \frac{1}{2}$.

Hence $r = \frac{p}{n} = -\frac{3}{2}$, $s = r + q = -1$, $t = s + q = -\frac{1}{2}$, $v = t + q = 0$, $w = v + q = \frac{1}{2}$, &c.

Then, by substituting these values in the general series, we have

$z^{-\frac{3}{2}} \times (a - b\ z^2 + c\ z^3)^{\frac{1}{2}} \times -2 - 2\sqrt{\frac{a - bz^2 + cz^3}{z^3}}$,

all the rest of the terms after the first vanishing. And because this expression is negative, it denotes the area on the other side of the ordinate.

Again, by comparing the latter form with the general series, we obtain $a = -b$, $b = 0$, $c = 3a$; $\alpha = c$, $\beta = -b$, $\gamma = 0$, $\delta = a$; $p = -1$, $n = -1$, $q = \frac{1}{2}$.

Hence $r = \frac{p}{n} = 1$, $s = r + q = \frac{3}{2}$, $t = s + q = 2$, &c.

Then substituting these values in the general series, we obtain $\sqrt{\frac{a - bz^2 + cz^3}{z^5}} \times$
$(\frac{b}{c} + \frac{3bA}{4cz} + \frac{5bB - 6a}{6cz^2} + \frac{7bC - 5aA}{8cz^3} + \frac{9bD - 7aB}{10cz^4} + \text{\&c})$
for the area in this case; where the law of the progression is manifest, and where A, B, C, &c, denote the whole coefficients of the first, second, third, &c, terms.

COROLLARY I.

When b, c, d, &c, are each equal to nothing, the curve will be simple, and the general expression for simple

ple areas becomes $\frac{az^p}{\alpha n} \times (\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \&c)^q \times$

$$+\frac{1}{r}$$

$$-\frac{s\beta A}{(r+1)\alpha}z^n$$

$$-\frac{t\gamma A+(s+1)\beta B}{(r+2)\alpha}z^{2n}$$

$$-\frac{v\delta A+(t+1)\gamma B+(s+2)\beta C}{(r+3)\alpha}z^{3n}$$

$$-\frac{w\varepsilon A+(v+1)\delta B+(t+2)\gamma C+(s+3)\beta D}{(r+4)\alpha}z^{4n}$$

&c.

COROLLARY II.

If the curve be only a trinomial, as $y = az^{p-1} \times (\alpha + \beta z^n + \gamma z^{2n})^{q-1}$, that is, δ, ε, &c, each equal to nothing, the area in the last corollary will become $\frac{az^p}{\alpha n} \times (\alpha + \beta z^n + \gamma z^{2n})^q \times (\frac{1}{r} - \frac{s\beta A}{(r+1)\alpha}z^n - \frac{(s+1)\beta B + t\gamma A}{(r+2)\alpha}z^{2n} - \frac{(s+2)\beta C + (t+1)\gamma B}{(r+3)\alpha}z^{3n} - \frac{(s+3)\beta D + (t+2)\gamma C}{(r+4)\alpha}z^{4n} - \&c)$.

But to find another expression of the area in a descending series, we have the ordinate y or $az^{p-1} \times (\alpha + \beta z^n + \gamma z^{2n})^{q-1} = az^{p-1+2nq-2n} \times (\gamma + \beta z^{-n} + \alpha z^{-2n})^{q-1}$; then, by comparing this with the original series, the area will come out $\frac{az^{p-2n}}{\gamma n} \times (\alpha + \beta z^n + \gamma z^{2n})^q \times (\frac{1}{t-2} - \frac{(s-2)\beta A}{(t-3)\gamma z^n} - \frac{(s-3)\beta B + (r-2)\alpha A}{(t-4)\gamma z^{2n}} - \frac{(s-4)\beta C + (r-3)\alpha B}{(t-5)\gamma z^{3n}} - \&c)$.

When any particular example is propoſed, if, after having ſubſtituted in the former or aſcending form, it come out an infinite ſeries, let the latter or deſcending form be tried; for ſometimes an area is quadrable by the one ſeries, when it is not ſo by the other.

EXAMPLE I.

Let the ordinate y be $= \frac{a}{\sqrt{z^6 - 2z^7 + 3z^8}}$; which reduced to form is $az^{-2-1} \times (1 - 2z + 3z^2)^{\frac{1}{2}-1}$.

Here then we have $a = a$; $\alpha = 1$, $\beta = -2$, $\gamma = 3$; $p = -2$, $n = 1$, $q = \frac{1}{2}$.

Whence $r = \frac{p}{n} = -2$, $s = r + q = -\frac{2}{3}$, $t = s + q = -1$, $v = t + q = -\frac{1}{2}$, $w = v + q = 0$, &c.

And by ſubſtituting theſe values in the aſcending ſeries, we obtain $az^{-2} \times (1 - 2z + 3z^2)^{\frac{1}{2}} \times (-\frac{1}{2} - \frac{3}{2}z)$ all the reſt of the terms vaniſhing, $= \frac{1 + 3z}{2zz} \times -a\sqrt{1 - 2z + 3zz}$ for the area required; and which therefore is quadrable.

But if the ſame values be ſubſtituted in the deſcending ſeries, we get the area expreſſed by the infinite ſeries

$$\frac{a\sqrt{1 - 2z + 3z^2}}{9z^4}\left(-1 - \frac{7A}{4z} - \frac{9B - 4A}{5z^2} - \frac{11C - 5B}{6z^3} \text{ \&c}\right) =$$

$$\frac{a\sqrt{1 - 2z + 3z^2}}{9z^4}\left(-1 + \frac{7}{4z} - \frac{9 \cdot 7 + 4 \cdot 4}{4 \cdot 5z^2} + \frac{11(9 \cdot 7 + 4 \cdot 4) + 5 \cdot 5}{4 \cdot 5 \cdot 6z^3} \text{ \&c}\right) =$$

$$\frac{a\sqrt{1 - 2z + 3z^2}}{9z^4} \times \left(-1 + \frac{7}{4z} - \frac{79}{4 \cdot 5z^2} + \frac{894}{4 \cdot 5 \cdot 6z^3} - \text{\&c}\right).$$

EX-

EXAMPLE II.

Suppose $y = \frac{azz}{2 + 3z + 3z^2|^{\frac{2}{3}}}$; which reduced to the proper form is $az^{3-1} \times (2 + 3z + 3z^2)^{\frac{1}{3}-1}$.

Here $a = a$; $\alpha = 2$, $\beta = 3$, $\gamma = 3$, $p = 3$, $n = 1$, $q = \frac{1}{3}$.

Then $r = \frac{p}{n} = 3$, $s = 3\frac{1}{3}$, $t = 3\frac{2}{3}$, $v = 4$, &c.

And, by the ascending series, the area will be expressed by the infinite series $\frac{1}{2}az^3\sqrt[3]{2 + 3z + 3z^2}$ $\times(\frac{1}{3} - \frac{10A}{8}z - \frac{13B + 11A}{10}z^2 - \frac{16C + 14B}{12}z^3 - \frac{19D + 17C}{14}z^4$ &c) $= az^3\sqrt[3]{2 + 3z + 3z^2} \times (\frac{1}{6} - \frac{10}{6.8} + \frac{42}{6.8.10} + \frac{728}{6.8.10.12}$ &c); and therefore it is not quadrable by this method.

But by the descending series the area is quadrable, and comes out $\frac{z-2}{5} \times a\sqrt{2 + 3z + 3z^2}$, all the terms after the second vanishing.

So that sometimes the area is quadrable by the one series, and sometimes by the other; but it is also sometimes quadrable by neither, and sometimes by both of them.

COROLLARY III.

In the case of a binomial $y = az^{p-1} \times \overline{\alpha + \beta z^n}|^{q-1}$, beside the letters which in the last corollary were equal to nothing, γ also will be nothing; and by supposing it to vanish in the ascending series in the last corollary, there will result $\frac{az^p}{\alpha n} \times (\alpha + \beta z^n)^q \times$ $(\frac{1}{r} - \frac{s\beta}{(r+1)\alpha}Az^n - \frac{(s+1)\beta}{(r+2)\alpha}Bz^{2n} - \frac{(s+2)\beta}{(r+3)\alpha}Cz^{3n} - \frac{(s+3)\beta}{(r+4)\alpha}Dz^{4n}$ &c. for the ascending series to be used in this case; where

where the several letters have the same value as before. And this series, it is evident, will always terminate when s is either nothing or a negative integer, and the number of terms will be one more than the number of units in s.

But the same ordinate $y = az^{p-1} \times (\alpha + \beta z^n)^{q-1}$ is $= az^{p-1+qn-n} \times (\beta + \alpha z^{-n})^{q-1}$. Then by writing, in the above ascending series, α for β, β for α, $p + qn - n$ for p, and $-n$ for $+n$, we shall have $\frac{az^{p-n}}{\beta n} \times (\alpha + \beta z^n)^q \times (\frac{1}{s-1} - \frac{(r-1)\alpha A}{(s-2)\beta z^n} - \frac{(r-2)\alpha B}{(s-3)\beta z^{2n}} - \frac{(r-3)\alpha C}{(s-4)\beta z^{3n}}$ &c) for the area expressed by a descending series; and which, it is evident, will terminate, and be quadrable, when r is any affirmative integer number.

EXAMPLE I.

If the ordinate be $y = \frac{azz}{bb + 2bcz^3 + c^2z^6} = \frac{azz}{(b+cz^3)^2}$ $= azz(b + cz^3)^{-2} = az^{3-1}(b + cz^3)^{-1-1}$.

We shall have $a = a$; $\alpha = b$, $\beta = c$; $p = 3$, $n = 3$, and $q = -1$. Hence $r = \frac{p}{n} = 1$, and $s = r + q = 1 - 1 = 0$; so that the curve is quadrable by both forms of the series. Thus,

By substituting in the ascending series, we shall have $\frac{az^3}{3c}(b + cz^3)^{-1} \times \frac{1}{1} = \frac{az^3}{3b(b + cz^3)}$ for the area on the one side of the ordinate.

And by using the descending series, we obtain $\frac{az^0}{3b}(b + cz^3)^{-1} \times -\frac{1}{1} = -\frac{a}{3c(b + cz^3)}$ for the area on the other side of the ordinate.

EX-

EXAMPLE II.

Suppose the ordinate be $y = \frac{az^3}{\sqrt{b + cz^2}} =$ $az^3 \times (b + cz^2)^{-\frac{1}{2}} = az^{4-1} \times (b + cz^2)^{\frac{1}{2}-1}$.

Then $a = a$; $\alpha = b$, $\beta = c$; $p = 4$, $n = 2$, and $q = \frac{1}{2}$. Hence $r = \frac{p}{n} = 2$, and $s = r + q = 2\frac{1}{2}$. So that it appears that the descending series will bring out a terminate, but the ascending one an interminate, area. Thus,

By substituting in the ascending series, we have $\frac{az^4\sqrt{b+cz^2}}{2b} \times (\frac{1}{2} - \frac{5cz^2}{2.3.2b} - \frac{5.7c^2z^4}{2.3.4.2^2b^2} - \frac{5.7.9c^3z^6}{2.3.4.5.2^3b^3} - \&c)$ for the area. Or by writing d for $2b$, it will be $\frac{az^4\sqrt{b+cz^2}}{d} \times (\frac{1}{2} - \frac{5cz^2}{2.3d} - \frac{5.7c^2z^4}{2.3.4d^2} - \frac{5.7.9c^3z^6}{2.3.4.5d^3} - \&c)$.

But by substituting in the descending series, the area is quadrable, and comes out $\frac{az^2\sqrt{b+czz}}{2c} \times (\frac{2}{3} - \frac{4b}{3czz}) = a\sqrt{b+czz} \times \frac{czz-2b}{3cc}$.

EXAMPLE III.

If the ordinate y be $= \sqrt{a(b+x)} = \sqrt{ab+ax} =$ $a^{\frac{1}{2}}x^0(b+x)^{\frac{1}{2}} = a^{\frac{1}{2}}x^{1-1}(b+x)^{\frac{3}{2}-1}$; which expresses the common parabola, a being the parameter, and $b+x$ the whole abscis, or distance of the ordinate from the vertex of the curve.

Here $a = a^{\frac{1}{2}}$; $\alpha = b$, $\beta = 1$, $z = x$; $p = 1$, $n = 1$, and $q = \frac{3}{2}$. Hence $r = \frac{p}{n} = 1$, and $s = r + q = \frac{5}{2}$. So that the curve is quadrable by the descending series,

ſeries, but not by the aſcending one. Thus,

By ſubſtituting in the deſcending ſeries, we obtain $a^{\frac{1}{2}}x^{0}(b+x)^{\frac{3}{2}} \times \frac{2}{3} = \frac{2}{3}\sqrt{a}(b+x)^{\frac{3}{2}}$ for the area.——And ſince this area vaniſhes only when $b+x$ or the whole abſciſs is equal to nothing, it denotes the area of the whole parabola. But when x is $=0$, this area becomes $\frac{2}{3}\sqrt{a} \times b^{\frac{3}{2}}$ for the area to the abſciſs b; hence $2\sqrt{a} \times \frac{(b+x)^{\frac{3}{2}} - b^{\frac{3}{2}}}{3}$ will be the area of the fruſtum included by the two ordinates anſwering to the two abſciſſes b and $b+x$. And, after the ſame manner, $2\sqrt{a} \times \frac{(b+x)^{\frac{3}{2}} - (b-x)^{\frac{3}{2}}}{3}$ will be found to denote the fruſtum contained by the two ordinates whoſe abſciſſes are $b+x$ and $b-x$.

But, by ſubſtituting in the aſcending ſeries, the area will come out the infinite ſeries

$$\frac{x\sqrt{a}}{b} \times (b+x)^{\frac{3}{2}} \times (1 - \frac{5x}{4b} + \frac{5\cdot7x^2}{4\cdot6b^2} - \frac{5\cdot7\cdot9x^3}{4\cdot6\cdot8b^3} + \&c).$$

As this ſeries vaniſhes when x is put equal to nothing, it expreſſes the area beginning where x begins, viz. the fruſtum included by the two ordinates whoſe abſciſſes are b and $b+x$; and therefore this ſeries is equal to $2\sqrt{a} \times \frac{(b+x)^{\frac{3}{2}} - b^{\frac{3}{2}}}{3}$, which was found above to expreſs the ſame area. Then, by making this quantity equal to the ſaid ſeries, and reducing, we obtain $\frac{2b}{3x} \times [1 - (\frac{b}{b+x})^{\frac{3}{2}}]$ for the value of the infinite ſeries $1 - \frac{5x}{4b} + \frac{5\cdot7x^2}{4\cdot6b^2} - \frac{5\cdot7\cdot9x^3}{4\cdot6\cdot8b^3} + \&c.$ And by ſuppoſing b and x in theſe laſt expreſſions

to

to be equal to each other, there will result $\frac{2\sqrt{2}-1}{3\sqrt{2}} = \frac{4-\sqrt{2}}{6}$ for the sum of the infinite series

$$1 - \frac{5}{4} + \frac{5.7}{4.6} - \frac{5.7.9}{4.6.8} + \frac{5.7.9.11}{4.6.8.10} - \&c;$$

or $\frac{1+\sqrt{2}}{3\sqrt{2}} = \frac{2+\sqrt{2}}{6} = \frac{5}{4} - \frac{5.7}{4.6} + \frac{5.7.9}{4.6.8} - \frac{5.7.9.11}{4.6.8.10} + \&c.$

Now by the binomial theorem the quantity

$$\frac{(b+x)^{\frac{3}{2}}}{b} = b^{\frac{1}{2}} \times (1 + \frac{3x}{2b} + \frac{3x^2}{2.4b^2} - \frac{1.3x^3}{2.4.6b^3} + \frac{1.3.5x^4}{2.4.6.8b^4} - \&c),$$

therefore the above quantity

$$\frac{(b+x)^{\frac{3}{2}}}{b} \times (1 - \frac{5x}{4b} + \frac{5.7x^2}{4.6b^2} - \&c), \text{ is}$$

$$= b^{\frac{1}{2}} \times (1 + \frac{x}{4b} - \frac{1x^2}{4.6b^2} + \frac{1.3x^3}{4.6.8b^3} - \frac{1.3.5x^4}{4.6.8.10b^4} + \&c),$$

and consequently the area becomes

$$a^{\frac{1}{2}}b^{\frac{1}{2}}x \times (1 + \frac{x}{4b} - \frac{1x^2}{4.6b^2} + \frac{1.3x^3}{4.6.8b^3} - \frac{1.3.5x^4}{4.6.8.10b^4} + \&c),$$

which must be $= \frac{2}{3}a^{\frac{1}{2}} \times [(b+x)^{\frac{3}{2}} - b^{\frac{3}{2}}]$;

so that, in general,

$$\frac{2}{3} \times \frac{(b+x)^{\frac{3}{2}} - b^{\frac{3}{2}}}{b^{\frac{1}{2}}x} = 1 + \frac{x}{4b} - \frac{1x^2}{4.6b^2} + \frac{1.3x^3}{4.6.8b^3} - \frac{1.3.5x^4}{4.6.8.10b^4} \&c).$$

If, in this equation, we take $x = +b$ and $-b$, we shall have

$$\tfrac{2}{3}(2\sqrt{2}-1) = 1 + \tfrac{1}{4} - \frac{1}{4.6} + \frac{1.3}{4.6.8} - \frac{1.3.5}{4.6.8.10} + \&c.$$

$$\text{and } \tfrac{2}{3} = 1 - \tfrac{1}{4} - \frac{1}{4.6} - \frac{1.3}{4.6.8} - \frac{1.3.5}{4.6.8.10} + \&c.$$

And by adding and subtracting these two, we have

$$\tfrac{1}{3} = \tfrac{1}{4} + \frac{1}{4.6} + \frac{1.3}{4.6.8} + \frac{1.3.5}{4.6.8.10} + \frac{1.3.5.7}{4.6.8.10.12} + \&c,$$

$$\tfrac{2}{3}\sqrt{2} = 1 - \frac{1}{4.6} - \frac{1.3.5}{4.6.8.10} - \frac{1.3.5.7.9}{4.6.8.10.12.14} - \&c,$$

$$\tfrac{2}{3}\sqrt{2} - 1 = \tfrac{1}{4} + \frac{1.3}{4.6.8} + \frac{1.3.5.7}{4.6.8.10.12} + \frac{1.3.5.7.9.11}{4.6.8.10.12.14.16} + \&c.$$

And,

And, further, if x be taken $= nb$, then, in general,

$$\frac{2}{3} \times \frac{(n+1)^{\frac{3}{2}} - 1}{n} = 1 + \frac{n}{4} - \frac{n^2}{4.6} + \frac{1.3n^3}{4.6.8} - \frac{1.3.5n^4}{4.6.8.10} + \&c.$$

COROLLARY IV.

In the caſe of a ſimple nomial
$y = az^{p-1} \times \alpha^{q-1} = a\alpha^{q-1}z^{p-1}$, ſuppoſing β in the laſt corollary to become nothing, the area will be $\frac{a\alpha^{q-1}z^p}{p} = \frac{mz^p}{p}$, putting $m = a\alpha^{q-1}$.

Or if inſtead of mz^{p-1} be written its value y, the area will be $\frac{yz}{p}$.

And the curve in this caſe is always quadrable; except when p is $= 0$; for then the area $\frac{mz^p}{p}$ becomes $\frac{mz^0}{0} = \frac{m}{0}$, viz. infinitely great.

To this caſe of areas belong all ſimple parabolic ſpaces, and the ſpaces lying between the curves and aſymptotes of all hyperbolas. And all ſuch are, evidently, to the parallelograms of the ſame baſe and altitude, as 1 to p.

EXAMPLE I.

The general equation to parabolas is

$p^m z^n = y^{m+n}$, or $y = p^{\frac{m}{m+n}} z^{\frac{n}{m+n}}$; where y is the ordinate, z the abſciſs, and p a given quantity.

Here then $m = p^{\frac{m}{m+n}}$, and $p = \frac{n}{m+n} + 1 = \frac{m+2n}{m+n}$; which value of p being of ſome magnitude, we conclude that all parabolas are quadrable, and the area will

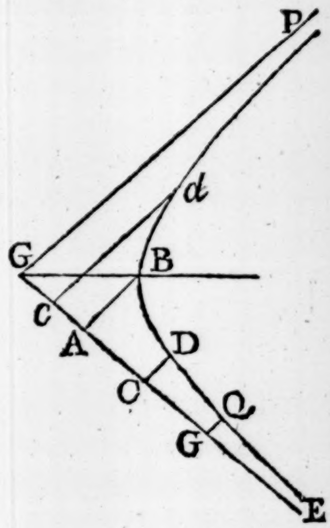
will be, by substituting in the two forms above,

either $\frac{m+n}{m+2n} \times p^{\frac{m}{m+n}} z^{\frac{m+2n}{m+n}}$, or $\frac{m+n}{m+2n} \times yz$.

So that the area of any parabola is to that of its circumscribing parallelogram, as $m+n$ is to $m+2n$.

In the common parabola, m and n are each equal to 1, or $pz = yy$, and the general area becomes $\frac{2}{3}yz$ or $\frac{2}{3}$ of the circumscribing parallelogram.

EXAMPLE II.



The general equation to hyperbolas is

$z^m y^n = p^{m+n}$, or $y = p^{\frac{m+n}{n}} z^{\frac{-m}{n}}$; where z is one asymptote GC, and y an ordinate CD drawn parallel to the other GP.

Here then $m = p^{\frac{m+n}{n}}$, and $p = 1 - \frac{m}{n} = \frac{n-m}{n}$; so that the space will always be quadrable, except when n and m are equal to each other, as in the case of the common or conic hyperbola; and by substituting in the two forms above, we obtain either

$\frac{np^{\frac{n+m}{n}} z^{\frac{n+m}{n}}}{n-m}$ or $\frac{nyz}{n-m}$ for the area GPdDC.

So that hyperbolic spaces are to their inscribed parallelograms, as $\frac{n}{n-m}$ to 1, or as n to $n-m$; and since when n is $= m$, as in the conic hyperbola, this ratio becomes that of n to 0; it appears that the said hyperbolic space GPdDC is in this case infinitely great;

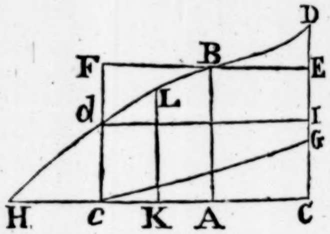
great; but in any other, of a finite magnitude, notwithstanding it is of an infinite length towards P.

COROLLARY V.

If in the original series, in the proposition, only β, γ, δ, &c, vanish, by becoming equal to nothing, the series for the area will become

$$\alpha^{q-1} z^p \times \left(\frac{a}{p} + \frac{bz^n}{p+n} + \frac{cz^{2n}}{p+2n} + \frac{cz^{3n}}{p+3n} + \&c\right).$$

And if here again p and n be supposed each equal to 1, and α^{q-1} be actually drawn into the series, and then the literal coefficients of the several terms be represented by A, B, C, &c, we shall obtain $Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \frac{1}{4}Dx^4 + \&c$, for the area corresponding to the ordinate $y = A + Bx + Cx^2 + Dx^3 + \&c$, putting x for z.



And by this series we might find a great variety of areas. But to reduce it to another series, converging quicker; if x be $=$ AC, $y =$ CD, as the above series will represent the area ABDC of the curve dBD; so likewise if y flow back from A to cd, making the ordinate $y = A - Bx + Cx^2 - Dx^3 + \&c$, x being now negative, the area ABdc will be
$Ax - \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 - \frac{1}{4}Dx^4 + \&c.$

Then, supposing the former x or AC, to be equal to this latter x or Ac, and adding the two series together, there will result
$2Ax + \frac{2}{3}Cx^3 + \frac{2}{5}Ex^5 + \&c$, for the area cdDC. Or if now z be put $= c$C $= 2x$, this area will be equal to $Az + \frac{Cz^3}{3 \cdot 4} + \frac{Ez^5}{5 \cdot 4^2} + \frac{Gz^7}{7 \cdot 4^3} + \&c$, where c may either be at the vertex of the curve, or not.

When

When x is $=0$, then $y = \text{A} = \text{AB}$; so that AB is always the value of A, or A denotes the middle ordinate of the area cdDC.

If FBE be drawn parallel to cC, and meet CD and cd in E and F; the rectangle cFEC will be equal to $c\text{C} \times \text{AB} = \text{A}z$, and consequently the difference between cdDC and cFEC will be $\frac{\text{C}z^3}{3 \cdot 4} + \frac{\text{E}z^5}{3 \cdot 4^2} +$ &c.

EXAMPLE I.

If dBD be a right line; its equation will be $y = \text{A} + \text{B}x$, B here denoting the ratio of the sine of the angle dBA to its cosine. Then, since the value of each of the letters C, D, E, &c, in the general formula, is nothing, the general area $\text{A}z + \frac{\text{C}z^3}{3 \cdot 4} +$ &c, will be only $\text{A}z =$ the rectangle $\text{CF} = \text{AB} \times c\text{C}$ $= \frac{dc + \text{DC}}{2} \times c\text{C}$ for the area $d\text{DC}c$, as in rule 6 prob. 3 sect. 1 part 2.

EXAMPLE II.

If dBD be a parabola, its equation will be $y = \sqrt{a \times (b + x)} = \sqrt{ab} \times (1 + \frac{x}{2b} - \frac{x^2}{2.4b^2} + \frac{1.3x^3}{2.4.6b^3} - \frac{1.3.5x^5}{2.4.6.8b^4} +$ &c, by extracting the root.

Here $\text{A} = \sqrt{ab}$, $\text{C} = -\frac{\text{A}}{2.4b^2}$, $\text{E} = -\frac{1.3.5\text{A}}{2.4.6.8b^4}$, &c, and, by substituting, the area cdDC will be $=$ $2\sqrt{ab} \times (x - \frac{x^3}{4.6b^2} - \frac{1.3.5x^5}{4.6.8.10b^4} - \frac{1.3.5.7.9x^7}{4.6.8.10.12.14b^6} -$ &c), $= \text{A}z \times (1 - \frac{z^2}{4.6.2^2b^2} - \frac{1.3.5z^4}{4.6.8.10.2\ b} -$ &c).

And therefore the rectangle Az or BA $\times$ cC, exceeds the parabolic area, by all the terms of the series after the first.

 But

But by example 3 to corollary 3, the same area cdDC is $= 2\sqrt{a} \times \frac{(b+x)^{\frac{3}{2}} - (b-x)^{\frac{3}{2}}}{3}$; conseq. $\frac{(b+x)^{\frac{3}{2}} - (b-x)^{\frac{3}{2}}}{3} = \sqrt{b} \times (x - \frac{x^3}{4.6b^2} - \frac{1.3.5x^5}{4.6.8.10b^4} - \frac{1.3.5.7.9x^7}{4.6.8.10.12.14b^6}$ &c). And when b and x are equal to each other in this equation, we obtain $\frac{2}{3}\sqrt{2} = 1 - \frac{1}{4.6} - \frac{1.3.5}{4.6.8.10} - \frac{1.3.5.7.9}{4.6.8.10.12.14} -$ &c; the same as was found, by a different method, in ex. 3 cor. 3.

When c is the vertex of the curve, then b is $= x$, and the series for the area cDC becomes AZ $\times (1 - \frac{1}{4.6} - \frac{1.3.5}{4.6.8.10} -$ &c); which also, by substituting for this last series its value $\frac{2}{3}\sqrt{2}$, as found above, will be $= \frac{2}{3}\sqrt{2}$AZ $= \frac{2}{3}\sqrt{2} \times$ the rectangle cFEC.

EXAMPLE III.

If the curve be a circle, and A were any point between the circumference and the center, the series would be very complex; but if A be the center, and r the radius, then will

y be $= \sqrt{rr - xx} = r \times (1 - \frac{x^2}{2r^2} - \frac{x^4}{2.4r^4} - \frac{1.3x^6}{2.4.6r^6} -$ &c, and consequently the area will be

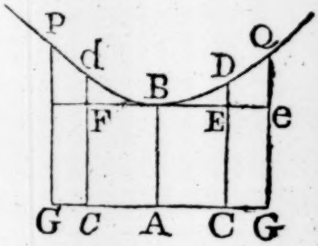
$2rx \times (1 - \frac{x^2}{2.3r^2} - \frac{x^4}{2.4.5r^4} - \frac{1.3x^6}{2.4.6.7r^6} - \frac{1.3.5x^8}{2.4.6.8.9r^8} -$ &c); which series converges very fast when x is small in respect of r.

When x becomes equal to r, we obtain $4rr \times (1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7} - \frac{1.3.5}{2.4.6.8.9} -$ &c, for the whole circle.

 Hence

Hence $1 - {}^{\cdot}785398$ &c, $= \frac{1}{2.3} + \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7}$ $+$ &c, as in pa. 104.

EXAMPLE. IV.



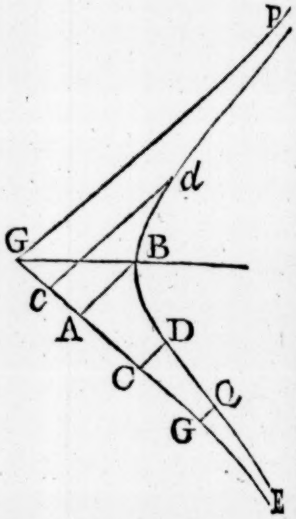
If dBD be an hyperbola; put its semi-transverse $AB = t$, semi-conjugate $AG = c$, abscis $AC = x$, and ordinate $CD = y$; then its equation will be $yy = tt \times \frac{cc + xx}{cc}$, or $y = t \times \sqrt{1 + \frac{xx}{cc}} = t \times (1 + \frac{x^2}{2c^2} - \frac{x^4}{2.4c^4} + \frac{1.3x^6}{2.4.6c^6} - \frac{1.3.5x^8}{2.4.6.8c^8}$ &c); and by substituting in the general series, the area $cdDC$ will be $2tx \times (1 + \frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6} -$ &c), the same with the series in the last example for the area of a circle, differing only in the signs of the second, fourth, sixth, &c, terms.

When x is $= c$, the series becomes $2tc \times (1 + \frac{1}{2.3} - \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7} -$ &c) for the area $GPQG$.

The rectangle $ABEC$ is $= tx$, which taken from the area $ABDC$, will leave $AB \times BE \times (\frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6} -$ &c) for the area BDE. So that the rectangle $ABEC$ is to the trilineal BDE, as 1 is to the series $\frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6} -$ &c. And when x or AC is $= c$ or AG, the rectangle $ABeG$ will be to the trilineal BeQ, as 1 is to $\frac{1}{2.3} - \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7} -$ &c.

 Parallel

Parallel to one asymptote GP of an hyperbola, draw ordinates BA, CD, to the other GG; B being the vertex of the curve: And put GA $= a$, AB $= b$, AC $= x$, and CD $= y$.



Then $a + x : a :: b : y = \frac{ab}{a+x} = b \times (1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3}$ &c.$)$ = the ordinate.

And by comparing this with the general series, we obtain $2bx \times (1 + \frac{x^2}{3a^2} + \frac{x^4}{5a^4} + \frac{x^6}{7a^6} +$ &c), for the area cdDC, taking Ac = AC.

And when $x = a$, or AC = AG, the above expression will become 2GA × AB × $(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$ &c) for the area GPBQG.

PROPOSITION II.

If there be Any Solid, in which all the Parallel Sections, either Right or Oblique to the Axe, are Like and Similar Figures; and if, when the Abscis, or Part of the Axe drawn through the Centers of the Parallel Sections, is represented by z, *the Value of the Section be expressed by Any Series of Terms involving* z, *and known quantities, after the same manner as the Ordinate is expressed in the last Proposition: Then will the Solidity be expressed by the same Quantity as the Area in the last Proposition: That is, supposing the relation between the Abscis and Section to be expressed by Any Equation of which one side is the Section, after the manner of the relation between the Abscis and Ordinate in the last Proposition; whenever the value of*

of the Ordinate agrees with that of the Section, then will the value of the Solidity be the same with that of the Area as found by the said Proposition.

DEMONSTRATION.

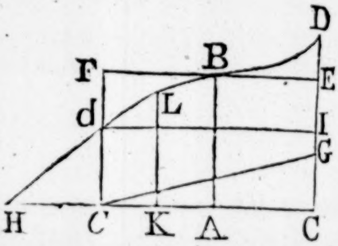
For, the fluxion of the solid being equal to the section drawn into the fluxion of the abscis, and the fluxion of the area equal to the ordinate drawn into the same fluxion of the abscis, whenever the ordinate and section agree, the area and solidity must likewise agree, since equal fluxions have equal fluents. *Q. E. D.*

SCHOLIUM.

All the examples that have been given to the corollaries of the last proposition, may be considered as examples to this, the quadrature in those being considered as the cubature here; and it is evident that curves will be cubable not only whenever they are quadrable, but they will often be cubable when the area cannot be expressed except in an infinite series, because the section of a solid, using the area of a given circle as a given number, is often a terminate expression, when the ordinate is denoted by an infinite series; and this is the case in the conic sections; for those curves, the parabola excepted, are not quadrable, yet all the solids generated by their revolution have finite expressions, as we have already found in the foregoing part of this work, and as will more generally appear in what follows; which I have set down not merely to shew that the same conclusions may be brought out by different means, but chiefly for the sake of some easy, curious, and general rules, with which this method of investigation so readily supplies us.

COROLLARY.

Thus, by applying here what is done in the last corollary of the last proposition, to the curves of the second kind, we shall obtain general and terminate expressions for their solidities. For, by prop. 2 sect. 4 part 3, the flowing section being always as $A + Bx + Cxx$, if p denote a given quantity, $p \times (a + Bx + Cxx)$ will denote the section itself; and consequently $px \times (A + \frac{1}{2}Bx + \frac{1}{3}Cx^2)$ or $p \times (Az + \frac{1}{12}Cz^3)$ will denote the solidity cdDC, generated by the section of a solid, in flowing from the place cd to the place CD, z being $= c$C.



Since pA represents the middle section AB, the first term pAz in the above expression will be equal to the cylinder cFBEC, whose base is equal to the middle section AB, and altitude cC $= z$; and consequently $\frac{1}{12}p$Cz^3 will always be the difference between the solid cdBDC and the cylinder cFBEC.

Or if there be taken CG : Cc :: $\surd$C : 1, and the three areas, viz. the curve cdBDC, the rectangle cFBEC, and the right-angled triangle cGC, be supposed to revolve together about the common axe cC; then will the difference between the solid generated by cdBDC, and the cylinder generated by cFBEC, be constantly equal to one-fourth of the cone generated by cGC.

The frust. generated by cdBDC is $\left\{\begin{array}{l}\text{greater than}\\ \text{equal to}\\ \text{or less than}\end{array}\right.$ the cylinder generated by cFBEC according as the value of C is $\left\{\begin{array}{l}\text{affirmative}\\ \text{nothing}\\ \text{or negative}\end{array}\right\}$ that is, according as the curve dBD is $\left\{\begin{array}{l}\text{an hyperbola}\\ \text{a parabola}\\ \text{or an ellipse.}\end{array}\right.$

In

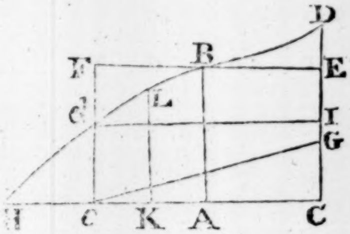
In the parabola C is $=$ 0, the equation to the curve being $yy = A + Bx$; and therefore the parabolic conoid, or any frustum of it, is equal to a cylinder of the same height with it, and whose end is equal to the middle section of the conoid, viz. the section parallel to and equally distant from its two ends.

In the hyperbola $\surd C$ is $= \frac{n}{m}$, putting m for the whole axe of which cC is the continuation, and n for its conjugate axe; and therefore the hyperbolic conoid, or its frustum exceeds the cylinder by one-fourth of a cone of which the radius of the base is to the common altitude cC of all the three solids, as n is to m.

But in the ellipse the value of C is $-\frac{nn}{mm}$, m and n being the axes as above; and therefore the semi-spheroid, or its frustum, is less than the cylinder by one-fourth of the said cone, of which the radius of the base is to the common altitude, as n is to m.

When the ellipse becomes a circle, m will be equal to n; and consequently the semi-sphere, or its frustum, will be less than the cylinder by one-fourth of the cone, of which the radius of the base is equal to the common altitude of all the solids.

When dBD is a right line, or the solid a cone, then the value of $\surd C$ or $\frac{CG}{Cc}$ will be equal to $\frac{DC}{CH}$, producing the right lines Dd, Cc to meet in H; consequently the triangles DHC, GcC are similar, and the conic frustum exceeds the cylinder by one-fourth of a similar cone; the solids being still all of the same altitude. The radius CG of the base of the said similar



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lar cone, will evidently be $= \frac{DC \times Cc}{CH}$, a fourth proportional to CH, CD and Cc: Or, if dI be drawn parallel to cC, the triangle dID will be equal to the triangle cCG in all reſpects, or CG $=$ ID $=$ DC $-$ cd, that is, the radius CG of the baſe of the ſimilar cone, is equal to the difference between the greateſt CD and leaſt cd radius of the fruſtum.——When the fruſtum becomes a complete cone, it is evident that it will exceed the cylinder by one-fourth of itſelf, or the cone will be to the cylinder as 4 to 3.

From the general manner of conſidering the generation of the fruſtum, in the beginning of this corollary, by the parallel motion of a flowing ſection, it is evident that the above properties will obtain in the ſame ſolids, whether the ends are perpendicular or oblique to the axe; and alſo that the general method will include the fruſtum of any pyramid, whether right or oblique; and ſuch a fruſtum will exceed the priſm, of the ſame altitude and whoſe baſe is equal to the middle ſection of the fruſtum, by one-fourth of a cone of the ſame altitude alſo, and of which the radius of the baſe is $\frac{DC \times Cc}{CH}$.

It may farther be obſerved, in general, that in the ſame or in ſimilar ſolids, when the altitude and the inclination of the ends to the axe are the ſame, the cone, or the difference between the fruſtum of the ſolid and the cylinder or priſm, will be conſtantly the ſame quantity, whatever the magnitude of the ends may be. When the altitude is conſtant, and the inclinations of the ends vary, the ſaid difference is reciprocally as the cube of the diameter of the generating plane which is conjugate to Cc the axe of the fruſtum, or diameter connecting the centers of its ends. When both the altitude and inclination vary, the difference is as the cube of the altitude directly, and cube of the

ſaid

said conjugate diameter reciprocally; but when they vary so as that the altitude is always reciprocally as that diameter, then the difference is a constant quantity.

PROPOSITION III.

In the Frustum of any Solid, generated by the Revolution of Any Conic Section about its Axe, if to the Sum of the two Ends be added four times the Middle Section, or section parallel to and equally distant from the two Ends, one-sixth of the Last Sum will be a Mean Area, and being drawn into the Altitude of the Solid, will produce the Content.

DEMONSTRATION.

By the corollary to the last prop. the content is $px \times (A + \frac{1}{2}Bx + \frac{1}{3}Cx^2) = px \times \frac{6A + 3Bx + 2Cx^2}{6}$; but by prop. 2 sect. 2 part 3, the one end pD^2 is $= p \times (A + Bx + Cx^2)$; then by writing $\frac{1}{2}x$ for x, the middle section will be $p\delta^2 = p \times (A + \frac{1}{2}Bx + \frac{1}{4}Cx^2)$; and by writing 0 for x, the other end will be $p\,d^2 = pA$; now it is evident that the sum of the two ends with four times the middle section, is equal to the numerator of the quantity expressing the content; consequently $px \times \frac{D^2 + 4\delta^2 + d^2}{6} = px \times \frac{6A + 3Bx + 2Cx^2}{6}$ will be the solidity of the frustum. *Q.E.D.*

COROLLARY I.

When the frustum becomes the complete solid, the less end vanishes, and the content is barely $px \times \frac{D^2 + 4\delta^2}{6}$.

COROL-

COROLLARY II.

This propoſition, it is evident, includes all fruſtums, as well as the complete ſolids, whether right or oblique, not only of the ſolids generated from the revolution of the conic ſections, but alſo of all pyramids, cones, and in ſhort of any ſolid whoſe parallel ſections are ſimilar figures. The ſame theorem may alſo be applied to the areas of all curves, whoſe equation is of this form $y = A + Bx + Cx^2$, calling D and d the extreme ordinates, and δ the middle one.

And of this form is the equation to the parabola dBD (figure to corollary 5 to prop. 1) CD being the axe of the curve, and putting cA $= x$, and AB $= y$; and conſequently any parabolic portion cdBA bounded by the curve dB, the two lines AB, cd being parallel to the axe CD, and the line Ac perpendicular to the ſame, is truly expreſſed by $\frac{AB + 4KL + cd}{6} \times Ac$, KL being the middle ordinate.

SECTION II.

OF THE APPROXIMATE QUADRATURE AND CUBATURE OF CURVES BY MEANS OF EQUIDISTANT ORDINATES OR SECTIONS.

PROPOSITION I.

If a Right Line AN *be divided into any Even Number of Equal Parts* AC, CE, EG, *&c; and at the points of diviſion be erected Perpendicular Ordinates* AB, CD, EF, *&c, terminated by Any Curve* BDF *&c; and if*

if A *be put for the sum of the Extreme or First and Last Ordinates* AB, NO; B *for the sum of the Even Ordinates* CD, GH, LM, *&c, viz. the second, fourth, sixth, &c;* *and* C *for the sum of all the rest* EF, IK, *&c, viz. the third, fifth, &c, or the Odd Ordinates, wanting the first and last: Then I say that the Common Distance* AC *or* CE, *&c, of the Ordinates, being drawn into the sum arising from the addition of* A, 4 *times* B, *and* 2 *times* C, *one-third of the Product will be the Area* ABON *very nearly.*

That is $\frac{A + 4B + 2C}{3} \times D =$ *the Area, putting* $D = AC$ *the Common Distance of the Ordinates.*

DEMONSTRATION.

If through the first three points B, D, F, of the proposed curve, a parabola be conceived to be drawn, having its axe parallel to the ordinates; the parabolic area, by the second corollary of the last proposition, will be $(AB + 4CD + EF) \times \frac{1}{6} AE = (AB + 4CD + EF) \times \frac{1}{3} AC$; but when the points B, D, F, are at no great distance from each other, the parabolic curve will nearly coincide with the curve proposed, and consequently the area of the one will be equal to the area of the other, very nearly; hence $(AB + 4CD + EF) \times \frac{1}{3} AC$ will be equal to the area ABFE, very nearly. After the same manner $(EF + 4GH + IK) \times \frac{1}{3} AC$, or EG, will be equal to the area EFKI; and $(IK + 4LM + NO) \times \frac{1}{3} AC =$ the area IKON; and so on.

Wherefore, by taking the sum of these areas, we shall have $(AB + 4CD + 2EF + 4GH + 2IK + 4LM + NO) \times \frac{1}{3} AC = (A + 4B + 2C) \times \frac{1}{3} AC$ for the whole area ABON, very nearly. *Q.E.D.*

COROL-

COROLLARY.

The ſame theorem will alſo obtain for the contents of all ſolids, by uſing the ſections perpendicular to the axe, inſtead of the ordinates, as will appear from the laſt propoſition of the laſt ſection; for the fruſtum of the ſolid there may be ſuppoſed to coincide in the extremes and middle with any other fruſtum, after the ſame manner as the parabola with any other curve. The propoſition is accurately true for all parabolic and right-lined areas, as alſo for all ſolids generated from the revolution of conic ſections or right lines, with all kinds of pyramids; and for all other kinds of areas, and ſolidities, it is a very good approximation.

SCHOLIUM.

It is evident that the greater the number of ordinates or ſections are uſed, the more accurately will the area or ſolidity be determined; but in a caſe of real practice, ſuch as caſk-gauging, three ſections are ſufficient; and becauſe of the ſimplicity and accuracy of this method, it is by far the moſt proper for practical gaugers of any thing that can well be deviſed; for by only taking the bung and head diameters, and a diameter in the middle between them, the ſum of the bung, head, and 4 times the middle circle, drawn into half the length of the caſk, will be ſix times the content, very nearly. And the ſame method may be uſed to good purpoſe in all caſes of ullaging either ſtanding or lying caſks, by taking the area at the top, bottom, and middle of the liquor.

EXAMPLE I.

Let it be required to find the area of the quadrant of a circle whoſe radius is 1.

Let

Let the radius be divided into 10 equal parts by 11 ordinates, the first being nothing, and the last equal to the radius, and the intermediate 9 ordinates the sines of the arcs whose versed sines are respectively $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}$, and $\frac{9}{10}$.

Now, by the property of the circle, we shall have
$\sqrt{1{\cdot}9 \times {\cdot}1} = \sqrt{{\cdot}19} = {\cdot}4358899$ the second ordinate,
$\sqrt{1{\cdot}8 \times {\cdot}2} = \sqrt{{\cdot}36} = {\cdot}6$ the 3d ordinate,
$\sqrt{1{\cdot}7 \times {\cdot}3} = \sqrt{{\cdot}51} = {\cdot}7141428$ the 4th,
$\sqrt{1{\cdot}6 \times {\cdot}4} = \sqrt{{\cdot}64} = {\cdot}8000000$ the 5th,
$\sqrt{1{\cdot}5 \times {\cdot}5} = \sqrt{{\cdot}75} = {\cdot}8660254$ the 6th,
$\sqrt{1{\cdot}4 \times {\cdot}6} = \sqrt{{\cdot}84} = {\cdot}9165151$ the 7th,
$\sqrt{1{\cdot}3 \times {\cdot}7} = \sqrt{{\cdot}91} = {\cdot}9539392$ the 8th,
$\sqrt{1{\cdot}2 \times {\cdot}8} = \sqrt{{\cdot}96} = {\cdot}9797959$ the 9th,
$\sqrt{1{\cdot}1 \times {\cdot}9} = \sqrt{{\cdot}99} = {\cdot}9949874$ the tenth,
the 11th, being $= 1$ the radius, as was before observed.

Hence $A = 0 + 1 = 1$,
$B =$ the 2d + 4th + 6th + 8th + 10th $= 3{\cdot}9649847$,
$C =$ the 3d + 5th + 7th + 9th $= 3{\cdot}296311$,
and $D = {\cdot}1$.

Consequently $(A + 4B + 2C) \times \frac{1}{3}D = 23{\cdot}45256 \times \frac{1}{30} = {\cdot}78175 =$ the area, pretty near the truth.

This area differs from the truth by more than in curves in general, on account of the obliquity of the curve to the ordinates near the beginning of the arc; but if by the same rule we find the arc belonging to the 9 greatest ordinates, which will be ${\cdot}70363$, and to it add the area of the semi-segment whose altitude is ${\cdot}2$ and base ${\cdot}6$, considering it as a parabola, viz. ${\cdot}6 \times {\cdot}2 \times \frac{2}{3} = {\cdot}08$, the sum ${\cdot}7836$ will be nearer the true area; and if ${\cdot}08175$ the true area of the segment had been added, we should have had ${\cdot}78538$ the area very exact.

EXAMPLE II.

Taking example 1 to prob. 5 of the parabola, in which the abſciſs of a parabola is 2, and the baſe or double ordinate 12, to find the area of the parabola.

Here, by taking three ordinates, of which the firſt and laſt are each nothing, and the middle one $= 2$, their common diſtance being 6; we ſhall have $A = 0$, $B = 2$, $C = 0$, and $D = 6$.

Conſequently $(A + 4B + 2C) \times \frac{1}{3}D = 8 \times 2 = 16$, the true area as before.

EXAMPLE III.

Given the lengths of five equidiſtant ordinates of an area, or the ſections of a ſolid, 10, 11, 14, 16, 16, and the length of the whole baſe 20; to find the area or the ſolidity.

Here $A = 10 + 16 = 26$, $B = 11 + 16 = 27$, $C = 14$, and $D = \frac{20}{4} = 5$.

Therefore $(A + 4B + 2C) \times \frac{1}{3}D = (26 + 108 + 28) \times \frac{5}{3} = \frac{162}{3} \times 5 = 54 \times 5 = 270$, the area or ſolidity required.

EXAMPLE IV.

Given eleven ordinates to an hyperbola between the aſymptotes, to find the area. Thus, taking the equation at the end of example 4 to corollary 5 to prop. 1 ſect. 1, in which any ordinate y is $= \frac{ab}{a+x} = \frac{1}{1+x}$; then ſuppoſing the firſt value of x to be nothing, and the laſt value equal to 1, and conſe-

quently

quently the common diſtance of the ordinates ·1 or $\frac{1}{10}$; we ſhall have for the ordinates

$\frac{1}{1+0}$, $\frac{1}{1 \times \cdot 1}$, $\frac{1}{1+\cdot 2}$, &c,

0: $\frac{10}{10}$, $\frac{10}{11}$, $\frac{10}{12}$, $\frac{10}{13}$, $\frac{10}{14}$, $\frac{10}{15}$, $\frac{10}{16}$, $\frac{10}{17}$, $\frac{10}{18}$, $\frac{10}{19}$, and $\frac{10}{20}$.

Here then A $= \frac{10}{10} + \frac{10}{20} = 1 + \frac{1}{2} = 1 \cdot 5$,
B $= \frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19} = 3 \cdot 4595393$,
C $= \frac{10}{12} + \frac{10}{14} + \frac{10}{16} + \frac{10}{18} = 2 \cdot 7281745\frac{5}{6}$,
and D $= \cdot 1$

Therefore (A + 4B + 2C) $\times \frac{1}{3}$D $= \cdot 69315021$ is the area required.

PROPOSITION II.

Given Any Equidiſtant Ordinates of a Curve, or Sections of a Solid, as a, b, c, d, e, f, &c; to find nearly a General Expreſſion for the Area or Solidity by the Terms a, b, c, &c, and the Baſe upon which they inſiſt.

INVESTIGATION.

The firſt differences of the terms a, b, c, d, e, f, &c, are

$b - a,\ c - b,\ d - c,\ e - d,\ f - e$, &c,

The dif. of theſe are

$c - 2b + a,\ d - 2c + b,\ e - 2d + c,\ f - 2e + d$, &c,

The dif. of theſe are

$d - 3c + 3b - a,\ e - 3d + 3c - b,\ f - 3e - 3d - c$, &c,

The dif. of theſe are

$e - 4d + 6c - 4b + a,\ f - 4e + 6d - 4c + b$, &c,

The dif. of theſe are

$f - 5e + 10d - 10c + 5b - a$, &c,

Then by putting $\alpha, \beta, \gamma, \delta, \varepsilon$, &c, for the firſt term of theſe differences reſpectively, and tranſpoſing, we ſhall have

$b =$

$$\begin{array}{l}
b = +a + \alpha \\
c = -a + 2b + \beta \\
d = +a - 3b + 3c + \gamma \\
e = -a + 4b - 6c + 4d + \delta \\
f = +a - 5b + 10c - 10d + 5e + \varepsilon \\
\&c. \qquad \&c.
\end{array}$$

Hence, by ſubſtituting the value of b in that of c; and the values of b and c in that of d; and thoſe of b, c, and d in that of e; and thoſe of b, c, d, and e in that of f; and ſo on; we ſhall obtain

$$\begin{array}{l}
a = a \\
b = a + \alpha \\
c = a + 2\alpha + \beta \\
d = a + 3\alpha + 3\beta + \gamma \\
e = a + 4\alpha + 6\beta + 4\gamma + \delta \\
f = a + 5\alpha + 10\beta + 10\gamma + 5\delta + \varepsilon \\
\&c. \qquad \&c.
\end{array}$$

Where it is evident that the coefficients in the value of any term, are the ſame with thoſe of a binomial raiſed to the power denoted by the place or number of the term, and conſequently the x term z will be $a + x\alpha + x.\frac{x-1}{3}\beta + x.\frac{x-1}{2}.\frac{x-2}{3}\gamma + x.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}\delta +$ &c. Which, by actually multiplying the factors together, &c, becomes $z = a$

$$\begin{array}{llll}
+ & \alpha x + & \frac{1}{2}\beta x^2 + & \frac{1}{2.3}\gamma x^3 + \\
- & \frac{1}{2}\beta x - & \frac{1+2}{2.3}\gamma x^2 - & \frac{1+2+3}{2.3.4}\delta x^3 - \\
+ & \frac{1.2}{2.3}\gamma x + & \frac{1.2+3.\overline{1+2}}{2.3.4}\delta x^2 + & \frac{1.2+3.\overline{1+2}+4.\overline{1+2+3}}{2.3.4.5}\varepsilon x^3 + \\
- & \frac{1.2.3}{2.3.4}\delta x - & \frac{1.2.3+4\times\overline{1.2+3.\overline{1+2}}}{2.3.4.5}\varepsilon x^2 & \&c \\
+ & \frac{1.2.3.4}{2.3.4.5}\varepsilon x & \&c & \\
 & \&c & &
\end{array}$$

$$\text{or } z = \begin{array}{llllll} a + & \alpha x + & \frac{1}{2}\beta x^2 + & \frac{1}{6}\gamma x^3 + & \frac{1}{24}\delta x^4 + & \frac{1}{120}\varepsilon x^5 \text{ \&c} \\ & - \frac{1}{2}\beta x - & \frac{1}{2}\gamma x^2 - & \frac{1}{4}\delta x^3 - & \frac{1}{12}\varepsilon x^4 & \text{\&c} \\ & + \frac{1}{3}\gamma x + & \frac{11}{24}\delta x^2 + & \frac{7}{24}\varepsilon x^3 & \text{\&c} & \\ & - \frac{1}{4}\delta x - & \frac{5}{12}\varepsilon x^2 & \text{\&c} & & \\ & + \frac{1}{5}\varepsilon x & \text{\&c} & & & \\ & \text{\&c} & & & & \end{array}$$

And by cor. 5 prop. 1 sect. 1, the area will be

$$L \times : \begin{array}{llllll} a + & \frac{1}{2}\alpha x + & \frac{1}{6}\beta x^2 + & \frac{1}{24}\gamma x^3 + & \frac{1}{120}\delta x^4 + & \frac{1}{720}\varepsilon x^5 \text{ \&c} \\ - & \frac{1}{4}\beta x - & \frac{1}{6}\gamma x^2 - & \frac{1}{16}\delta x^3 - & \frac{1}{60}\varepsilon x^4 & \text{\&c} \\ + & \frac{1}{6}\gamma x + & \frac{11}{72}\delta x^2 + & \frac{7}{96}\varepsilon x^3 & \text{\&c} & \\ - & \frac{1}{8}\delta x - & \frac{5}{36}\varepsilon x^2 & \text{\&c} & & \\ + & \frac{1}{10}\varepsilon x & \text{\&c} & & & \\ & \text{\&c} & & & & \end{array}$$

Where L represents the length of the base.

Which area will be had accurately true when the differences are continued till one order of them become equal to nothing, for then the series will break off and terminate; that is, an area is quadrable whenever one of the orders of the differences of its equidistant ordinates consists of a series of nothings, or any other equals, or of a series of arithmeticals; but if an order of the differences never become equal to nothing, the area will be expressed by an infinite series.

When the terms or ordinates are taken near to one another, the differences will the sooner become equal to nothing, or nearly so; and if any order of differences, and consequently its succeeding ones, be rejected as inconsiderable, we shall have an approximate value of the area, and that the nearer to the truth as the more of the differences α, β, γ, δ, ε, &c, are used.

Thus, if there be only one ordinate, or if α, β, γ, &c, be rejected, the area will be aL.

If there be two ordinates, or β, γ, δ, &c, be rejected, x will be $= 1$, and the area will be

$$(a + \tfrac{1}{2}\alpha) \times L = \frac{a+b}{2} \times L.$$

If there be three ordinates, or γ, δ, ε, &c, be rejected, x will be $= 2$, and the area will be $(a + \alpha + \frac{2}{3}\beta - \frac{1}{2}\beta) \times L = (a + \alpha + \frac{1}{6}\beta) \times L = (a + 4b + c) \times \frac{1}{6}L$.

If there be four ordinates, or δ, ε, &c, be rejected, x will be $= 3$, and the area will be

$$\left.\begin{array}{r} a + \frac{3}{2}\alpha + \frac{3}{2}\beta + \frac{9}{8}\gamma \\ - \frac{3}{4}\beta - \frac{3}{2}\gamma \\ + \frac{1}{2}\gamma \end{array}\right\} \begin{array}{l} \times L = (a + \frac{3}{2}\alpha + \frac{3}{4}\beta + \frac{1}{8}\gamma) \times L \\ \\ = (a + 3b + 3c + d) \times \frac{1}{8}L. \end{array}$$

And thus by putting x equal to every number successively, we shall have the following table of areas, answering to the respective number of ordinates set opposite to them; of which every expression is more accurate than the preceding ones, and in which A represents the sum of the first and last terms, B the sum of the second and last but one, C the sum of the third and last but two, &c, and the last of the letters denotes the double of the middle term when the number of terms is odd, also L denotes the length of the whole base, or the distance between the first and last terms.

No

N° of ordin.	Areas.
1	AL
2	$\frac{A}{2} \times L$
3	$\frac{A + 2B}{6} \times L$
4	$\frac{A + 3B}{8} \times L$
5	$\frac{7A + 32B + 6C}{90} \times L$
6	$\frac{19A + 75B + 50C}{288} \times L$
7	$\frac{41A + 216B + 27C + 136D}{840} \times L$
8	$\frac{751A + 3577B + 1323C + 2989D}{17280} \times L$
9	$\frac{989A + 5888B - 928C + 10496D - 2270E}{28350} \times L$
&c	&c

EXAMPLE.

Taking the third example to the last proposition, in which are given the five perpendiculars 10, 11, 14, 16, 16, and the distance between the first and last $= 20$; we shall have $A = 10 + 16 = 26$, $B = 11 + 16 = 27$, $C = 14 \times 2 = 28$, and $L = 20$.

Hence by the rule for five ordinates $\frac{7A + 32B + 6C}{90} \times L = \frac{182 + 864 + 168}{90} \times 20 = \frac{1214}{9} \times 2 = 269\frac{7}{9}$ the area as before, nearly.

SCHOLIUM.

When there are many ordinates given, the case may be reduced to fewer, by adding together the first and last ordinates, the second and last but one, the third and last but two, and so on, and considering the sums as a new set of terms upon a base equal to half

half that of the former. Or there may be added together the two first and two last terms into one sum, then the four terms next to these, and so on; or we may add three at the beginning to three at the end, then the next six, and so on; always diminishing the base in proportion to the number of terms that are added into each sum. And the content will be nearly the same in each case.

EXAMPLE.

Taking the fourth example to the last proposition, in which are given the eleven ordinates $\frac{10}{10}$, $\frac{10}{11}$, $\frac{10}{12}$, $\frac{10}{13}$, $\frac{10}{14}$, $\frac{10}{15}$, $\frac{10}{16}$, $\frac{10}{17}$, $\frac{10}{18}$, $\frac{10}{19}$, $\frac{10}{20}$, and the distance of the first and last $= 1$; we shall have $\frac{10}{10} + \frac{10}{20} = 1{\cdot}5$, $\frac{10}{11} + \frac{10}{19} = 1{\cdot}4354069$, $\frac{10}{12} + \frac{10}{18} = 1{\cdot}3888889$, $\frac{10}{13} + \frac{10}{17} = 1{\cdot}357466$, $\frac{10}{14} + \frac{10}{16} = 1{\cdot}3392857$, and $\frac{10}{15} \times 2 = 1\frac{1}{3}$.

Hence $A = 1{\cdot}5 + 1\frac{1}{3} = \frac{5}{26}$,
$B = 1{\cdot}4354069 + 1{\cdot}3392857 = 2{\cdot}7746926$,
$C = 1{\cdot}3888889 + 1{\cdot}357466 = 2{\cdot}7463549$,
and $L = \frac{1}{2}$.

Then, by the rule for six ordinates, we shall have $\frac{19A + 75B + 50C}{288} \times L = {\cdot}6931476$, which is much nearer to the truth than the number found by the former rule, the true number being ${\cdot}69314718$, and is the hyperbolic logarithm of 2.

SEC-

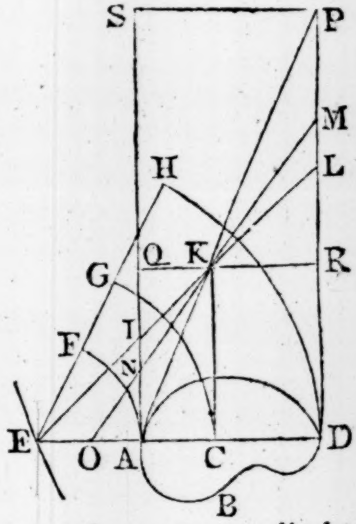
SECTION III.

OF THE RELATION BETWEEN THE AREAS AND SOLIDITIES OF FIGURES AND THE CENTERS OF GRAVITY OF THEIR GENERATING LINES AND PLANES.

PROPOSITION I.

If any Line, Right or Curved, or Any Plane Figure, whether it be bounded by Right Lines or Curves, revolve about an Axe in the Plane of the Figure; the Surface or Solid generated will be respectively equal to the Surface or Solid whose base is the Given Line or Figure, and its Height equal to the Arc described by the Center of Gravity of the said Generating Line or Figure; and consequently the Content will be found by drawing the Generating Line or Figure into the Arc described by its Center of Gravity.

DEMONSTRATION.



Let AFHD be the figure generated by the given line or plane ABD; through C the center of gravity of which draw DCAE perpendicular to the axe of revolution, and meeting HGFE in E; and let every point of the base be reduced to AD by means of perpendiculars to it.

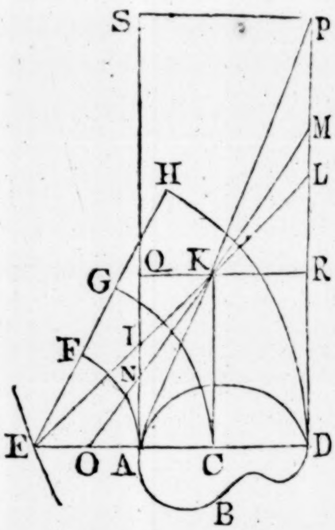
The figure AFHD generated, is equal to all the AF, CG, DH, &c. But, by similar figures, all the AF, CG, DH, &c, are as all the

 EA,

EA, EC, ED, &c; and, by mechanics, the ſum of all the EA, EC, ED, &c, is equal to as many times EC; therefore the ſum of all the AF, CG, DH, &c, is equal to as many times CG, or equal to AD × CG; that is, the figure AFHD is equal to ABD × CG, the baſe drawn into the line deſcribed by its center of gravity. *Q. E. D.*

COROLLARY I.

From E draw EIKL cutting the upright priſmatic figure erected upon the given baſe ABD, ſo as that any perpendicular AI may be equal to its correſponding arc AF. Then will the figure AILD be equal to the figure AFHD.



For, by ſimilar figures, all the AF, CG, DH, &c, are as all the AI, CK, DL, &c, each to each; and as one of each are equal, therefore they are all equal, each to each; viz. all the AI, CK, DL, &c, equal to all the AF, CG, DH, &c; that is, the figure AILD equal to the figure AFHD.

COROLLARY II.

Through K draw MKNO; and the figure ANMD will be equal to the figure AIKLD, or equal to the figure AFHD.

For, by the laſt corollary, ANMD is equal to the figure deſcribed by the baſe AD revolving about O, till the arc deſcribed by C be equal to CK; which, by the propoſition, is equal to AD × CK or AD × CG.

COROL-

COROLLARY III.

Hence all the upright figures AQKRD, AIKLD, ANKMD, AKPD, &c, of the same base, and bounded at the top by lines or planes cutting the upright sides, and passing through the extremity K of the line CK erected upon the center of gravity of the base, are equal to one another; and the value of each will be equal to the base drawn into the line CK.

Hence also all figures, described by the revolution of the same line or plane about different centers or axes, will be equal to one another, when the arcs described by the center of gravity are equal. But if those arcs be not equal, the figures generated will be as the arcs. And in general, the figures generated, will be to one another, as the revolving lines or planes drawn into the arcs described by their respective centers of gravity.

COROLLARY IV.

Moreover, the opposite parts NIK, MLK, of any two of these figures, are equal to each other.

COROLLARY V.

The figure ASPD is to the figure APD, as AS to CK; or, by similar triangles, they will be as AD to AC.

For ASPD is equal to AD × AS, and APD is equal to AD × CK.

COROLLARY VI.

If the line or plane be supposed to be at an infinite distance from the center about which it revolves, the figure generated will be an upright surface or prism, the altitude being the line described by the center of gravity; so that the base drawn into the said line will

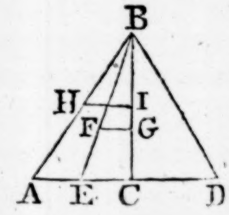
be equal to the baſe drawn into the altitude, as it ought for all upright figures, whoſe ſections parallel to the baſe are all equal to each other.

EXAMPLE I.

If a right line, or a parallelogram, revolve about a line perpendicular to the length, there will be deſcribed a ring either ſuperficial or ſolid; and as the center of gravity of the deſcribing line, or parallelogram, is in the middle of them, the general rule will become the ſame with rule 3 ſect. 1 part 3, and the rule at prob. 2 ſect. 3 part 3.

When the center of revolution is in the end of the line, the line will deſcribe a circle whoſe radius is the ſaid deſcribing line, and whoſe circumference is double the circumference deſcribed by the center of gravity; conſequently the radius drawn into half the circumference, will be the area of the circle.

EXAMPLE II.



If the right-angled triangle ABC revolve about the perpendicular BC, and deſcribe the cone ABD.

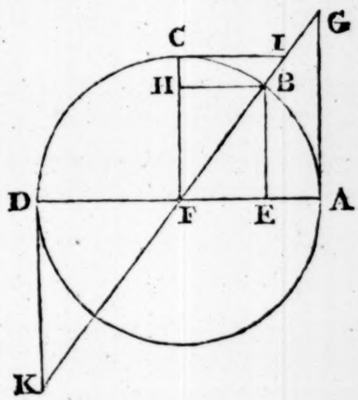
Draw BE to biſect AC, and take EF equal to one-third of BE; and F will be the center of gravity of the triangle ABC, as is well known. Draw FG parallel to AC.——Then the line deſcribed by F will be the circumference of a circle whoſe radius is FG; and, by the general rule, the cone will be equal to the triangle ABC $\times$ GF $\times$ $8n$, putting $n = \cdot785398$ &c, or $=$ AC $\times \frac{1}{2}$ CB $\times \frac{2}{3}$ CE $\times 8n$ $=$ AC $\times \frac{1}{2}$ CB $\times \frac{1}{3}$ CA $\times 8n =$ AC$^2 \times \frac{1}{3}$ CB $\times 4n =$ AD^2 $\times \frac{1}{3}$ CB $\times n =$ the baſe drawn into one-third of the altitude, as it ought.

Again,

Again, from H, the middle of AB, draw HI parallel to AC; then is H the center of gravity of AB, and consequently the surface described by AB will be AB × circumference whose radius is HI = AB × half the circumference whose radius is AC = the side drawn into half the circumference of the base = the surface of the cone, as it ought.

EXAMPLE III.

Let the semi-circle DCA revolve about the diameter AD, and describe the surface of a sphere.



If there be taken DC : FC :: FC : FH $= \frac{FC^2}{DC} = \frac{4rr}{c}$, putting r for the radius, and c for the whole circumference; H will be the center of gravity of the arc DCA, and consequently $r : c ::$ FH $: 4r =$ the line or circumference described by H the center of gravity; and, by the general rule, DCA $\times 4r = \frac{1}{2}c \times 4r = 2rc =$ the surface of the sphere = the circumference into the diameter, as it ought.

And for the solidity of the sphere, we shall have first $\frac{1}{2}c : 2r :: \frac{2}{3}r : \frac{8rr}{3c} =$ the distance FH of the center of gravity of the semi-circle DCAD from the diameter AD, which is two-thirds of the distance of the

the center of gravity of the arc DCA from the ſame diameter DA, in the former caſe; conſequently the line deſcribed by the center of gravity in this caſe will be two thirds of that in the former; but the deſcribing line in the former caſe is to the deſcribing ſpace in this, as 1 is to $\frac{1}{2}r$, therefore $1 : \frac{2}{3} \times \frac{1}{2}r ::$ ſurface of the ſphere : ſolidity $= \frac{1}{3}r \times$ ſurface.

COROLLARY.

The circumference of the circle, whoſe radius is the diſtance of the center of gravity of the ſemi-circumference of any circle from its center, is equal to four times the radius of that circle.

EXAMPLE IV.

For the ſolidity of the parabolic ſpindle, putting $b =$ the baſe and $a =$ the altitude or axe of the generating parabola, and $n = \cdot 785398$, as before.

It is known that $\frac{2}{3}a$ is the diſtance of the center of gravity from the baſe, and conſequently $\frac{16}{5}a =$ the line deſcribed by the center of gravity; but $\frac{2}{3}ab$ is $=$ the revolving area; therefore $\frac{16}{5}a \times \frac{2}{3}ab = \frac{32}{15}aab$ will be the content, which is $\frac{8}{15}$ of the circumſcribed cylinder.

EXAMPLE V.

For the paraboloid. Making the notation as in the laſt example, $\frac{3}{8}b$ will be the diſtance of the center of gravity of the ſemi-parabola from the axe, conſequently $\frac{3}{8}b \times 8n \times \frac{2}{3}ab = 2abbn =$ the ſolidity $=$ half the circumſcribed cylinder.

PART

PART V.

SECTION I.

OF LAND SURVEYING.

IT is ſuppoſed that land-meaſuring firſt gave riſe to geometry, which has ſince been gradually riſing to the height at which we at preſent view it. Since the diviſion of common grounds has become ſo frequent in England, ſurveying has been univerſally taught and practiſed throughout the nation. I ſhall here give a ſhort account of two or three of the moſt uſeful inſtruments, before we enter upon the meaſurements themſelves.

CHAPTER I.

The Deſcription and Uſe of the moſt uſeful Inſtruments for Surveying.

I. OF THE CHAIN.

Land is meaſured with a chain, called Gunter's chain, of 4 poles or 22 yards in length, which conſiſts of 100 equal links, each link being $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches long, that is nearly 8 inches or $\frac{1}{3}$ of 2 feet.

An acre of land, is equal to 10 ſquare chains, that is, 10 chains in length and 1 chain in breadth. Or it is 220×22 or 4840 ſquare yards. Or it is 40×4 or 160 ſquare poles. Or it is 1000×100 or 100000 ſquare links. Theſe being all the ſame quantity.

Alſo, an acre is divided into 4 parts called roods, and

and a rood into 40 parts called perches, which are ſquare poles, or the ſquare of a pole of $5\frac{1}{2}$ yards long, or the ſquare of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 ſquare links. So that the diviſions of land meaſure will be thus :

625	ſq. links	=	1 pole or perch
40	perches	=	1 rood
4	roods	=	1 acre

The length of lines, meaſured with a chain, are ſet down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore after the content is found, it will be in ſquare links; then cut off five of the figures on the right-hand for decimals, and the reſt will be acres. Thoſe decimals are then multiplied by 4 for roods, and the decimals of theſe again by 40 for perches.

EXAMPLES.

Suppoſe the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

```
    792
    385
  -----
   3960
  6336
 2376
 -------       ac  ro  p
 3·04920  Anſ.  3   0   7
       4
 -------
 ·19680
      40
 -------
 7·87200
 -------
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2. OF THE PLAIN TABLE.

This instrument consists of a plain rectangular board of any convenient size, the center of which, when used, is fixed by means of screws to a three-legged stand, having a ball and socket, or joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belongs,

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a sheet of paper upon the table. The one side of this frame is usually divided into equal parts, for drawing lines across the table, parallel or perpendicular to the sides; and the other side of the frame is divided into 360 degrees from a center which is in the middle of the table; by means of which the table is to be used as a theodolite, &c.

2. A needle and compass screwed into the side of the table, to point out the directions, and to be a check upon the sights.

3. An index, which is a brass two-foot scale, with either a small telescope, or open sights erected perpendicularly upon the ends. These sights, and one edge of the index are in the same plane, and that edge is called the fiducial edge of the index.

Before you use this instrument, take a sheet of paper which will cover it, and wet it to make it expand; then spread it flat upon the table, pressing down the frame upon the edges, to stretch it and keep it fixed there; and when the paper is become dry, it will by contracting again, stretch itself smooth and flat from any cramps or unevenness. Upon this paper is to be drawn the plan or form of the thing measured.

In using this instrument, begin at any part of the ground you think the most proper, and make a point upon a convenient part of the paper or table, to represent that point of the ground; then fix in that point one

one leg of the compaſſes, or a fine ſteel pin, and apply to it the fiducial edge of the index, moving it round till through the ſights you perceive ſome remarkable object, as the corner of a field, &c, and from the ſtation point draw a line with the point of the compaſſes along the fiducial edge of the index; then ſet another object or corner, and draw its line; do the ſame by another, and ſo on till as many objects are ſet as may be thought neceſſary. Then meaſure from your ſtation towards as many of the objects as may be neceſſary, and no more, taking the requiſite offsets to corners or crooks in the hedges, &c, and lay the meaſures down upon their reſpective lines upon the table. Then, at any convenient place, meaſured to, fix the table in the ſame poſition, and ſet the objects which appear from thence, &c, as before; and thus continue till your work is finiſhed, meaſuring ſuch lines as are neceſſary, and determining as many as you can by interſecting lines of direction drawn from different ſtations.

And in theſe operations obſerve the following particular cautions and directions.

1. Let the lines upon which you make ſtations be directed towards objects as far diſtant as poſſible; and when you have ſet any ſuch object, go round the table and look through the ſights from the other end of the index, to ſee if any other remarkable object lie directly oppoſite; if there be not ſuch an one, endeavour to find another forward object, ſuch as ſhall have a remarkable backward oppoſite one, and make uſe of it rather than the other; becauſe the back object will be of uſe in fixing the table in the original poſition either when you have meaſured too near to the forward object, or when it may be hid from your ſight at any neceſſary ſtation by intervening hedges, &c.

2. Let the ſaid lines upon which the ſtations are taken, be purſued as far as you conveniently can; for that

that will be the means of preserving more accuracy in the work.

3. At each station it will be necessary to prove the truth of it; that is, whether the table be straight in the line towards the object, and also whether the distance be rightly measured and laid down on the paper. —To know if the table be set down straight in the line; lay the index upon the table in any manner, and move the table about till through the sights you perceive either the fore or back object; then, without moving the table, go round it and look through the sights by the other end of the index, to see if the other object can be perceived; if it be, the table is in the line; if not, it must be shifted to one side, according to your judgment, till through the sights both objects can be seen.—The aforesaid operation only informs you if the station be straight in the line; but to know if it be in the right part of the line, that is, if the distance has been rightly laid down; fix the table in the original position, by laying the index along the station line, and turning the table about till the fore and back objects appear through the sights, and then also will the needle point at the same degree as at first; then lay the index over the station point and any other point on the paper representing an object which can be seen from the station; and if the said object appear straight through the sights, the station may be depended on as right; if not, the distance should be examined and corrected till the object can be so seen. And for this very useful purpose, it is advisable to have some high object or two, which can be seen from the most part of the ground, accurately laid down on the paper from the beginning of the survey, to serve continually as proof objects.

When, from any station, the fore and back objects cannot both be seen, the agreement of the needle with one of them may be depended on for placing the table straight

ſtraight on the line, and for fixing it in the original poſition.

Of ſhifting the Paper on the Plain Table.

When one paper is full, and you have occaſion for more; draw a line in any manner through the fartheſt point of the laſt ſtation line, to which the work can be conveniently laid down; then take the ſheet off the table, and fix another on, drawing a line upon it, in a part the moſt convenient for the reſt of the work; then fold or cut the old ſheet by the line drawn on it, apply the edge to the line on the new ſheet, and, as they lie in that poſition, continue the laſt ſtation line upon the new paper, placing upon it the reſt of the meaſure, beginning at where the old ſheet left off. And ſo on from ſheet to ſheet.

When the work is done, and you would faſten all the ſheets together into one piece, or rough plan, the aforeſaid lines are to be accurately joined together, as when the lines were transferred from the old ſheets to the new ones.

But it is to be noted, that if the ſaid joining lines, upon the old and new ſheet, have not the ſame inclination to the ſide of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to reſpect ſtill the ſame degree of the compaſs, the eaſieſt way of drawing the lines in the ſame poſition, is to draw them both parallel to the ſame ſides of the table, by means of the equal diviſions marked on the other two ſides.

3. OF THE THEODOLITE.

The theodolite is a brazen circular ring, divided into 360 degrees, and having an index with ſights, or a teleſcope, placed upon the center, about which the index is moveable; alſo a compaſs fixed to the center, to point out courſes and check the ſights; the whole being

being fixed by the center upon a stand of a convenient height for use.

In using this instrument, an exact account, or field-book, of all measures and things necessary to be remarked in the plan, must be kept, from which to make out the plan upon your return home from the ground.

Begin at such part of the ground, and measure in such directions, as you judge most convenient; taking angles or directions to objects, and measuring such distances as appear necessary, under the same restrictions as in the use of the plain table. And it is safest to fix the theodolite in the original position at every station by means of fore and back objects, and the compass, exactly as in using the plain table; registering the number of degrees cut off by the index when directed to each object; and, at any station, placing the index at the same degree as when the direction towards that station was taken from the last preceding one, to fix the theodolite there in the original position, after the same manner as the plain table is fixed in the original position, by laying its index along the line of the last direction.

The best method of laying down the aforesaid lines of direction, is to describe a pretty large circle, quarter it, and lay upon it the several numbers of degrees cut off by the index in each direction; then, by means of a parallel ruler, draw, from station to station, lines parallel to lines drawn from the center to the respective points in the circumference.

4. OF THE CROSS.

The cross consists of two pair of sights set at right angles to each other, upon a staff having a sharp point at the bottom to stick in the ground.

The cross is very useful to measure small and crooked pieces of ground. The method is to measure a base or chief line, usually in the longest direction of

 the

the piece from corner to corner; and while meaſuring it, finding the places where perpendiculars would fall upon this line from the ſeveral corners and bends in the boundary of the piece, with the croſs, by fixing it, by trials, upon ſuch parts of the line as that through one pair of the ſights both ends of the line may appear, and through the other pair you can perceive the correſponding bends or corners; and then meaſuring the lengths of the ſaid perpendiculars.

REMARKS.

Beſides the fore-mentioned inſtruments, which are moſt commonly uſed, there are ſome others; as the circumferentor, which reſembles the theodolite in ſhape and uſe; and the ſemi-circle, for taking angles, &c. But of all the inſtruments for meaſuring, the plain table is certainly the beſt; not only becauſe it may be uſed as a theodolite or ſemi-circle, by turning uppermoſt that ſide of the frame which has the 360 degrees upon it; but becauſe it is, in its own proper uſe, by much the eaſieſt, ſafeſt, and moſt accurate for the purpoſe; for by planning every part immediately upon the ſpot, as ſoon as meaſured, there is not only ſaved a great deal of writing in the field-book, but every thing can alſo be planned more eaſily and accurately while it is in view, than it can be afterwards from a field-book, in which many little things muſt be either neglected or miſtaken; and beſides, the opportunities which the plain table afford of correcting your work, or proving if it be right, at every ſtation, are ſuch advantages as can never be balanced by any other method. But although the plain table be the moſt generally uſeful inſtrument, it is not *always* ſo; there being many caſes in which ſometimes one inſtrument is the propereſt, and ſometimes another; nor is that ſurveyor maſter of his buſineſs who cannot in any caſe diſtinguiſh which is the fitteſt inſtrument or method, and

and use it accordingly: nay, sometimes no instrument at all, but barely the chain itself, is the best method, particularly in regular open fields lying together; and even when you are using the plain table, it is often of advantage to measure such large open parts with the chain only, and from those measures lay them down upon the table.

The perambulator is used for measuring roads, and other great distances on level ground, and by the sides of rivers. It has a wheel of $8\frac{1}{4}$ feet, or half a pole, in circumference, upon which the machine turns; and the distance measured is pointed out by an index, which is moved round by clock work.

Levels, with telescopic or other sights, are used to find the level between place and place, or how much one place is higher or lower than another.

An offset-staff is a very useful and necessary instrument, for measuring the offsets and other short distances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten small arrows, or rods of iron or wood, are used to mark the end of every chain length, in measuring lines. And sometimes pickets, or staves with flags, are set up as marks or objects of direction.

Various scales are also used in protracting and measuring on the plan or paper; such as plane scales, line of chords, protractor, compasses, reducing scales, parallel and perpendicular rulers, &c. Of plane scales, there should be several sizes, as a chain in 1 inch, a chain in $\frac{3}{4}$ of an inch, a chain in $\frac{1}{2}$ an inch, &c. And of these, the best for use, are those that are laid on the edges of the ivory scale, to prick off distances by, without compasses.

THE FIELD BOOK.

In surveying with the plain table, a field-book is not used, as every thing is drawn on the table immediately

when it is meaſured. But in ſurveying with the theodolite, or any other inſtrument, ſome ſort of a field-book muſt be uſed, to write down in it a regiſter or account of all that is done and occurs relative to the ſurvey in hand.

This book every one contrives and rules as he thinks fitteſt for himſelf. The following is a ſpecimen of a form very generally uſed. It is ruled into 3 columns: the middle, or principal column, is for the ſtations, angles, bearings, diſtances meaſured, &c; and thoſe on the right and left are for the offsets on the right and left, which are ſet againſt their correſponding diſtances in the middle column; as alſo for ſuch remarks as may occur, and be proper to note in drawing the plan, &c.

Here ⊙ 1 is the firſt ſtation, where the angle or bearing is 105° 25′. On the left, at 73 links in the diſtance or principal line, is an offset of 92; and at 610 an offset of 24 to a croſs hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the firſt line, the 0 denotes its terminating in the hedge. And ſo on for the other ſtations.

Draw a line under the work, at the end of every ſtation line, to prevent confuſion.

Form

Form of the Field-book.

Offsets and Remarks on the left.	Stations, Bearings, and Distances.	Offsets and Remarks on the right.
	⊙ 1	
	105° 25′	
	00	25 corner
92	73	
	248	Brown's hedge
crofs a hedge 24	610	35
	954	00
	⊙ 2	
	53° 10′	
	00	00
houfe corner 51	25	21
	120	29 a tree
34	734	40 a ftile
	⊙ 3	
	67° 20′	
	61	35
a brook 30	248	
	639	16 a fpring
foot path 16	810	
crofs hedge 18	973	20 a pond

But in fmaller furveys and meafurements, a very good way of fetting down the work, is, to draw, by the eye on a piece of paper, a figure refembling that

which is to be meaſured; and ſo write the dimenſions, as they are found, againſt the correſponding parts of the figure. And this method may be practiſed to a conſiderable extent, even in the larger ſurveys.

CHAPTER II.

THE PRACTICE OF SURVEYING.

THIS part contains the ſeveral works proper to be done in the field, or the ways of meaſuring by all the inſtruments, and in all ſituations.

PROBLEM I.

To Meaſure a Line or Diſtance.

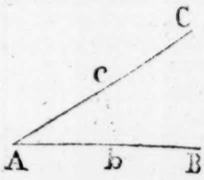
To meaſure a line on the ground with the chain: Having provided a chain, with 10 ſmall arrows, or rods, to ſtick one into the ground, as a mark, at the end of every chain; two perſons take hold of the chain, one at each end of it, and all the 10 arrows are taken by one of them, who is to go foremoſt, and is called the leader; the other being called the follower, for diſtinction's ſake.

A picket, or ſtation ſtaff, being ſet up in the direction of the line to be meaſured, if there do not appear ſome marks naturally in that direction; the follower ſtands at the beginning of the line, holding the ring at the end of the chain in his hand, while the leader drags forward the chain by the other end of it, till

till it is ſtretched ſtraight, and laid or held level, and the leader directed, by the follower waving his hand, to the right or left, till the follower ſee him exactly in a line with the mark or direction to be meaſured to; there both of them ſtretching the chain ſtraight, and ſtooping and holding it level, the leader having the head of one of his arrows in the ſame hand by which he holds the end of the chain, let him there ſtick one of them down with it while he holds the chain ſtretched. This done, he leaves the arrow in the ground, as a mark for the follower to come to, and advances another chain forward, being directed in his poſition by the follower ſtanding at the arrow, as before; as alſo by himſelf now, and at every ſucceeding chain's length, by moving himſelf from ſide to ſide, till he brings the follower and the back mark into a line. Having then ſtretched the chain, and ſtuck down an arrow, as before, the follower takes up his arrow, and they advance again in the ſame manner another chain-length. And thus they proceed till all the 10 arrows are employed, and are in the hands of the follower; and the leader, without an arrow, is arrived at the end of the 11th chain-length. The follower then ſends or brings the 10 arrows to the leader, who puts one of them down at the end of his chain, and advances with the chain as before. And thus the arrows are changed from the one to the other at every 10 chains length, till the whole line is finiſhed; when the number of changes of the arrows ſhews the number of tens, to which the follower adds the arrows he holds in his hand, and the number of links of another chain over to the mark or end of the line. So if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is ſet down in links thus 3645.

PROBLEM II.

To take Angles and Bearings.



Let B and C be two objects, or two pickets ſet up perpendicular; and let it be required to take their bearings, or the angle formed between them at any ſtation A.

1. *With the Plain Table.*

The table being covered with a paper, and fixed on its ſtand; plant it at the ſtation A, and fix a fine pin, or a point of the compaſſes, in a proper part of the paper, to repreſent the point A: Cloſe by the ſide of this pin lay the fiducial edge of the index, and turn it about, ſtill touching the pin, till one object B can be ſeen through the ſights: then by the fiducial edge of the index draw a line. In the very ſame manner draw another line in the direction of the other object C. And it is done.

2. *With the Theodolite, &c.*

Direct the fixed ſights along one of the lines, as AB, by turning the inſtrument about till you ſee the mark B through theſe ſights; and there ſcrew the inſtrument faſt. Then turn the moveable index about till, through its ſights, you ſee the other mark C. Then the degrees cut by the index, upon the graduated limb or ring of the inſtrument, ſhews the quantity of the angle.

3. *With the Magnetic Needle and Compaſs.*

Turn the inſtrument, or compaſs, ſo, that the north end of the needle point to the flower-de-luce. Then direct the ſights to one mark, as B, and note the degrees cut

cut by the needle. Then direct the ſights to the other mark c, and note again the degrees cut by the needle. Then their ſum or difference, as the caſe is, will give the quantity of the angle BAC.

4. *By Meaſurement with the Chain, &c.*

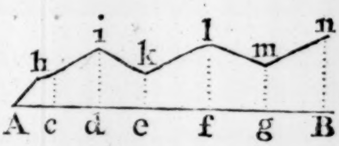
Meaſure one chain length, or any other length, along both directions, as to b and c. Then meaſure the diſtance b c, and it is done.—This is eaſily transferred to paper, by making a triangle A b c with theſe three lengths, and then meaſuring the angle A as in Practical Geometry, prob. XI.

PROBLEM III.

To Meaſure the Offsets.

A h i k l m n being a crooked hedge, or river, &c. From A meaſure in a ſtraight direction along the ſide of it to B. And in meaſuring along this line AB obſerve when you are directly oppoſite any bends or corners of the hedge, as at c d, e, &c; and from thence meaſure the perpendicular offsets c, h, d i, &c, with the offset-ſtaff, if they are not very large, otherwiſe with the chain itſelf; and the work is done. And the regiſter, or field-book, may be as follows:

Offs. left.		Baſe line AB.	
	0	⊙	A
ch	62	45	AC
di	84	220	Ad
ek	70	340	Ae
fl	98	510	Af
gm	57	634	Ag
Bn	91	785	AB



Note. When the offsets are not very large, their places

places c, d, e, &c, on the baſe line, can be very well determined by the eye, eſpecially when aſſiſted by laying down the offset-ſtaff in a croſs or perpendicular direction. But when theſe perpendiculars are very large, find their poſitions by the croſs, or by the inſtrument which you happen to be uſing, in this manner: As you meaſure along AB, when you come about c, where you judge a perpendicular will ſtand, plant your inſtrument in the line, and turn the index till the marks A and B can be ſeen through both the ſights, looking both backward and forward; then look along the croſs ſights, or the croſs line on the index; and if it point directly to the corner or bend h, the place of c is right. Otherwiſe, move the inſtrument backward or forward on the line AB, till the croſs line points ſtraight to h. This being found, ſet down the diſtance meaſured from A to c; then meaſure the offset c h, and ſet it down oppoſite the former, and on the left hand ſide.

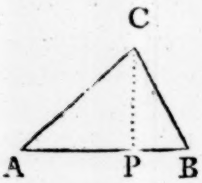
Then proceed forward in the line AB, till you arrive oppoſite another corner, and determine the place d of the perpendicular as before. And ſo on throughout the whole length.

PROBLEM IV.

To Survey a Triangular Field ABC.

1. By the Chain.

AP	794
AB	1321
PC	826



Having ſet up marks at the corners, which is to be done in all caſes where there are not marks naturally; meaſure

measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting down the distance AP. Then complete the distance AB by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area from them is easily found. Or having the place P of the perpendicular, the triangle is easily constructed.

Or, measure all the three sides with the chain, and note them down. From which the content is easily found, or the figure constructed.

2. *By taking one or more of the Angles.*

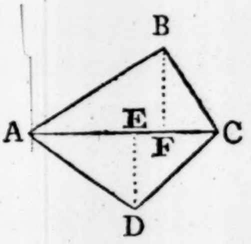
Measure two sides AB, AC, and the angle A between them. Or measure one side AB, and the two adjacent angles A and B. From either of these ways the figure is easily planned; then by measuring the perpendicular CP on the plan, and multiplying it by half AB, you have the content.

PROBLEM V.

To measure a Four-sided Field.

1. *By the Chain.*

AE	214	210	DE
AF	362	306	BF
AC	592		

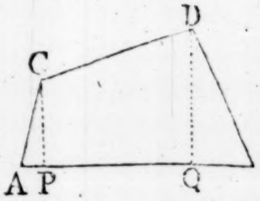


Measure along either of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last problem; or else the sides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

Other-

Otherwise by the Chain.

AP	110	352	PC
AQ	745	595	QD
AB	1110		



Measure on the longest side, the distances AP, AQ, AB; and the perpendiculars PC, QD.

2. *By taking one or more of the Angles.*

Measure the diagonal AC (see the first fig. above), and the angles CAB, CAD, ACB, ACD.—Or measure the four sides, and any one of the angles as BAD.

Thus		Or thus	
AC	591	AB	486
CAB	37° 20′	BC	394
CAD	41 15	CD	410
ACB	72 25	DA	462
ACD	54 40	BAD	78° 35′

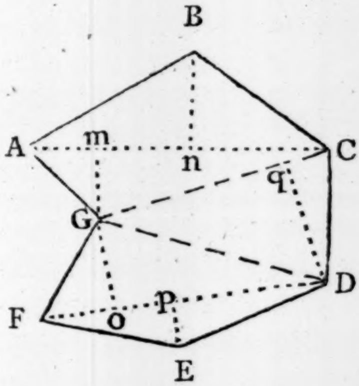
PROBLEM VI.

To Survey any Field by the Chain only.

Having set up marks at the corners, where necessary, of the proposed field ABCDEFG. Walk over the ground, and consider how it can best be divided into triangles and trapeziums; and measure them separately as in the last two problems. And in this way it will be proper to divide it into as few separate triangles, and as many trapeziums as may be, by drawing diagonals from corner to corner; and so as that all the perpendiculars may fall within the figure. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the first, beginning at A, measure the diagonal AC, and

and the two perpendiculars GM, BN. Then the base GC, and the perpendicular Dq. Lastly the diagonal DF, and the two perpendiculars pE, OG. All which measures write against the corresponding parts of a rough figure drawn to resemble the figure to be surveyed, or set them down in any other form you choose.

AM 135	130 mG
AN 410	180 nB
AC 550	
cq 152	230 qD
CG 440	
FO 206	120 OG
FP 288	80 pE
FD 520	



Or thus.

Measure all the sides AB, BC, CD, DE, EF, FG, and GA; and the diagonals AC, CG, GD, DF.

Otherwise.

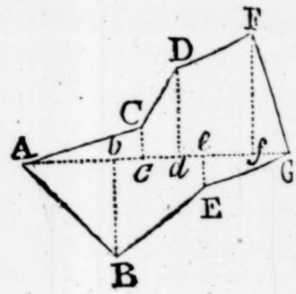
Many pieces of land may be very well surveyed, by measuring any base line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by those means divided into several triangles and trapezoids, all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed if the base line fall within it; if not, the sum of the parts which are without being taken from the sum of the whole which are both within and without, will leave the area of the figure proposed.

In pieces that are not very large, it will be sufficiently exact to find the points, in the base line, where the

the ſeveral perpendiculars will fall, by means of the *croſs*, and from thence meaſuring to the corners for the lengths of the perpendiculars.—And it will be moſt convenient to draw the line ſo as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and meaſuring along the line AG the diſtances and perpendiculars, on the right and left, are as below.

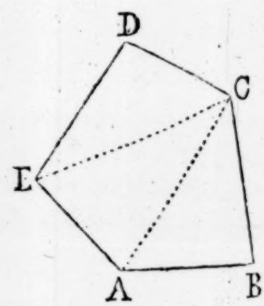
Ab	315	350	bB
Ac	440	70	cC
Ad	585	320	dD
Ae	610	50	eE
Af	990	470	fF
AG	1020	0	



PROBLEM VII.

To Survey any Field with the Plain Table.

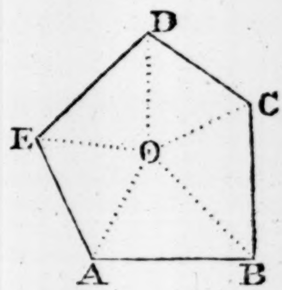
1. *From one Station.*



Plant the table at any angle, as C, from whence all the other angles, or marks ſet up, can be ſeen; and turn the table about till the needle point to the flower-de-luce; and there ſcrew it faſt. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the ſights you ſee the mark D; and by the edge of the index draw a dry or obſcure line: then meaſure the diſtance CD, and lay that diſtance down on the line CD. Then turn the index about the point C, till the mark

mark E be ſeen through the ſights, by which draw a line, and meaſure the diſtance to E, laying it on the line from C to E. In like manner determine the poſitions of CA and CB, by turning the ſights ſucceſſively to A and B; and lay the lengths of thoſe lines down. Then connect the points with the boundaries of the field, by drawing the black lines CD, DE, EA, AB, BC.

2. *From a Station within the Field.*



When all the other parts cannot be ſeen from one angle, chooſe ſome place O within; or even without, if more convenient, from whence the other parts can be ſeen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index ſucceſſively to O, turning it round with the ſights to each angle A, B, C, D, E, drawing dry lines to them by the edge of the index, then meaſuring the diſtances OA, OB, &c, and laying them down upon thoſe lines. Laſtly draw the boundaries AB, BC, CD, DE, EA.

3. *By going round the Figure.*

When the figure is a wood or water, or from ſome other obſtruction you cannot meaſure lines acroſs it; begin at any point A, and meaſure round it, either within or without the figure, and draw the directions of all the ſides thus: Plant the table at A, turn it with the needle to the north or flower-de-luce, fix it and mark the point A. Apply the index to A, turning it till you can ſee the point E, there draw a line; and then the point B, and there draw a line: then meaſure theſe lines, and lay them down from A to E and B.

4 Next

Next move the table to B, lay the index along the line AB, and turn the table about till you can ſee the mark A, and ſcrew faſt the table; in which poſition alſo the needle will again point to the flower-de-luce, as it will do indeed at every ſtation when the table is in the right poſition. Here turn the index about B till through the ſights you ſee the mark C; there draw a line, meaſure BC, and lay the diſtance upon that line after you have ſet down the table at C. Turn it then again into its proper poſition, and in like manner find the next line CD. And ſo on quite round by E to A again. Then the proof of the work will be the joining at A: for if the work is all right, the laſt direction EA on the ground, will paſs exactly through the point A on the paper; and the meaſured diſtance will alſo reach exactly to A. If theſe do not coincide, or nearly ſo, ſome error has been committed, and the work muſt be examined over again.

PROBLEM VIII.

To Survey a Field with the Theodolite, &c.

1. *From one Point or Station.*

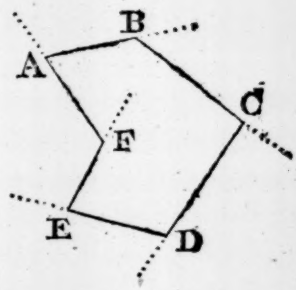
When all the angles can be ſeen from one point, as the angle E (firſt fig. to laſt prob.) place the inſtrument at E, and turn it about till, through the fixed ſights, you ſee the mark B, and there fix it. Then turn the moveable index about till the mark A is ſeen through the ſights, and note the degrees cut on the inſtrument. Next turn the index ſucceſſively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD. Laſtly, meaſure the lines CB, CA, CE, CD; and enter the meaſures in a field-book, or rather againſt the correſponding parts of a rough figure drawn by gueſs to reſemble the field.

2. *From*

2. *From a Point within or without.*

Plan the instrument at o, (last fig.) and turn it about till the fixed sights point to any object, as A; and there screw it fast. Then turn the moveable index round till the sights point successively to the other points E, D, C, B, noting the degrees cut off at each of them; which gives all the angles round the point o. Lastly, measure the distances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

3. *By going round the Field.*



By measuring round, either within or without the field, proceed thus. Having set up marks at B, C, &c, near the corners as usual, plant the instrument at any point A, and turn it till the fixed index be in the direction AB, and there screw it fast: then turn the moveable index to the direction AF; and the degrees cut off will be the angle A. Measure the line AB, and plant the instrument at B, and there in the same manner observe the angle A. Then measure BC, and observe the angle C. Then measure the distance CD, and take the angle D. Then measure DE, and take the angle E. Then measure EF, and take the angle F. And lastly measure the distance FA.

To prove the work; add all the inward angles A, B, C, &c, together, and when the work is right, their sum will be equal to twice as many right angles as the figure has sides, wanting 4 right angles. And when there is an angle, as F, that bends inwards, and you measure the external angle, which is less than two right angles, subtract it from 4 right angles, or 360

degrees, to give the internal angle greater than a ſemicircle or 180 degrees.

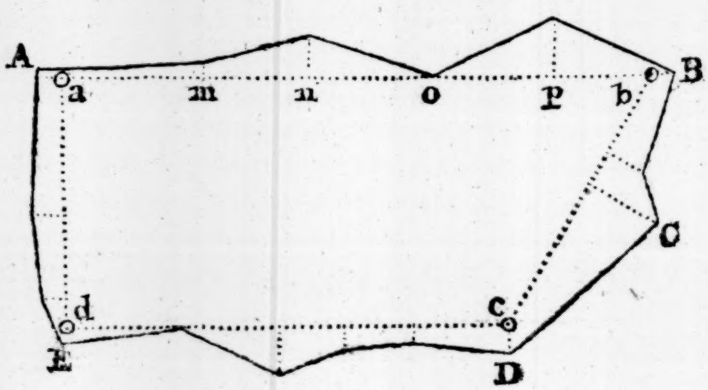
Otherwiſe.

Inſtead of obſerving the internal angles, you may take the external angles, formed without the figure by producing the ſides further out. And in this caſe, when the work is right, their ſum altogether will be equal to 360 degrees, But when one of them, as F, runs inwards, ſubtract it from the ſum of the reſt, to leave 360 degrees.

PROBLEM IX.

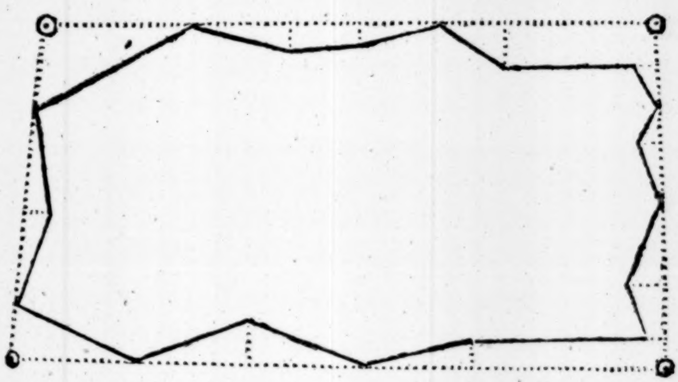
To Survey a Field with Crooked Hedges, &c.

With any of the inſtruments meaſure the lengths and poſitions of imaginary lines running as near the ſides of the field as you can; and in going along them meaſure the offsets in the manner before taught; and you will have the plan on the paper in uſing the plain table, drawing the crooked hedges through the ends of the offsets; but in ſurveying with the theodolite, or other inſtrument, ſet down the meaſures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.



So, in surveying the piece ABCDE, set up marks a, b, c, d, dividing it into as few sides as may be. Then begin at any station a, and measure the lines ab, bc, cd, da, and take their positions, or the angles a, b, c, d; and in going along the lines measure all the offsets, as at m, n, o, p, &c, along every station line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, &c; then measure without, as in the figure here below.



PROBLEM X.

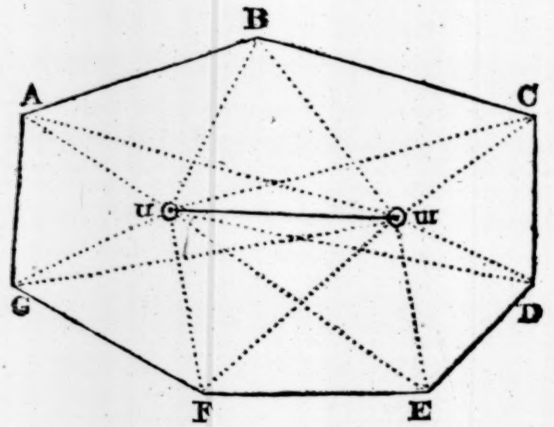
To Survey a Field or any other Thing, by Two Stations.

This is performed by choosing two stations, from whence all the marks and objects can be seen, then measuring the distance between the stations, and at each station taking the angles formed by every object, from the station line or distance.

The two stations may be taken either within the bounds, or in one of the sides, or in the direction of two of the objects, or quite at a distance, and without the bounds of the objects, or part to be surveyed.

In this manner, not only grounds may be surveyed, without even entering them, but a map may be taken of the principal parts of a country, or the chief places

of a town, or any part of a river or coaſt ſurveyed, or any other inacceſſible objects; by taking two ſtations, on two towers, or two hills, or ſuch like.



When the plain table is uſed; plant it at one ſtation m, draw a line m n on it, along which lay the edge of the index, and turn the table about till the ſights point directly to the other ſtation; and there ſcrew it faſt. Then turn the ſights round m ſucceſſively to all the objects ABC, &c, drawing a dry line by the edge of the index at each, as mA, mB, mc, &c. Then meaſure the diſtance to the other ſtation, there plant the table, and lay that diſtance down on the ſtation line from m to n. Next lay the index by the line nm, and turn the table about till the ſights point to the other ſtation m, and there ſcrew it faſt. Then direct the ſights ſucceſſively to all the objects A, B, C, &c, as before, drawing lines each time, as nA, nB, nc, &c; and their interſection with the former lines will give the places of all the objects, or corners, A, B, C, &c.

When the theodolite, or any other inſtrument for taking angles, is uſed; proceed in the ſame way, meaſuring the ſtation diſtance mn, planting the inſtrument firſt at one ſtation, and then at another; then

then placing the fixed sights in the direction mn, and directing the moveable sights to every object, noting the degrees cut off at each time. Then, these observations being planned, the intersections of the lines will give the objects as before.

When all the objects, to be surveyed, cannot be seen from two stations; then three stations may be used, or four, or as many as is necessary; measuring always the distance from one station to another; placing the instrument in the same position at every station, by means described before; and from each station observing or setting every object that can be seen from it, by taking its direction or angular position, till every object be determined by the intersection of two or more lines of direction, the more the better. And thus may very extensive surveys be taken, as of large commons, rivers, coasts, countries, hilly grounds, and such like.

PROBLEM XI.

To Survey a Large Estate.

If the estate be very large, and contain a great number of fields, it cannot well be done by surveying all the fields singly, and then putting them together; nor can it be done by taking all the angles and boundaries that inclose it. For in these cases, any small errors will be so multiplied, as to render it very much distorted.

1. Walk over the estate two or three times, in order to get a perfect idea of it, and till you can carry the map of it tolerably in your head. And to help your memory, draw an eye draught of it on paper, or at least, of the principal parts of it, to guide you.

2. Choose two or more eminent places in the estate, for your stations, from whence you can see all the principal parts of it: and let these stations be as far

diſtant from one another as poſſible; as the fewer ſtations you have to command the whole, the more exact your work will be: and they will be fitter for your purpoſe, if theſe ſtation lines be in or near the boundaries of the ground, and eſpecially if two lines or more proceed from one ſtation.

3. Take what angles, between the ſtations, you think neceſſary, and meaſure the diſtances from ſtation to ſtation, always in a right line: theſe things muſt be done, till you get as many angles and lines as are ſufficient for determining all your points of ſtation. And in meaſuring any of theſe ſtation diſtances, mark accurately where theſe lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c, and where any remarkable object is placed, by meaſuring its diſtance from the ſtation line, and where a perpendicular from it cuts that line; and always mind, in any of theſe obſervations, that you be in a right line, which you will know by taking backſight and foreſight, along your ſtation line. And thus as you go along any main ſtation line, take offsets to the ends of all hedges, and to any pond, houſe, mill, bridge, &c, omitting nothing that is remarkable. And all theſe things muſt be noted down; for theſe are your data, by which the places of ſuch objects are to be determined upon your plan. And be ſure to ſet marks up at the interſections of all hedges with the ſtation line, that you may know where to meaſure from, when you come to ſurvey theſe particular fields, which muſt immediately be done, as ſoon as you have meaſured that ſtation line, whilſt they are freſh in memory. By this means all your ſtation lines are to be meaſured, and the ſituation of all places adjoining to them determined, which is the firſt grand point to be obtained. It will be proper for you to lay down your work upon paper every night, when you go home, that you may ſee how you go on.

4. As

4. As to the inner parts of the estate, they must be determined in like manner, by new station lines: for, after the main stations are determined, and every thing adjoining to them, then the estate must be subdivided into two or three parts by new station lines; taking inner stations at proper places, where you can have the best view. Measure these station lines as you did the first, and all their intersections with hedges, and all offsets to such objects as appear. Then you may proceed to survey the adjoining fields, by taking the angles that the sides make with the station line, at the intersections, and measuring the distances to each corner, from the intersections. For every station line will be a basis to all the future operations; the situation of all parts being entirely dependant upon them; and therefore they should be taken as long as possible; and it is best for them to run along some of the hedges or boundaries of one or more fields, or to pass through some of their angles. All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide till at last you come to single fields; repeating the same work for the inner stations, as for the outer ones, till all be done: and close the work as often as you can, and in as few lines as possible. And that you may choose stations the most conveniently, so as to cause the least labour, let the station lines run as far as you can along some hedges, and through as many corners of the fields, and other remarkable points, as you can. And take notice how one field lies by another; that you may not misplace them in the draught.

5. An estate may be so situated, that the whole cannot be surveyed together; because one part of the estate cannot be seen from another. In this case, you may divide it into three or four parts, and survey the parts separately, as if they were lands belonging to different persons; and at last join them together.

6. As it is neceſſary to protract or lay down your work as you proceed in it, you muſt have a ſcale of a due length to do it by. To get ſuch a ſcale, you muſt meaſure the whole length of the eſtate in chains; then you muſt conſider how many inches long the map is to be; and from theſe you will know how many chains you muſt have in an inch; then make your ſcale, or chooſe one already made, accordingly.

7. The trees in every hedge row muſt be placed in their proper ſituation, which is ſoon done by the plain table; but may be done by the eye without an inſtrument; and being thus taken by gueſs, in a rough draught, they will be exact enough, being only to look at; except it be ſuch as are at any remarkable places, as at the ends of hedges, at ſtiles, gates, &c, and theſe muſt be meaſured. But all this need not be done till the draught is finiſhed. And obſerve in all the hedges, what ſide the gutter or ditch is on, and to whom the fences belong.

8. When you have long ſtations, you ought to have a good inſtrument to take angles with; and the plain table may very properly be made uſe of, to take the ſeveral ſmall internal parts, and ſuch as cannot be taken from the main ſtations, as it is a very quick and ready inſtrument.

PROBLEM XII.

To Survey a County, or Large Tract of Land.

1. Chooſe two, three, or four eminent places for ſtations; ſuch as the tops of high hills or mountains, towers, or church ſteeples, which may be ſeen from one another; and from which moſt of the towns, and other places of note, may alſo be ſeen. And let them be as far diſtant from one another as poſſible. Upon theſe places raiſe beacons, or long poles, with flags of different

different colours flying at them; ſo as to be viſible from all the other ſtations.

2. At all the places, which you would ſet down in the map, plant long poles with flags at them of ſeveral colours, to diſtinguiſh the places from one another; fixing them upon the tops of church ſteeples, or the tops of houſes, or in the centres of leſſer towns.

But you need not have theſe marks at many places at once, as ſuppoſe half a ſcore at a time. For when the angles have been taken, at the two ſtations, to all theſe places, the marks may be moved to new ones; and ſo ſucceſſively to all the places you want. Theſe marks then being ſet up at a convenient number of places, and ſuch as may be ſeen from both ſtations; go to one of theſe ſtations, and with an inſtrument to take angles, ſtanding at that ſtation, take all the angles between the other ſtation, and each of theſe marks, obſerving which is blue, which red, &c, and which hand they lie on; and ſet all down with their colours. Then go to the other ſtation, and take all the angles between the firſt ſtation, and each of the former marks, and ſet them down with the others, each againſt his fellow with the ſame colour. You may, if you can, alſo take the angles at ſome third ſtation, which may ſerve to prove the work, if the three lines interſect in that point, where any mark ſtands. The marks muſt ſtand till the obſervations are finiſhed at both ſtations; and then they muſt be taken down, and ſet up at freſh places. And the ſame operations muſt be performed, at both ſtations, for theſe freſh places; and the like for others. Your inſtrument for taking angles muſt be an exceeding good one, made on purpoſe with teleſcopic ſights; and of three, four, or five feet radius. A circumferentor is reckoned a good inſtrument for this purpoſe.

3. And though it is not abſolutely neceſſary to meaſure any diſtance, becauſe any ſtationary line being

laid down from any ſcale, all the other lines will be proportional ~~to it~~; yet it is better to meaſure ſome of the lines to aſcertain the diſtances of places in miles; and to know how many geometrical miles there are in any length; and from thence to make a ſcale to meaſure any diſtance in miles. In meaſuring any diſtance, it will not be exact enough to go along the high roads; by reaſon of their turnings and windings, and hardly ever lying in a right line between the ſtations, which muſt cauſe infinite reductions, and create endleſs trouble to make it a right line; for which reaſon it can never be exact. But a better way is to meaſure in a right line with a chain, between ſtation and ſtation, over hills and dales or level fields, and all obſtacles. Only in caſe of water, woods, towns, rocks, banks, &c, where onecannot paſs, ſuch parts of the line muſt be meaſured by the methods of inacceſſible diſtances; and beſides, allowing for aſcents and deſcents, when we meet with them. And a good compaſs that ſhews the bearing of the two ſtations, will always direct you to go ſtraight, when you do not ſee the two ſtations; and in your progreſs, if you can go ſtraight, you may take offsets to any remarkable places, likewiſe note the interſection of your ſtationary line with all roads, rivers, &c.

4. And from all your ſtations, and in your whole progreſs, be very particular in obſerving ſea coaſts, river mouths, towns, caſtles, houſes, churches, windmills, watermills, trees, rocks, ſands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, ſluices, floodgates, locks, &c; and in general all things that are remarkable.

5. After you have done with your firſt and main ſtation lines, which command the whole county; you muſt then take inner ſtations, at ſome places already determined; which will divide the whole into ſeveral partitions: and from theſe ſtations you muſt determine

the places of as many of the remaining towns as you can. And if any remain in that part, you muſt take more ſtations, at ſome places already determined; from which you may determine the reſt. And thus we muſt go through all the parts of the county, taking ſtation after ſtation, till we have determined all we want. And in general the ſtation diſtances muſt always paſs through ſuch remarkable points as have been determined before, by the former ſtations.

6. Laſtly, the poſition of the ſtation line you meaſure, or the point of the compaſs it lies on, muſt be determined by aſtronomical obſervation. Hang up a thread and plummet in the ſun, over ſome part of the ſtation line, and obſerve when the ſhadow runs along that line, and at that moment take the ſun's altitude; then having his declination, and the latitude, the azimuth will be found by ſpherical trigonometry. And the azimuth is the angle the ſtation line makes with the meridian; and therefore a meridian may eaſily be drawn through the map. Or a meridian may be drawn through it by hanging up two threads in a line with the pole ſtar, when he is juſt north, which may be known from aſtronomical tables. Or thus; obſerve the ſtar Alioth, or that in the rump of the great bear, being that next the ſquare; or elſe Caſſiopeia's hip; I ſay, obſerve by a line and plummet when either of theſe ſtars and the pole ſtar come into a perpendicular; and at that time they are due north. Therefore two perpendicular lines being fixed at that moment, towards theſe two ſtars, will give the poſition of the meridian.

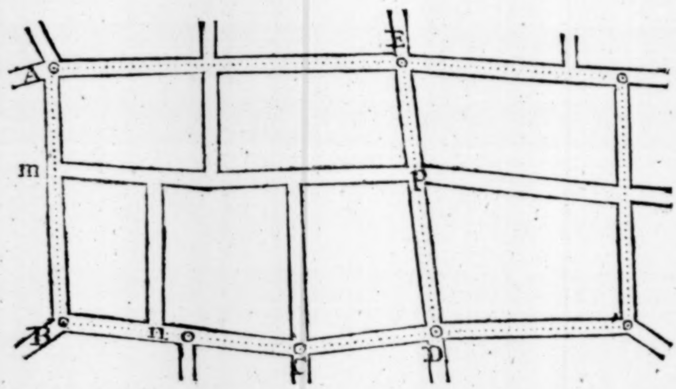
PROBLEM XIII.

To Survey a Town or City.

This may be done with any of the inſtruments for taking angles, but beſt of all with the plain table, where

where every minute part is drawn while in ſight. It is proper alſo to have a chain of 50 feet long, divided into 50 links, and an offset-ſtaff of 10 feet long.

Begin at the meeting of two or more of the principal ſtreets, through which you can have the longeſt proſpects, to get the longeſt ſtation lines. There having fixed the inſtrument, draw lines of direction along thoſe ſtreets, uſing two men as marks, or poles ſet in wooden pedeſtals, or perhaps ſome remarkable places in the houſes at the further ends, as windows, doors, corners, &c. Meaſure theſe lines with the chain, taking offsets with the ſtaff, at all corners of ſtreets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houſes, &c. Then remove the inſtrument to another ſtation along one of theſe lines; and there repeat the ſame proceſs as before. And ſo on till the whole is finiſhed.



Thus, fix the inſtrument at A, and draw lines in the direction of all the ſtreets meeting there; and meaſure AB, noting the ſtreet on the left at m. At the ſecond ſtation B, draw the directions of the ſtreets meeting there; meaſure from B to C, noting the places of the ſtreets at n and o as you paſs by them. At the 3d ſtation C take the direction of all the ſtreets meeting

meeting there, and measure CD. At D do the same, and measure DE, noting the place of the cross streets at p. And in this manner go through all the principal streets. This done, proceed to the smaller and intermediate streets; and lastly to the lanes, alleys, courts, yards, and every part that it may be thought proper to represent.

CHAPTER III.

Of Planning, Casting-up, and Dividing.

PROBLEM I.

To Lay down the Plan of any Survey.

IF the survey was taken with a plain table, you have a rough plan of it already on the paper which covered the table. But if the survey was with any other instrument, a plan of it is to be drawn from the measures that were taken in the survey, and first of all a rough plan upon paper.

To do this, you must have a set of proper instruments, for laying down both lines and angles, &c; as scales of various sizes, the more of them, and the more accurate, the better; scales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal scales are best for the lines, because they extend to three figures, or chains and links, which are hundredth parts of chains. But in using the diagonal scale, a pair of compasses must be employed to take off the lengths of the principal lines very accurately. But a scale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of those offsets upon the station line; which is done at only one application of the edge of the

the ſcale to that line, and then pricking off all at once the diſtances along it. Angles are to be laid down either with a good ſcale of chords, which is perhaps the moſt accurate way; or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

Very particular directions for laying down all ſorts of figures cannot be neceſſary in this place, to any perſon who has learned practical geometry, and the conſtruction of figures, and the uſe of his inſtruments. It may therefore be ſufficient to obſerve, that all lines and angles muſt be laid down on the plan in the ſame order in which they were meaſured in the field, and in which they are written in the field-book; laying down firſt the angles for the poſition of lines, then the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themſelves, all with dry or obſcure lines; then a black line drawn through the extremities of all the offsets, will be the hedge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or cloſe properly, proceed next to the ſmaller objects, till you have entered every thing that ought to appear in the plan, as houſes, brooks, trees, hills, gates, ſtiles, roads, lanes, mills, bridges, woodlands, &c, &c.

The north ſide of a map or plan is commonly placed uppermoſt, and a meridian ſomewhere drawn, with the compaſs or flower-de-luce pointing north. Alſo, in a vacant part, a ſcale of equal parts or chains muſt be drawn, and the title of the map in conſpicuous characters, and embelliſhed with a compartment. All hills muſt be ſhadowed, to diſtinguiſh them in the map. Colour the hedges with different colours; repreſent hilly grounds by broken hills and valleys; draw ſingle dotted lines for foot-paths, and double

double ones for horſe or carriage roads. Write the name of each field and remarkable place within it, and, if you chooſe, its content in acres, roods, and perches.

In a very large eſtate, or a county, draw vertical and horizontal lines through the map, denoting the ſpaces between them by letters, placed at the top, and bottom, and ſides, for readily finding any field or other object, mentioned in a table.

In mapping counties, and eſtates that have uneven grounds of hills and valleys, reduce all oblique lines, meaſured up hill and down hill, to horizontal ſtraight lines, if that was not done during the ſurvey, before they were entered in the field-book, by making a proper allowance to ſhorten them. For which purpoſe there is commonly a ſmall table engraven on ſome of the inſtruments for ſurveying.

PROBLEM. II.

To Caſt up the Contents of Fields.

1. Compute the contents of the figures, whether triangles, or trapeziums, &c, by the proper rules for the ſeveral figures laid down in meaſuring; multiplying the lengths by the breadths, both in links; the product is acres after you have cut off five figures on the right, for decimals; then bring theſe decimals to roods and perches, by multiplying firſt by 4, and then by 40. An example of which is given in the deſcription of the chain, page 508.

2. In ſmall and ſeparate pieces, it is uſual to caſt up their contents from the meaſures of the lines taken in ſurveying them, without making a correct plan of them.

Thus,

Thus, in the triangle in prob. iv, page 522, where we had AP = 794, and AB = 1321

	1321
PC =	826
	7926
	2642
	10568
2)	10·91146
	5·45573
	4
	1·82292
	40
	32·91680

ac r p
Anſ. 32 1 33 nearly

Or the firſt example to prob. v, page 523, thus:

AE	214	210	ED
AF	362	306	FB
AC	592		
		516	ſum of perp.
		592	AC
		1032	
		4644	
		2580	
		3·05472	
		4	
		·21888	
		40	
		8·75520	

ac r p
Anſ. 3 0 8

Or

Or the 2d example to the same prob. v, thus:

AP	110	352	PC
AQ	745	595	QD
AB	1110		

PC 352	PC 352	QD 595
AP 110	QD 595	QB 365
2APC 38720	sum 947	2975
	PQ 635	3570
	4735	1785
	2841	217175=2QDB
	5682	601345=2PCDQ
	2PCDQ 601345	38720=2APC
		2) 8·57240=dou. the whole
		4·2862
		4
		1·1448 Ans. ac 4 r 1 p 5
		40
		5·7920

3. In pieces bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most accurately measured separately as small trapezoids. Thus, for the example to prob. III, p. 521, where

AC	45	62	ch
Ad	220	84	di
Ae	340	70	ek
Af	510	98	fl
Ag	634	57	gm
AB	785	91	BN

Then

Then

AC 45	ch 62	di 84	ek 70	fl 98	gm 57
ch 62	di 84	ek 70	fl 98	gm 57	BN 91
90	146	154	168	145	148
270	cd 175	de 120	ef 170	fg 124	gB 151
2790	730	18480	11760	580	148
	1022		168	290	740
	146			145	148
	25550		28560	17980	22348

```
      2790
     25550
     18480
     28560
     17980
     22348
   -------
2) 1·15708
    ·57854   Content   ac  r  p
         4              0  2  12
   -------
   2·31416
        40
   -------
  12·56640
```

4. Sometimes such pieces as that above, are computed by finding a mean breadth, by dividing the sum of the offsets by the number of them, accounting that for one of them where the boundary meets the station line, as at A; then multiply the length AB by that mean breadth.

Thus:

Thus :

00	785	AB
62	66	mean breadth
84	———	
70	4710	ac r p
98	4710	Content 0 2 2 by this method, which is 10 perches too little.
57	———	
91	·51810	
———	4	
7) 462	———	
66	2·07240	And this method is always erroneous, except when the offsets stand at equal distances from one another.
———	40	
	———	
	2·89600	

5. But in larger pieces, and whole estates, consisting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents quite independent of the measures of the lines and angles that were taken in surveying. For then new lines are drawn in the fields in the plan, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale from which it was drawn, and so multiplied together for the contents. In this way the work is very expeditiously done, and sufficiently correct; for such dimensions are taken, as afford the most easy method of calculation; and, among a number of parts, thus taken and applied to a scale, it is likely that some of the parts will be taken a small matter too little, and others too great; so that they will, upon the whole, in all probability, very nearly balance one another. After all the fields, and particular parts, are thus computed separately, and added all together into one sum, calculate the whole estate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these

alſo together. Then if this ſum be equal to the former, or nearly ſo, the work is right; but if the ſums have any conſiderable difference, it is wrong, and they muſt be examined, and recomputed, till they nearly agree.

A ſpecimen of dividing into one triangle, or one trapezium, which will do for moſt ſingle fields, may be ſeen in the examples to the laſt problem; and a ſpecimen of dividing a large tract into ſeveral ſuch trapeziums and triangles, in prob. VI of chapter II of Surveying, page 524, where a piece is ſo divided, and its dimenſions taken and ſet down; and again at prob. VI of Menſuration of Surfaces, where the contents of the ſame piece are computed.

6. But the chief ſecret in caſting up, conſiſts in finding the contents of pieces bounded by curved, or very irregular lines, or in reducing ſuch crooked ſides of fields or boundaries to ſtraight lines, that ſhall incloſe the ſame or equal area with thoſe crooked ſides, and ſo obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked ſides to ſtraight ones, is very eaſily and accurately performed thus: Apply the ſtraight edge of a thin, clear piece of lanthorn-horn to the crooked line, which is to be reduced, in ſuch a manner, that the ſmall parts cut off from the crooked figure by it, may be equal to thoſe which are taken in: which equality of the parts included and excluded, you will preſently be able to judge of very nicely by a little practice: then with a pencil draw a line by the ſtraight edge of the horn. Do the ſame by the other ſides of the field or figure. So ſhall you have a ſtraight ſided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the curved figure propoſed.

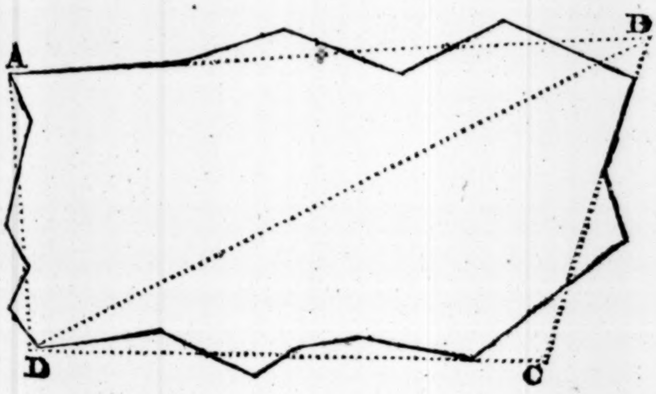
Or, inſtead of the ſtraight edge of the horn, a horſe-

hair

hair may be applied acroſs the crooked ſides in the ſame manner; and the eaſieſt way of uſing the hair, is to ſtring a ſmall ſlender bow with it, either of wire, or cane, or whale-bone, or ſuch like ſlender ſpringy matter; for, the bow keeping it always ſtretched, it can be eaſily and neatly applied with one hand, while the other is at liberty to make two marks by the ſide of it, to draw the ſtraight line by.

EXAMPLE.

Thus, let it be required to find the contents of the ſame figure as in prob. IX of the laſt chapter, page 530, to a ſcale of 4 chains to an inch.



Draw the four dotted ſtraight lines AB, BC, CD, DA, cutting off equal quantities on both ſides of them, which they do as near as the eye can judge: ſo is the crooked figure reduced to an equivalent right-lined one of four ſides ABCD. Then draw the diagonal BD, which by applying a proper ſcale to it, meaſures 1256. Alſo the perpendicular, or neareſt diſtance, from A to this diagonal, meaſures 456; and the diſtance of C from it, is 428.

Then

```
Then     456
         428
        ----
         884
        1256
        ----
        5024
       10048
      10048
      -------
   2) 11·10304
       5·55152
             4
      -------
       2·20608
            40
      -------                 ac  ro  p
       8·24320   Content       5   2  8
      -------
```

And thus the content of the trapezium, and consequently of the irregular figure, to which it is equal, is easily found to be 5 acres, 2 roods, 8 perches.

PROBLEM III.

To Transfer a Plan to another Paper, &c.

After the rough plan is completed, and a fair one is wanted; this may be done, either on paper or vellum, by any of the following methods.

FIRST METHOD.

Lay the rough plan upon the clean paper, and keep them always pressed flat and close together, by weights laid upon them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them asunder, and connect the pricked

pricked points on the clean paper, with lines; and it is done. This method is only to be practised in plans of such figures as are small and tolerably regular, or bounded by right lines.

SECOND METHOD.

Rub the back of the rough plan over with black lead powder; and lay the said black part upon the clean paper, upon which the plan is to be copied, and in the proper position. Then with the blunt point of some hard substance, as brass, or such like, trace over the lines of the whole plan; pressing the tracer so much as that the black lead under the lines may be transferred to the clean paper; after which take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink, &c.—Or, instead of blacking the rough plan, you may keep constantly a blacked paper to lay between the plans.

THIRD METHOD.

Another way of copying plans, is by means of squares. This is performed by dividing both ends and sides of the plan, which is to be copied, into any convenient number of equal parts, and connecting the corresponding points of division with lines; which will divide the plan into a number of small squares. Then divide the paper, upon which the plan is to be copied, into the same number of squares, each equal to the former when the plan is to be copied of the same size, but greater or less than the others, in the proportion in which the plan is to be increased or diminished, when of a different size. Lastly, copy into the clean squares, the parts contained in the corresponding squares, of the old plan; and you will have the copy either of the same size, or greater or less in any proportion.

FOURTH METHOD.

A fourth way is by the inſtrument called a pentagraph, which alſo copies the plan in any ſize required.

FIFTH METHOD.

But the neateſt method of any is this. Procure a copying frame or glaſs, made in this manner; namely, a large ſquare of the beſt window glaſs, ſet in a broad frame of wood, which can be raiſed up to any angle, when the lower ſide of it reſts on a table. Set this frame up to any angle before you, facing a ſtrong light; fix the old plan and clean paper together with ſeveral pins quite around, to keep them together, the clean paper being laid uppermoſt, and upon the face of the plain to be copied. Lay them, with the back of the old plan, upon the glaſs, namely, that part which you intend to begin at to copy firſt; and, by means of the light ſhining through the papers, you will very diſtinctly perceive every line of the plan through the clean paper. In this ſtate then trace all the lines on the paper with a pencil. Having drawn that part which covers the glaſs, ſlide another part over the glaſs, and copy it in the ſame manner. And then another part. And ſo on till the whole be copied.

Then, take them aſunder, and trace all the pencil-lines over with a fine pen and Indian ink, or with common ink.

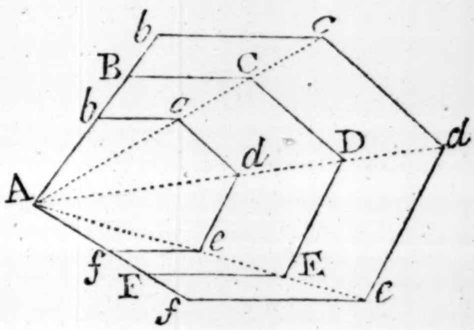
And thus you may copy the fineſt plan, without injuring it in the leaſt.

When the lines, &c, are copied upon the clean paper or vellum, the next buſineſs is to write ſuch names, remarks, or explanations as may be judged neceſſary; laying down the ſcale for taking the lengths of any parts, a flower-de-luce to point out the direction,

tion, and the proper title ornamented with a compartment; and illustrating or colouring every part in such manner as shall seem most natural, such as shading rivers or brooks with crooked lines, drawing the representations of trees, bushes, hills, woods, hedges, houses, gates, roads, &c, in their proper places; running a single dotted line for a foot path, and a double one for a carriage road; and either representing the bases or the elevations of buildings, &c.

PROBLEM IV.

To change a Figure from One Scale to Another.



From one angle A draw lines AC, AD, AE, &c, to all the other angles of the figure given; then augment or diminish one side AB till A*b* be to AB in the given proportion of the scales; and by means of a parallel ruler, draw *bc* parallel to BC and meeting AC in *c*, and in the same manner *cd* parallel to CD, *de* parallel to DE, *ef* parallel to EF; so shall A*bcdef*A be the figure required.

CHAPTER V.

Of the Division of Lands.

In the division of commons, after the whole is surveyed and cast up, and the proper quantities to be allowed for roads, &c, deducted, divide the net quantity remaining among the several proprietors, by the rule of Fellowship, in proportion to the real value

value of their eſtates, and you will thereby obtain their proportional quantities of the land. But as this diviſion ſuppoſes the land, which is to be divided, to be all of an equal goodneſs, you muſt obſerve that if the part in which any one's ſhare is to be marked off, be better or worſe than the general mean quality of the land, then you muſt diminiſh or augment the quantity of his ſhare in the ſame proportion *.

PROBLEM I.

It is required to divide any given quantity of ground, or its value, into any given number of parts, and in proportion as any given numbers.

Divide the given piece, or its value, as in the rule of Fellowſhip, by dividing the whole content or value by the ſum of the numbers expreſſing the proportions of the ſeveral ſhares, and multiplying the quotient ſeverally by the ſaid proportional numbers for the reſpective ſhares required, when the land is all of the ſame quality. But if the ſhares be of different qualities,

* Or, which comes to the ſame thing, divide the ground among the claimants in the direct ratio of the value of their claims, and the inverſe ratio of the quality of the ground allotted to each; that is in proportion to the quotients ariſing from the diviſion of the value of each perſon's eſtate, by the number which expreſſes the quality of the ground in his ſhare.

But theſe regular methods cannot always be put in practice; ſo that, in the diviſion of commons, the uſual way is, to meaſure ſeparately all the land that is of different values, and add into two ſums the contents and the values; then, by the firſt part of the following problem 1, the value of every claimant's ſhare is computed, by dividing the whole value among them in proportion to their eſtates; and, laſtly, by the 2d problem, a quantity is laid out for each perſon, that ſhall be of the value of his ſhare before found.

then divide the numbers expressing the proportions or values of the shares, by the numbers which express the qualities of the land in each share; and use the quotients instead of the former proportional numbers.

EXAMPLE I.

If the total value of a common be 2500 pounds, it is required to determine the values of the shares of the three claimants A, B, C, whose estates are of these values 10000, and 15000, and 25000 pounds.

The estates being in proportion as the numbers 2, 3, 5, whose sum is 10, we shall have $2500 \div 10 = 250$; which being severally multiplied by 2, 3, 5, the products 500, 750, 1250, are the values of the shares required.

EXAMPLE II.

It is required to divide 300 acres of land among A, B, C, D, E, F, G, and H, whose claims upon it are respectively in proportion as the numbers 1, 2, 3, 5, 8, 10, 15, 20.

The sum of these proportional numbers is 64, by which dividing 300, the quotient is 4a 2r 30p, which being multiplied by each of the numbers 1, 2, 3, 5, &c, we obtain for the several shares as below:

	AC.	R.	P.
A =	4	2	30
B =	9	1	20
C =	14	0	10
D =	23	1	30
E =	37	2	00
F =	46	3	20
G =	70	1	10
H =	93	3	00
Sum =	300	0	00

EXAMPLE III.

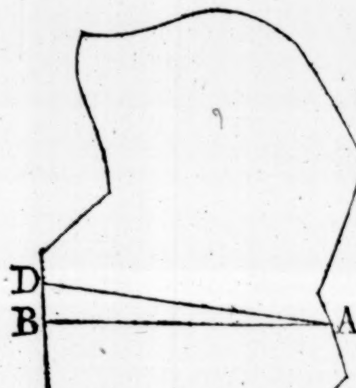
It is required to divide 780 acres among A, B, and C, whose estates are 1000, 3000, and 4000 pounds a year; the ground in their shares being worth 5, 8, and 10 shillings the acre respectively.

Here their claims are as 1, 3, 4; and the qualities of their land are as 5, 8, 10; therefore their quantities must be as $\frac{1}{5}$, $\frac{3}{8}$, $\frac{2}{5}$, or, by reduction, as 8, 15, 16. Now the sum of these numbers is 39; by which dividing the 780 acres, the quotient is 20; which being multiplied severally by the three numbers 8, 15, 16, the three products are 160, 300, 320, for the shares of A, B, C, respectively.

PROBLEM II.

To Cut off from a Plan a Given Number of Acres, &c, by a Line drawn from Any Point in the Side of it.

Let A be the given point in the annexed plan, from which a line is to be drawn cutting off suppose 5ac. 2r. 14p.



Draw AB cutting off the part ABC as near as can be judged equal to the quantity proposed; and let the true quantity of ABC, when calculated, be only 4ac. 3r. 20p. which is less than 5ac. 2r. 14p. the true quantity, by 0 ac. 2r. 34p. or 71250 square links. Then measure AB, which suppose = 1234 links, and divide 71250, by 617 the half of it, and the quotient 115 links will be the altitude of the triangle to be added, and whose base is

is AB. Therefore if upon the center B, with the radius 115, an arc be described; and a line be drawn parallel to AB, touching the arc, and cutting BD in D; and if AD be drawn, it will be the line cutting off the required quantity ADCA.

NOTE. If the first piece had been too much, then D must have been set below B.

In this manner the several shares of commons, to be divided, may be laid down upon the plan, and transferred from thence to the ground itself.

Also for the greater ease and perfection in this business, the following problems may be added.

PROBLEM III.

From an Angle in a Given Triangle, to draw Lines to the Opposite Side, dividing the Triangle into any Number of Parts, which shall be in any assigned Proportion to each other.

Divide the base into the same number of parts, and in the same proportion, by problem 1; then from the several points of division draw lines to the proposed angle, and they will divide the triangle as required *.

EXAMPLE.

Let the triangle ABC, of 20 acres, be divided into five parts, which shall be in proportion to the numbers 1, 2, 3, 5, 9; the lines of division to be drawn from A to CB, whose length is 1600 links.

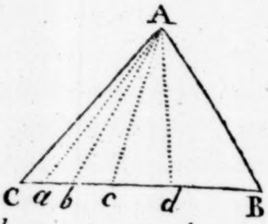
---

* DEMONSTRATION.

For the several parts are triangles of the same altitude, and which therefore are as their bases, which bases are taken in the assigned proportion.

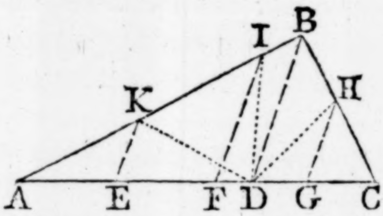
Here

Here $1 + 2 + 3 + 5 + 9 = 20$, and $1600 \div 20 = 80$; which being multiplied by each of the proportional numbers, we have 80, 160, 240, 400, and 720. Therefore I make $ca = 80$, $ab = 160$, $bc = 240$, $cd = 400$, and $dB = 720$; then by drawing the lines Aa, Ab, Ac, Ad, the triangle is divided as required.



PROBLEM IV.

From Any Point in one Side of a Given Triangle, to draw Lines to the other Two Sides, dividing the Triangle into Any Number of Parts which shall be in Any Assigned Ratio.



From the given point D, draw DB to the angle opposite the side AC in which the point is taken; then divide the same side AC into as many parts AE, EF, FG, GC, and in the same proportion with the required parts of the triangle, like as was done in the last problem; and from the points of division draw lines EK, FI, GH, parallel to the line BD, and meeting the other sides of the triangle in K, I, H; lastly, draw KD, ID, HD, so shall ADK, KDI, ID, HB, HDC be the parts required*.

The example to this will be done exactly as the last.

* DEMONSTRATION.

The triangles ADK, KDI, IDB, being of the same height, are as their bases AK, KI, IB; which, by means of the parallels EK, FI, DB, are as AE, EF, FD; in like manner, the triangles CDH, HDB are to each other as CG, GD: but the two triangles IDB, BDH, having the same base BD, are to each other as the distances of I and H from BD, or as FD to DG; consequently the parts DAK, DKI, DIBH, DHC are to each other as AE, EF, FG, GC.

A COL-

A COLLECTION OF QUESTIONS IN SURVEYING.

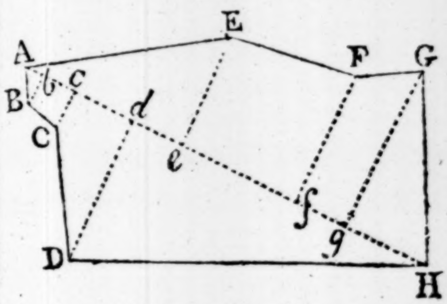
CHAPTER V.

QUESTION I.

REQUIRED the area and plot of a figure from the following field-book.

Note, That a cipher in the place of a perpendicular, denotes that there the base line touches an angular point. So here the ciphers for the first and last perpendiculars, shew that the base line begins and ends at an angle. Also R denotes right, and L left.

Field-Book.	
Base line	Perpendicular
0	0
19	42 R
80	62 R
184	235 R
307	184 L
470	220 L
556	270 L
697	0



Ans. 1ac. 3r. 8·908p.

QUESTION II.

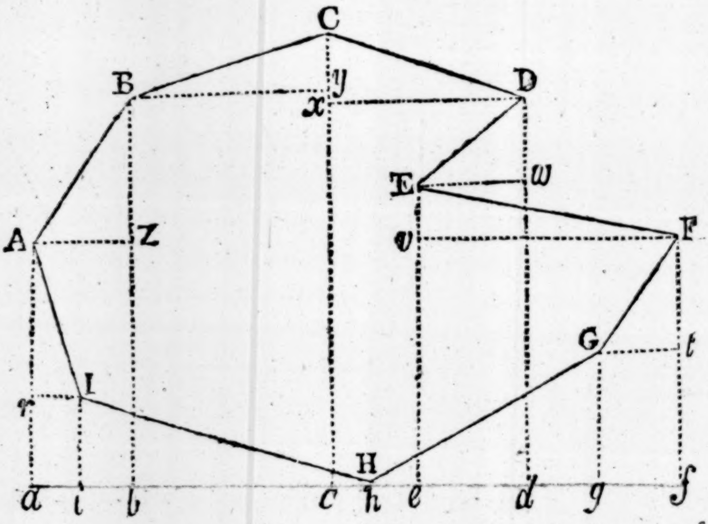
Beginning at the westmost station A, of a large tract of land, and going round towards the north, suppose the lengths of the lines and the angles formed by them and the meridians, to be taken thus: AB = 1550 links, and its

its direction N, $35\frac{1}{2}^{\circ}$ E, that is, $35\frac{1}{2}^{\circ}$ from the north towards the east; then proceeding to the 2d station B, I find the direction of BC to be N E, $72\frac{3}{4}^{\circ}$, and its length = 1870 links. In the same manner I find all the other sides and their directions as expressed in order in the 2d and 3d columns of the following table; the 1st column containing only the number and mark of each station.

Required the plan and content of this piece.

Stations		Angles		Sides	
1.	A	NE	$35\frac{1}{2}^{\circ}$	1550	AB
2.	B	NE	$72\frac{3}{4}$	1870	BC
3.	C	SE	$70\frac{3}{4}$	1870	CD
4.	D	SW	53	1245	DE
5.	E	SE	$83\frac{1}{4}$	2410	EF
6.	F	SW	$31\frac{1}{4}$	1520	FG
7.	G	SW	$62\frac{3}{4}$	2260	GH
8.	H	NW	$73\frac{1}{2}$	2730	HI
9.	I	NW	$17\frac{5}{12}$	1456	IA

Answer
145ac. 1r. 39·5776p.



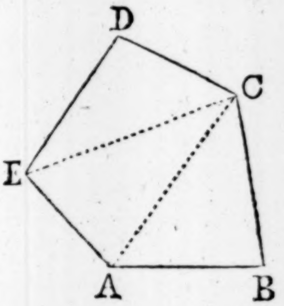
Note, A line is wanted to be drawn along the bottom of the figure, through the points *a i b c* H *e d g f*.

QUES-

QUESTION III.

In a pentangular field, beginning with the south side, and measuring round towards the east first, the 1st or south side was = 2735 links, the 2d = 3115, the 3d = 2370, the 4th = 2925, and the 5th = 2220; also the diagonal from the 1st angle to the 3d was 3800, and that from the 3d to the 5th was 4010: Required the figure and area.

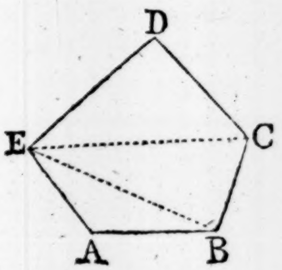
	ac	r	p
Answer	117	2	39·1408



QUESTION IV.

In going round a five-sided field ABCDE, the sides and angles were thus: The side AB = 1940 links, and the angle B 110° 30′; the side BC = 1555, and the angle C 117° 45′; the side CD = 2125, and the angle D 91° 20′; and the side DE = 2741: Required the figure and the content.

	ac	r	p
Answer	66	2	24



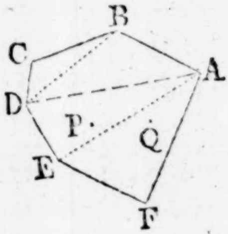
QUESTION V.

In a field I took two stations P and Q, at the distance of 10 chains from each other:

From P I took the angles			And from Q the angles		
QPA	=	21° 20′	PQD	=	10° 40′
APB	=	49 10	DQC	=	18 30
BPC	=	57 12	CQB	=	42 00
CPD	=	29 40	BQA	=	67 05
DPE	=	64 25	AQF	=	137 00
EPF	=	79 16	FQE	=	62 52

Required the figure and area.

	ac	r	p
Answer	50	3	39·3136



QUESTION VI.

From a station within a field of five sides I measured the distances to the several corners, beginning with that on the west, and measuring round towards the north, viz. 1st distance 7345 links, the 2d = 5980, the 3d = 6495, the 4th = 6015, and the 5th = 7050; also the angles formed by these distances in the same order were $71\frac{1}{2}$, $55\frac{3}{4}$, $49\frac{1}{4}$, and $81\frac{1}{2}$, degrees. What is the area? Ans. 979ac. 2r. 35·921p.

QUESTION VII.

From a place O, near the middle of a field ABCDEF, from which I could see all the angles, I measured the distances to the several corners, and observed the quantities of the angles formed at O by those distances, as below.

Distances	Angles
OA = 4315 links	AOB = 60° 30′
OB = 2982	BOC = 47 40
OC = 3561	COD = 49 50
OD = 5010	DOE = 57 10
OE = 4618	EOF = 64 15
OF = 3606	FOA = 80 35

What is the area? Ans. 412ac. 1r. 17·3224p.

QUESTION VIII.

Having made choice of a station within a piece of land, I measured from thence the several bearings and distances of the corners as below: Required the area.

	Bearings	Distances
1st	NE	3540
2d	N $\frac{1}{2}$ E	4785
3d	NWbW	3915
4th	SWbS	4125
5th	SSE 7° E	2030
6th	Ebs $3\frac{1}{2}$ E	2945

Ans. 319ac. 3r. 1·193p.

SECTION II.

OF CASK GAUGING.

THE meaning of the word *Gauging* is restricted to the measuring of casks, and other things falling under the cognizance of the officers of the excise;

excise; and it has received its name from a gauge or rod used by the practitioners of the art.

The business being performed, or the calculations made, commonly by means of the instruments, called the gauging or diagonal rod, and the sliding rule or gauging rule, it will be necessary to treat of these instruments, which I shall do as below.

CHAPTER I.

The Description and Use of the Sliding Rule.

This is a square rule, having consequently four sides or faces, three of which are furnished with sliding pieces running in grooves. The lines upon them are mostly logarithmic ones, or distances which are proportional to the logarithms of the numbers placed at the ends of them; which kind of lines were placed upon rules, by Mr. *Edmund Gunter*, for expeditiously performing arithmetical operations; in which business he used a pair of compasses for taking the several logarithmic distances: but instead of the compasses, sliding pieces were added, by Mr. *Thomas Everard*, as being more convenient and certain in practice.

Upon the first face are three lines, namely, two marked A, B, for multiplying and dividing; and the third, MD, for malt depth, because it serves to gauge malt. The middle one B is upon the slider, and is a kind of double line, being marked at both the edges of the slider, for applying it to both the lines A and MD. These three lines are all of the same radius, or distance from 1 to 10, each containing twice the length of the radius. A and B are placed and numbered exactly alike, each beginning at 1, which may be either 1, or 10, or 100, &c, or ·1, or ·01, or ·001, &c; but whatever it is, the middle division 10, will be 10 times as much,

much, and the last division 100 times as much. But 1 on the line MD is opposite 215, or more exactly 2150·4 on the other lines, which number 2150·4 denotes the cubic inches in a malt bushel, and its divisions numbered retrograde to those of A and B. Upon these two lines are also several other marks and letters: thus, on the line A are MB, for malt bushel, at the number 2150·4; and A for ale, at 282, the cubic inches in an ale gallon; and upon the line B is W, for wine, at 231, the cubic inches in a wine gallon; also *si*, for square inscribed, at ·707, the side of a square inscribed in a circle whose diameter is 1; *se*, for square equal, at ·886, the side of a square which is equal to the same circle; and C for circumference, at 3·1416, the circumference of the same circle.

Upon the second face, or that opposite the first, are a slider and four lines, marked D, C, D, E, at one end, and root, square, root, cube, at the other; the lines C and E containing respectively the squares and cubes of the opposite numbers on the lines D, D; the radius of D being double to that of A, B, C, and triple to that of E: so that whatever the first 1 on D denotes, the first on C is the square of it, and the first on E the cube of it,; so if D begin with 1, C and E will begin with 1; but if D begin with 10, C will begin with 100, and E with 1000; and so on. Upon the line C are marked O*c* at ·0796, for the area of the circle whose circumference is 1; and O*d* at ·7854 for the area of the circle whose diameter is 1. Also, upon the line D are WG, for wine gauge, at 17·15; and AG for ale gauge, at 18·95; and MR, for malt round, at 52·32; these three being the gauge points for round or circular measure, and are found by dividing the square roots of 231, 282, and 2150·4 by the square root of ·7854. also MS, for malt square, are marked at 46·37, the malt gauge point for square measure being the square root of 2150·4.

Upon the third face are three lines, one upon a ſlider marked N; and two on the ſtock, marked SS and SL, for ſegment ſtanding and ſegment lying, which ſerve for ullaging ſtanding and lying caſks.

And upon the fourth, or oppoſite face, are a ſcale of inches, and three other ſcales, marked ſpheroid or 1ſt variety, 2d variety, 3d variety; the ſcale for the 4th, or conic variety, being on the inſide of the ſlider in the third face. The uſe of theſe lines is to find the mean diameters of caſks.

Beſides all thoſe lines, there are two others on the inſides of the two firſt ſliders, being continued from the one ſlider to the other. The one of theſe is a ſcale of inches, from $12\frac{1}{2}$ to 36; and the other is a ſcale of ale gallons between the correſponding numbers ·435 and 3·61; which form a table to ſhew, in ale gallons, the contents of all cylinders whoſe diameters are from $12\frac{1}{2}$ to 36 inches, their common altitude being 1 inch.

As the ſliding rule is for performing, very expeditiouſly, any operations of multiplication, diviſion, and extraction of roots, which may be required by any precept propoſed in words, &c; ſo the manner of making theſe operations will appear in the following problems.

PROBLEM I.

To find the Product of Two Given Numbers, by the Sliding Rule.

RULE. To either of the given numbers on A ſet 1 on B, then againſt the other number on B is the product on A.

EXAMPLE 1. Required the product of 12 and 25.

By placing 1 on B under 12 on A, above 25 on B ſtands 300 on A; which is the product required.

NOTE. When the 1 on B has been ſet to the one factor

factor on A, if it happen that the other factor on B fall beyond the division, on either A or B, divide it by 10, or 100, &c, till the quotient found on B fall under some division on the line A, and multiply this said division by the same 10, or 100, &c, for the product required.

EXAMPLE II. So when 250 is to be multiplied by 56: Having set 1 on B to 250 on A, although 56 be found on B, it is beyond the end of A; therefore dividing it by 10, I find that opposite to the quotient 5·6 on B, is the division 1400 on A; which being multiplied by 10, we obtain 14000 for the product required.

EXAMPLE III. But if 250 were to be multiplied by 1120: Having set 1 to 250 as before, 1120 is beyond the end of B, but being divided by 100, opposite to the quotient 11·2 on B I find 2800 on A, which being multiplied by 100, we have 280000 for the product required.

PROBLEM II.

To find the Quotient of Two Numbers.

RULE. Set 1 on B to the divisor on A, then against the dividend on A, is the quotient on B.

EXAMPLE I. To divide 300 by 25. Having set 1 on B to 25 on A, opposite 300 on A I find 12 on B, the quotient required.

NOTE. When the dividend falls beyond the end of the line A, let it be divided by 10, 100, or some other power of 10 till it fall within the line, and use the quotient instead of it, multiplying the result by the same power of 10 as before.

EXAMPLE II. So if 14000 must be divided by 56. Having set 1 to 56, the dividend cannot be found on A till it be divided by 100, the quotient being 140, opposite to which I find 2·5 on B, which being

multiplied by 100 we obtain 250 for the quotient required.

PROBLEM III.

To work the Rule-of-Three on the Sliding Rule: or having Three Numbers given, to find a Fourth, which shall be to the Third as the Second is to the First.

RULE. Set the first term on B, to either the second or third on A; then against the remaining term on B, stands the fourth term required on A.

EXAMPLE. If 8 yards of cloth cost 24 shillings, what will 96 yards cost at the same rate?

Having set 8 on B to 24 on A, opposite 96 on B, I find, on A, 288 shillings, or 14l 8s, which is the answer.

PROBLEM IV.

To Extract the Square Root by the Sliding Rule.

RULE. The first 1 on C standing against the first 1 on D, on the stock, opposite the given number on C is its root on D.

EXAMPLE. To find the side of a square, which shall be equal to a triangle, or circle, &c, whose area is 225; or, to extract the root of 225.

Here opposite 225 on C stands 15 on D, which is the answer required.

PROBLEM V.

To Extract the Cube Root by the Sliding Rule.

RULE. The line D upon the slide being set straight with E; find the given number on E, and opposite to it will be its cube root on D.

EXAM-

Example. To find the side of a cube equal to any other solid whose content is 3375; or to find the cube root of 3375.

Here opposite 3375 on E, stands 15 on D, which is the answer required.

Note. It is evident that the same lines, as are used in these two last problems, will serve to find the square or the cube of any given number, by taking the given number on the contrary lines.

PROBLEM VI.

To find a Mean Proportional between Two Given Numbers.

Rule. Set one of the given numbers on C to the like or same number on D, then against the other given number on C, is the number required on D.

Example. To find the side of a square whose area shall be equal to that of a parallelogram whose length is 9, and its breadth 4 feet; or, to find a mean proportional between 4 and 9.

Having set 4 on C to 4 on D, against 9 on C stands 6 on D, which is the number sought.

PROBLEM VII.

To find a Number which shall be to a Given Number, in a given Duplicate Proportion; or having given Three Numbers, to find a Fourth, which shall be to the Third, as the Square of the Second is to the Square of the First.

Rule. Set the third number on C to the first on D, then against the second on D, will be found, on C, the fourth required.

Exam-

Example. If the area of a parallelogram, or any other figure, be 120; it is required to find the area of a similar figure, their like dimensions or sides being as 2 to 3.

Similar figures being as the squares of their like dimensions, by setting 120 on C to 2 on D, against 3 on D, stands 270 on C, for the number sought.

PROBLEM VIII.

To find a Number which shall be to a Given Number, in a given Subduplicate Proportion; or having given Three Numbers, to find a Fourth, which shall be to the Third, as the Root of the Second is to the Root of the First.

Rule. Set the first number on C to the third on D, then against the second on C, will be found the fourth on D.

Example. The side of a regular figure is 2, and its area 120; it is required to find the side of a similar figure whose area is 270.

The roots of the areas of similar figures being as their sides, we must find a number which shall be to 2, as the root of 270, is to the root of 120. Therefore, having set 120 on C to 2 on D, against 270 on C, will be found 3 on D, which is the number sought.

PROBLEM IX.

To find a Number in a given Triplicate Proportion to a Number given; or, having Three Numbers given, to find a Fourth, which shall be to the Third, as the Cube of the Second, is to the Cube of the First.

Rule. Set the first number on the slide D, to the third

third number on E, then against the second on D, is the fourth required on E.

EXAMPLE. If a cask, whose length is 40 inches, contain 100 gallons, what will be the content of a similar cask whose length is 36 inches?

Similar solids being as the cubes of their like sides, the content required must be to 100 gallons, as 36^3 is to 40^3. Therefore setting 40 on D to 100 on E, against 36 on D, will be found 72·9 gallons on E, which is the content required.

PROBLEM X.

To find a Number in a given Subtriplicate Proportion to a Given Number; or, having Three Numbers given, to find a Fourth, which shall be to the Third, as the Cube Root of the Second, is to the Cube Root of the First.

RULE. Set the third number on D, to the first on E, then against the second on E, will stand the fourth on D.

EXAMPLE. What is the length of a cask whose content is 72·9 gallons, supposing the length of a similar cask to be 40 inches, and its content 100 gallons?

Since the dimensions of similar solids are as the cube roots of their contents, we must find a number which shall be to 40, as the cube root of 72·9 is to the cube root of 100. Therefore, having set 40 on D to 100 on E, against 72·9 on E, will be found 36 on D, which is the length required.

PRO-

PROBLEM XI.

The Length and Breadth of a Parallelogram being given, to find its Area in Malt Bushels by the Line MD.

RULE. Set either of the given dimensions on B, to the other on MD, then against 1 on A, is the required area on B.

EXAMPLE. How many malt bushels can be contained on every inch of the depth of a cistern, whose length is 180, and breadth 72 inches?

By setting 72 on B, to 180 on MD, against 1 on A, will appear nearly 6 bushels on B, which is the quantity required.

PROBLEM XII.

To find, by the Line MD, *the Malt Bushels which may be contained in a Couch, Floor, or Cistern, whose length, breadth, and depth are given.*

RULE. Set one of the dimensions on B to another on MD, then against the third on A, will appear the content on B.

EXAMPLE. Required the number of bushels in the cistern whose length is 230, breadth 58·2, and depth 5·4 inches.

Having set 230, on B to 5·4 on MD, against 58·2 on A, I find 33·6 bushels on B, which is the content nearly.

The use of the other parts or marks on the rule will appear in the examples farther on.

CHAPTER II.

Of the Gauging or Diagonal Rod.

The diagonal rod is a square rule having four sides or

or faces, being generally four feet long, and folding together by means of joints.

It takes its name from its use in measuring the diagonals of casks, and computing the contents from the said diagonal only. Where it may be noted, that by the diagonal of a cask, is meant the line from the bung to the intersection of the head with the stave opposite to it, and is commonly the longest line that can be drawn from the middle of the bung to any part within the cask.

And, accordingly, upon one face of the rule is a scale of inches for taking the measure of the diagonal; to which is adapted the areas, in ale gallons, of circles to the corresponding diameters, like the lines on the under sides of the three slides in the sliding rule.

Upon the opposite face are two scales of ale and wine gallons, expressing the contents of casks having the corresponding diagonals; and these are the lines which chiefly constitute the difference between this instrument and the sliding rule; for all the other lines upon it are the same with those on that instrument, and are to be used in the same manner.

EXAMPLE.

Let it be required to find the content of a cask whose diagonal measures 34·4 inches, which agrees with the cask in the following chapter, whose head and bung diameters are 32 and 24, and length, 40 inches; for if to the square of 20, half the length, be added the square of 28, half the sum of the diameters, the square root of the sum will be 34·4 nearly.

Now to this diagonal 34·4, corresponds, upon the rule, the content $90\frac{3}{4}$ ale or 111 wine gallons; which differs from all the contents, in the next chapter, obtained by considering the cask as belonging

to

to each of the four propoſed varieties; being indeed a kind of medium among them all, and falling in between the ſecond and third variety; and ſo anſwering to the moſt common form of caſks.

CHAPTER III.

Of Caſks conſidered as divided into ſeveral Varieties.

According to the cuſtom of moſt writers on this ſubject, I ſhall diſtinguiſh caſks into four forms or varieties, viz.

1. The middle fruſtum of a ſpheroid,
2. The middle fruſtum of a parabolic ſpindle,
3. The two equal fruſtums of a paraboloid,
4. The two equal fruſtums of a cone.

I omit here the middle fruſtums of circular, elliptic, and hyperbolic ſpindles, on account of the difficulty of their rules, which renders them unfit for the purpoſe of practical gauging. And indeed ſome of the above four forms are of very little real uſe; for very few, if any, caſks are to be met with which will hold ſo much as the firſt form, or ſo little as the third or fourth; ſo that the ſecond is the moſt generally, if not the only uſeful one, of the four varieties.

Note, 282 cubic inches make one ale gallon,
231 — — wine gallon,
2150·42 — — a malt buſhel.

It is alſo to be noted that the dimenſions are ſuppoſed to be inches, in the following rules.

PRO-

PROBLEM I.

To find the Content of a Cask of the First or Spheroidal Variety.

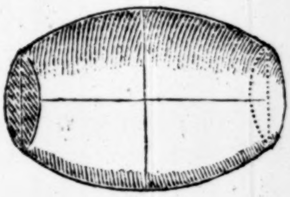
*To the ſquare of the head diameter add double the ſquare of the bung diameter, and multiply the ſum by the length of the caſk. Then let the product be mult. by $\cdot 0009\frac{1}{4}$, or divided by 1077 for ale gallons, or mult. by $\cdot 0011\frac{1}{3}$, or divided by 882 for wine gallons

BY THE SLIDING RULE.

Set the length on C to 32·82, for ale, or to 29·7, for wine, on D, and on D find the bung and head diameters, noting the numbers oppoſite to them on C, then if the latter of theſe two numbers be added to the double of the former, the ſum will be the meaſure in gallons.

EXAMPLE.

Required the content of a ſpheroidal caſk, whoſe bung and head diameters are 32 and 24, and length 40 inches.



By

* For, by prob. 12 ſect. 5 part 3, the content in inches is $(2B^2 + H^2) \times \frac{1}{3} L n$, which being divided by 282 and 231, becomes $\frac{2B^2 + H^2}{1077\cdot 157} \times L$ or $(2B^2 + H^2) \times \cdot 00092837L$ in the one caſe, and $\frac{2B^2 + H^2}{882\cdot 355} \times L$ or $(2B^2 + H^2) \times \cdot 00113333$ in the other; B being the bung and H the head diameter, and L the length of the caſk.——And in working by the ſliding rule, it need only be remarked that 32·82 and 29·7 are the roots of 1077·157 and 882·355.

By the Pen. Here $(2 \times 32^2 + 24^2) \times 40 \times \left\{ \begin{array}{l} \cdot 0009\frac{1}{4} \\ \cdot 0011\frac{1}{3} \end{array} \right\} = \left\{ \begin{array}{l} 97\cdot 44 \text{ ale} \\ 118\cdot 95 \text{ wine} \end{array} \right\}$ gallons, the content required.

By the Sliding Rule. Having ſet 40 on C to 32·82 on D, againſt 32 and 24 on D, ſtand 38 and 21·3, as near as can be judged, on C; then $2 \times 38 + 21\cdot 3 = 76 + 21\cdot 3 = 97\cdot 3$ ale gallons.

And having ſet 40 on C to 29·7 on D, againſt 32 and 24 on D, ſtand 46·5 and 26·1 on C; then $2 \times 46\cdot 5 + 26\cdot 1 = 93 + 26\cdot 1 = 119\cdot 1$ wine gallons.

PROBLEM II.

To find the Content of a Caſk of the Second or Parabolic Spindle form.

*To the ſquare of the head diameter, add double that of the bung diameter, and from the ſum take $\frac{2}{5}$ or $\frac{4}{10}$ of the ſquare of the difference of the ſaid diameters; then multiply the remainder by the length, and the product multiplied or divided by the ſame numbers as in the rule to the laſt problem, will give the content.

BY THE SLIDING RULE.

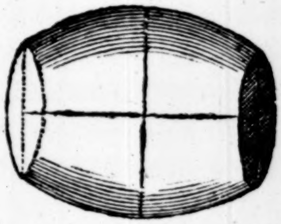
As in the laſt problem, ſet the length on C to 32·82 or 29·7 on D, and on D find both the bung and head diameters, and alſo their difference, taking out the three numbers oppoſite to them on C; then if to twice the firſt be added the ſecond, and $\frac{4}{10}$ of the third

* For, by prob. 18 ſect. 6 part 3, the content in inches is $\frac{8B^2 + 4BH + 3H^2}{15} \times Ln = \frac{Ln}{3} \times \left(\frac{10B^2 + 4BH + 5H^2}{5} - \frac{2B^2 + 2H^2}{5}\right) = \frac{1}{3}Ln \times (2B^2 + H^2 - \frac{2}{5} \times (B - H)^2)$, and $\frac{1}{3}n$ will give the ſame numbers as in the laſt problem.

third be taken from the sum, the remainder will be the content.

EXAMPLE.

Required the content of a cask of the second variety, whose bung and head diameters are 32 and 24, and length 40 inches.



By the Pen. Here $(2 \times 32^2 + 24^2 - \frac{2}{5} \times 8^2) \times 40 \times \left\{\begin{matrix} \cdot 0009\frac{1}{4} \\ \cdot 0011\frac{1}{3} \end{matrix}\right\} = \left\{\begin{matrix} 96\cdot 49 \text{ ale} \\ 117\cdot 79 \text{ wine} \end{matrix}\right\}$ gallons, the content required.

By the Sliding Rule. Having set 40 on C to 32·82 on D, against 32, 24, and 8 on D, stand 38, 21·3, and 2·4; then $2 \times 38 + 21\cdot 3 - \frac{2}{5} \times 2\cdot 4 = 76 + 21\cdot 3 - 0\cdot 9 = 96\cdot 4$ ale gallons.

And having set 40 on C to 29·7 on D, against 32, 24, and 8 on D, stand 46·5, 26·1, and 2·9; then $2 \times 46\cdot 5 + 26\cdot 1 - \frac{2}{5} \times 2\cdot 9 = 93 + 26\cdot 1 - 1\cdot 2 = 117\cdot 9$ wine gallons.

PROBLEM III.

To find the Content of a Cask of the Third or Paraboloidal Variety.

To the square of the bung diameter add the square of the head diameter, and multiply the sum by the length; then if the product be

mult. by $\left\{\begin{matrix} \cdot 0014 \\ \cdot 0017 \end{matrix}\right\}$ or div. by $\left\{\begin{matrix} 718 \\ 588 \end{matrix}\right\}$ for $\left\{\begin{matrix} \text{ale} \\ \text{wine} \end{matrix}\right\}$,

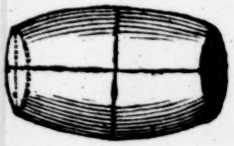
the product or quotient will be the content required.

BY THE SLIDING RULE.

Set the length on C to $\left\{\begin{array}{c}26\cdot8\\24\cdot25\end{array}\right\}$ on D $\left\{\begin{array}{l}\text{for ale}\\\text{for wine}\end{array}\right\}$; then find the bung and head diameters on D, noting the two opposite numbers on C, whose sum will be the content required*.

EXAMPLE.

Required the content of a cask of the third variety, whose bung and head diameters are 32 and 24, and length 40 inches.



By the Pen. Here $(32^2 + 24^2) \times 40 \times \left\{\begin{array}{c}\cdot0014\\\cdot0017\end{array}\right\} = \left\{\begin{array}{l}89\cdot1 \text{ ale}\\108\cdot8 \text{ wine}\end{array}\right\}$ gallons, the content.

By the Sliding Rule. Having set 40 on C to 26·8 on D, against 32 and 24 on D, stand 57·3 and 32; whose sum is 89·3 ale gallons.

And having set 40 on C to 24·25 on D, against 32 and 24 on D, stand 69·8 and 39·1, whose sum is 108·9 wine gallons.

PROBLEM IV.

To find the Content of a Cask of the Fourth or Conical Variety.

To three times the square of the sum of the di-

* For, by prob. 12 sect. 6 part 3, the content in inches is $\frac{B^2 + H^2}{2} \times Ln$; and $\frac{n}{2 \times 282} = \frac{1}{718\cdot105} = \cdot00139255$, and $\frac{n}{2 \times 231} = \frac{1}{588\cdot233} = \cdot0017$.

Also the numbers 26·8 and 24·25 are the roots of the numbers 718·105 and 588·233.

ameters, add the square of the difference of the diameters; then if the sum be

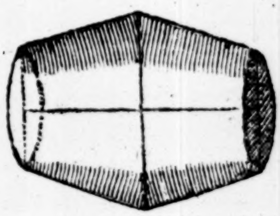
mult. by $\left\{\begin{matrix}\cdot00023\frac{1}{5}\\ \cdot00028\frac{1}{3}\end{matrix}\right\}$ or div. by $\left\{\begin{matrix}4308\\ 3529\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{ale}\\ \text{wine}\end{matrix}\right.$ the product or quotient will be the content required.

BY THE SLIDING RULE.

Set the length on C to $\left\{\begin{matrix}65\cdot64\\ 59\cdot41\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{ale}\\ \text{wine}\end{matrix}\right\}$ on D, and on D find the sum and the difference of the diameters, noting their opposite numbers on C; then if the second be added to three times the first, the sum will be the content *.

EXAMPLE.

Required the content of a cask of the fourth variety, whose bung and head diameters are 32 and 24, and length 40 inches.



By the Pen. Here $(3 \times 56^2 + 8^2) \times 40 \times \left\{\begin{matrix}\cdot00023\frac{1}{5}\\ \cdot00028\frac{1}{3}\end{matrix}\right\} = \left\{\begin{matrix}87\cdot9342 \text{ ale}\\ 107\cdot348 \text{ wine}\end{matrix}\right\}$ gallons, the content.

By the Sliding Rule. Having set 40 on C to 65·64 on D, against 56 and 8 on D, stand 29·1 and 0·6; then $3 \times 29\cdot1 + 0\cdot6 = 87\cdot3 + 0\cdot6 = 87\cdot9$ ale gallons.

---

* For, by prob. 8 sect. 1 part 3, the content in inches is $\frac{B^2 + BH + H^2}{3} \times Ln = \frac{Ln}{12} \times [3 \times (B + H)^2 + (B - H)^2]$; where $\frac{n}{12 \times 282} = \frac{1}{4308\cdot628} = \cdot00023209$, and $\frac{n}{12 \times 231} = \frac{1}{3529\cdot42} = \cdot00028333$. Also 65·64 and 59·41 are the roots of the numbers 4308·628 and 3529·42.

And, having set 40 on C to 59·41 on D, against 56 and 8 on D, stand 35·6 and 0·7; then $3 \times 35·6 + 0·7 = 107·5$ wine gallons.

CHAPTER IV.

Of Gauging Casks by their Mean Diameters.

PROBLEM I.

To find the Mean Diameter of a Cask of Any of the Four Varieties, having Given the Bung and Head Diameters.

*Divide the head diameter by the bung diameter, and find the quotient in the first column of the following table, marked Qu. Then if the bung diameter be multiplied by the number upon the same line with it, and in the column answering to the proper variety, the product will be the true mean diameter, or the diameter of a cylinder of the same content with the cask proposed, cutting off four figures for decimals.

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* *The Investigation of the numbers of this table.*

By the 3d part it appears that if the bung diameter be represented by 1, and the head diameter by h, then the content for the 1st, 2d, 3d, and 4th varieties will be respectively ln drawn into

$$\frac{2+hh}{3},\quad \frac{8+4h+3hh}{15},\quad \frac{1+hh}{2},\quad \frac{1+h+hh}{3};$$

but the content must also be equal to ln drawn into the square of the mean diameter; and consequently the said mean diameter, or multiplier in the table, will be respectively equal to

$$\sqrt{\frac{2+hh}{3}},\quad \sqrt{\frac{8+4h+3hh}{15}},\quad \sqrt{\frac{1+hh}{2}},\quad \sqrt{\frac{1+h+hh}{3}}.$$

Then by writing, in these forms, the several values of h, viz. ·50, ·51, ·52, &c, there will result the corresponding numbers of the table.

Qu.

Qu	1 Var	2 Var	3 Var	4 Var	Qu	1 Var	2 Var	3 Var	4 Var
50	8660	8465	7905	7637	76	9270	9227	8881	8827
51	8680	8493	7937	7681	77	9296	9258	8924	8874
52	8700	8520	7970	7725	78	9324	9290	8967	8922
53	8720	8548	8002	7769	79	9352	9320	9011	8970
54	8740	8576	8036	7813	80	9380	9352	9055	9018
55	8760	8605	8070	7858	81	9409	9383	9100	9066
56	8781	8633	8104	7902	82	9438	9415	9144	9114
57	8802	8662	8140	7947	83	9467	9446	9189	9163
58	8824	8690	8174	7992	84	9496	9478	9234	9211
59	8846	8720	8210	8037	85	9526	9510	9280	9260
60	8869	8748	8246	8082	86	9556	9542	9326	9308
61	8892	8777	8282	8128	87	9586	9574	9372	9357
62	8915	8806	8320	8173	88	9616	9606	9419	9406
63	8938	8835	8357	8220	89	9647	9638	9466	9455
64	8962	8865	8395	8265	90	9678	9671	9513	9504
65	8986	8894	8433	8311	91	9710	9703	9560	9553
66	9010	8924	8472	8357	92	9740	9736	9608	9602
67	9034	8954	8511	8404	93	9772	9768	9656	9652
68	9060	8983	8551	8450	94	9804	9801	9704	9701
69	9084	9013	8590	8497	95	9836	9834	9753	9751
70	9110	9044	8631	8544	96	9868	9867	9802	9800
71	9136	9074	8672	8590	97	9901	9900	9851	9850
72	9162	9104	8713	8637	98	9933	9933	9900	9900
73	9188	9135	8754	8685	99	9966	9966	9950	9950
74	9215	9166	8796	8732	100	10000	10000	10000	10000
75	9242	9196	8838	8780					

EXAMPLE.

Supposing the diameters to be 32 and 24, it is required to find the mean diameter for each variety.

Dividing 24 by 32 we obtain ·75, which being found in the column of quotients, opposite thereto stand the numbers

$$\left\{\begin{matrix}\cdot 9242\\ \cdot 9196\\ \cdot 8838\\ \cdot 8780,\end{matrix}\right\} \text{which being each multiplied by 32, produce respectively} \left\{\begin{matrix}29\cdot 5744\\ 29\cdot 4272\\ 28\cdot 2816\\ 28\cdot 0960\end{matrix}\right\} \text{for the corresponding mean diameters required.}$$

BY THE SLIDING RULE.

Find the difference between the bung and head diameters upon the fourth face of the rule, or inside of the third slider, and opposite thereto is, for each variety, a number to be added to the head diameter, for the mean diameter required.

So in the above example, against 8, the difference of the diameters, are found the numbers

5·60	which be-	29·60	for the respective mean
5·10	ing added	29·10	diameters: all of which
4·56	to 24, there	28·56	are too great except the
4·12	result	28·12	2d, which is too little.

So that this method does not give the true mean diameter.

PROBLEM II.

To find the Content of a Cask by the Mean Diameter on the Sliding Rule.

Set the length on C to the * gauge point, 18·95 for ale, or 17·15 for wine, on D; then against the mean diameter on D, is the content on C.

EXAMPLE.

If the bung diameter be 32, the head 24; and the length 40 inches.

Having found the mean diameters as in the last problem, and set 40 on C to 18·95 or 17·15 on D, against

* The above gauge points are found thus, viz. $\sqrt{\dfrac{282}{\cdot 785398}} = 18\cdot 95$, and $\sqrt{\dfrac{231}{\cdot 785398}} = 17\cdot 15$.

$$\text{against}\begin{Bmatrix}29{\cdot}57\\29{\cdot}43\\28{\cdot}28\\28{\cdot}10\end{Bmatrix}\text{on D is}\begin{Bmatrix}97{\cdot}4\\96{\cdot}5\\89{\cdot}1\\88{\cdot}0\end{Bmatrix}\text{or}\begin{Bmatrix}119{\cdot}5\\118{\cdot}0\\108{\cdot}8\\107{\cdot}3\end{Bmatrix}$$

on c, as near as can be judged; which agree nearly with the contents determined in the preceding chapter.

CHAPTER V.

Of the Gauging of all Casks in general by means of Four Dimensions, viz. the Length, the Bung and Head Diameters, and the Diameter taken in the Middle between the Bung and Head.

GENERAL PROBLEM.

To find the Content of Any Cask in Ale or Wine Gallons, by Four Dimensions.

*Add into one sum, the square of the bung diameter, the square of the head diameter, and the square of

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* This rule is taken from prob. 3 sect. 1 part 4; the number $\cdot 0005\frac{2}{3}$ being $= \frac{\cdot 785398}{6 \times 231}$ very nearly, and $\frac{\cdot 785398}{6 \times 282} = \cdot 0004641844 = \cdot 0004\frac{2}{3}$ nearly $= \cdot 000\frac{51}{11}$ more nearly.

And with regard to the operation by the sliding rule, it may be observed that 42 is $= \sqrt{\frac{6 \times 231}{\cdot 785398}}$, and $46{\cdot}4 = \sqrt{\frac{6 \times 282}{\cdot 785398}}$.

The above rule, by the said prob. 3 sect. 1 part 4, was proved to be accurately true, not only for all the four different varieties of casks, but also for all casks and solids generated from any conic section whatever; and although the cask be not precisely in the form of any such curve, the rule will give the content very near the truth; so that whatever be the form of the cask, we may in all cases be pretty sure of the content to within $\frac{1}{10}$ of a gallon, or perhaps less, supposing the dimensions to be truly taken. So that more perfect than this, both with respect to truth and expedition, nothing can be expected, or indeed wished for, in gauging: which

of double the middle diameter, and multiply that sum by the length of the cask; then the product

$$\left.\begin{matrix}\text{mult.}\\ \text{by}\end{matrix}\right\}\begin{matrix}\cdot0004\frac{7}{11}\\ \cdot0005\frac{2}{3}\end{matrix}\left\{\begin{matrix}\text{for ale}\\ \text{for wine}\end{matrix}\right\}\begin{matrix}\text{will give the}\\ \text{content.}\end{matrix}$$

BY THE SLIDING RULE.

Set the length on C to $\left\{\begin{matrix}46\cdot4 \text{ for ale}\\ 42\cdot0 \text{ for wine}\end{matrix}\right\}$ on D; and find both the bung, head and middle diameters on

---

which makes me hope that one day this method will come into general use with the practitioners in the excise; and till then, I am fully persuaded that much of their practice must be mere guess work. The pretended difficulty of taking the middle diameter may perhaps deter some from using this method; but there cannot be any real difficulty in taking this diameter, except when a wooden hoop may happen to be at the part where the diameter ought to be taken; but that will very rarely happen.

To find the fourth dimension or diameter in the middle between the bung and head; upon one side of a square rule let be drawn a scale of quarter inches, numbered both ways from nothing; and let the middle or 0 division be applied to the bung or middle of the cask, as in the margin, parallel or nearly parallel to the axe, and in the direction of the staff; then whatever number of inches are in the whole length of the cask, it is evident that, from the nature of the scale, the same division, or number of quarter inches, will be opposite to the part where the middle diameter must be taken, which here suppose to be 10; and at 10, on each part of the rule, measure the distance *rs*; then if the sum of *rs* and *rs* be taken from the bung diameter, there will remain the required middle diameter, excepting the allowance for the thickness of the staff, which must be subtracted.

The

on D, noting the three numbers againſt them on C; then the ſum of the firſt and ſecond with 4 times the third will be the content required.

EXAMPLE

Let the length of a caſk be 40 inches, the bung diameter 32, the head 24, and middle diameter 30·2 inches

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The following collection of caſks happened in real practice; and their dimenſions were carefully taken; but their contents were computed by a ſliding rule, and ſo may not all be preciſely true.

From hence it appears that a ſpheroidal caſk is a mere imaginary thing, the contents of real caſks being leſs than is aſſigned to them by that form; as indeed they ought, from the nature of the curves; for a ſpheroidal caſk would be leaſt curved in the middle, and the moſt at the ends; whereas a real caſk is the leaſt curved at the ends, if it be any thing curved there at all; and indeed there is reaſon to think it is not, as will appear in chap. 7.

Caſks gauged by four dimenſions.

Numb	Length	Head diam.	Bung diam.	Mid. diam.	Ale gallons	Diff. leſs than ſphr.	Numb	Length	Head diam.	Bung diam.	Mid. diam.	Ale gallons	Diff. leſs than ſphr.
1	28·3	23·2	27·7	26·3	53·6	1·0	19	48·8	24·2	32·1	29·4	114·8	5·5
2	29·8	22·2	26·0	24·8	50·2	1·1	20	51·2	23·3	31·0	28·2	111·3	5·8
3	30·8	23·2	27·5	26·1	57·7	1·1	21	49·3	23·8	32·6	29·5	117·0	6·1
4	32·2	24·5	30·1	28·4	70·6	1·3	22	48·0	28·2	33·8	31·4	137·3	6·1
5	30·0	24·7	29·2	27·6	62·6	1·8	23	45·2	26·6	33·2	30·4	115·6	6·5
6	32·5	23·8	28·2	26·8	63·6	1·9	24	51·6	36·6	41·6	39·6	223·3	6·7
7	34·3	26·3	33·5	31·1	90·4	2·9	25	44·2	28·1	36·4	33·3	134·6	6·8
8	34·5	26·4	33·0	30·7	89·0	3·0	26	57·0	32·7	42·0	39·1	236·6	6·8
9	41·0	26·3	32·2	30·2	102·2	3·0	27	51·0	33·1	38·1	35·7	181·0	8·0
10	37·0	26·1	31·8	29·9	90·3	3·1	28	51·5	33·3	40·0	37·2	197·0	8·7
11	44·5	34·4	40·8	38·8	183·8	3·2	29	54·0	34·8	44·8	41·5	253·5	8·8
12	47·0	26·3	33·8	31·4	126·3	3·5	30	50·0	34·3	40·5	37·7	197·6	9·4
13	34·2	27·2	33·8	31·4	92·3	3·8	31	49·0	29·5	36·0	33·0	148·1	9·4
14	47·0	25·3	32·0	29·7	113·1	4·3	32	51·0	33·5	39·2	36·4	189·6	9·7
15	45·5	30·7	38·0	35·5	157·0	4·7	33	51·0	33·4	39·8	37·0	194·0	9·8
16	44·6	24·7	32·2	29·6	106·6	4·7	34	55·5	30·6	40·6	37·0	207·2	10·2
17	48·6	24·2	32·1	29·4	114·4	4·9	35	45·6	28·0	34·6	32·4	134·8	12·0
18	46·0	25·7	34·7	31·7	125·3	5·5	36	55·0	35·8	48·0	43·2	282·2	17·8

inches nearly $= \sqrt{912}$, which is taken upon the supposition that the cask is spheroidal.

Then $(32^2 + 24^2 + 4 \times 912) \times 40 \times \cdot0004\frac{7}{11} = 97\cdot44$ ale gallons; or multiplied by $\cdot0005\frac{2}{3}$, gives $118\cdot95$ wine gallons, for the content, the same as at prob. 1 chap 3.

By the Sliding Rule. Having set 40 on C to 46·4 on D, against 32, 24, and 30·2 on D, stand 19, 10·5, and 17 on C; then $19 + 10\cdot5 + 4 \times 17 = 97\cdot50$, ale gallons.

And, having set 40 on C to 42 on D, against 32, 24, and 30·2 on D, stand 23·2, 13, and 20·7 on C; then $23\cdot2 + 13 + 4 \times 20\cdot7 = 119$ wine gallons, for the content as before, nearly.

CHAPTER VI.

A New, Easy, and Expeditious Method of computing the Content of a Cask from Three Dimensions only, of Whatever Variety it may be.

*In the column of diameters, or middle column, in

* The method of gauging here proposed, is no other than the rule for one form of solids, corrected so as to make it agree, in content, with the most common or general form of casks: and, for the more expedition, the squares of all diameters, within certain limits, are divided by the proper constant divisor, and the quotients arranged in the table opposite to their corresponding diameters; that in practice nothing more may be required than to multiply these quotients by any assigned length of a cask, for the content in gallons.

Now it hath been found that wine hogsheads generally contain about a gallon and a quarter less than the content assigned to them by the rule for spheroidal casks; to correct the rule for that

in the following table, find the bung and head diameters, taking out the number on the left of the head diameter, and the number on the right of the bung diameter; then the sum of those two numbers multiplied by the length of the cask will give the content

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that form of casks, then, we must increase the divisor 882·355 in the proportion of $61\frac{3}{4}$ to 63; by which proportion we obtain 900, very nearly, to be used instead of 882·355, and then the rule will become $\frac{H^2 + 2B^2}{900} \times L$ for the content in gallons.

Then as the least diameter of a wine hogshead is about 19 inches, and the greatest bung of a pipe about 35, to all diameters, in inches and tenths, within those limits, I have computed the values of the quantities $\frac{H^2}{900}$ and $\frac{2B^2}{900}$, and disposed them in the columns titled head and bung areas; which it is evident need only be taken out of the table, for any example of bung and head diameters, and multiply their sum by the length for the content. So that we have here a general and easy rule by which any person, in half a minute, may compute the content of any cask whose bung and head diameters, and length are given, and that generally nearer to the truth than by any rule now commonly used.

I shall here insert the method by which I made this table; for although I have as above explained the principle upon which the table is founded, viz. that the numbers in it consist of every diameter squared, and then divided by 900, I did not find the numbers in that manner, but by the following method in perhaps a tenth part of the time required by the former.

Since the 1st, 2d, 3d, &c, numbers are

$$\frac{19{\cdot}1^2}{900} = \frac{19^2}{900} + \frac{3{\cdot}81 \times 1}{900} \text{ or } + \frac{3{\cdot}81}{900} = A,$$

$$\frac{19{\cdot}2^2}{900} = \frac{19^2}{900} + \frac{3{\cdot}82 \times 2}{900} = A + \frac{3{\cdot}83}{900} = A + {\cdot}0042\dot{5} = B,$$

$$\frac{19{\cdot}3^2}{900} = \frac{19^2}{900} + \frac{3{\cdot}83 \times 3}{900} = B + \frac{3{\cdot}85}{900} = B + {\cdot}0042\dot{7} = C,$$

&c,

it is evident that the differences of the terms form an arithmetical series, whose common difference is $\frac{{\cdot}02}{900}$ or ${\cdot}0000\dot{2}$, and therefore by writing

tent in wine gallons. And the wine gallons being multiplied by 77, and the product divided by 94, the quotient will be the ale gallons ; or multiply by 9 and divide by 11 for ale gallons, because $\frac{231}{282} = \frac{77}{94} = \frac{9}{11}$ very nearly.

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writing down a column of differences, and to the first term adding the first difference, to obtain the second term : then to the second term adding the second difference, for the third term; and so on as appears in the margin ; the terms will evidently be found in a very expeditious manner : also by doubling these head areas, we obtain the corresponding bung areas.——In the above computations the last figures, having a point above them, are infinite repetends.

Diff.	Areas.
	$4011\dot{1}$
425	40536
427	40964
430	41394
432	41826
434	42261
436	42697
438	43136

Farther, it is evident that in this method, the gauge point will be 30 or $\sqrt{900}$, which is to be used, in the rule for spheroidal casks, instead of 29·7, and the correct content will be obtained.

EX-

Head area	Diameter	Bung area	Head aread	Diameter	Bung area	Head area	Diameter	Bung area
·4054	19·1	·8107	·6615	24·4	1·3230	·9801	29·7	1·9602
·4096	19·2	·8192	·6669	24·5	1·3339	·9867	29·8	1·9734
·4139	19·3	·8278	·6724	24·6	1·3448	·9933	29·9	1·9867
·4183	19·4	·8364	·6779	24·7	1·3558	1·0000	30·0	2·0000
·4226	19·5	·8450	·6834	24·8	1·3668	1·0067	30·1	2·0134
·4270	19·6	·8537	·6889	24·9	1·3778	1·0134	30·2	2·0268
·4314	19·7	·8624	·6944	25·0	1·3888	1·0201	30·3	2·0402
·4356	19·8	·8712	·7000	25·1	1·4000	1·0268	30·4	2·0537
·4400	19·9	·8800	·7056	25·2	1·4112	1·0336	30·5	2·0772
·4444	20·0	·8889	·7112	25·3	1·4224	1·0404	30·6	2·0808
·4489	20·1	·8978	·7168	25·4	1·4337	1·0472	30·7	2·0944
·4534	20·2	·9068	·7225	25·5	1·4450	1·0540	30·8	2·1081
·4579	20·3	·9158	·7282	25·6	1·4564	1·0609	30·9	2·1218
·4624	20·4	·9248	·7339	25·7	1·4678	1·0678	31·0	2·1356
·4669	20·5	·9339	·7396	25·8	1·4792	1·0747	31·1	2·1494
·4715	20·6	·9430	·7453	25·9	1·4907	1·0816	31·2	2·1632
·4761	20·7	·9522	·7511	26·0	1·5022	1·0885	31·3	2·1771
·4807	20·8	·9614	·7569	26·1	1·5138	1·0955	31·4	2·1910
·4853	20·9	·9707	·7627	26·2	1·5254	1·1025	31·5	2·2050
·4900	21·0	·9800	·7685	26·3	1·5371	1·1095	31·6	2·2190
·4947	21·1	·9894	·7744	26·4	1·5488	1·1165	31·7	2·2331
·4994	21·2	·9988	·7803	26·5	1·5606	1·1236	31·8	2·2472
·5041	21·3	1·0082	·7862	26·6	1·5724	1·1307	31·9	2·2614
·5088	21·4	1·0177	·7921	26·7	1·5842	1·1378	32·0	2·2756
·5136	21·5	1·0272	·7980	26·8	1·5961	1·1449	32·1	2·2898
·5184	21·6	1·0368	·8040	26·9	1·5080	1·1520	32·2	2·3041
·5232	21·7	1·0464	·8100	27·0	1·6200	1·1592	32·3	2·3184
·5280	21·8	1·0561	·8160	27·1	1·6320	1·1664	32·4	2·3328
·5329	21·9	1·0658	·8220	27·2	1·6441	1·1736	32·5	2·3472
·5378	22·0	1·0756	·8281	27·3	1·6562	1·1808	32·6	2·3617
·5427	22·1	1·0854	·8342	27·4	1·6684	1·1881	32·7	2·3762
·5476	22·2	1·0952	·8403	27·5	1·6806	1·1954	32·8	2·3908
·5525	22·3	1·1051	·8464	27·6	1·6928	1·2027	32·9	2·4054
·5575	22·4	1·1150	·8525	27·7	1·7051	1·2100	33·0	2·4200
·5625	22·5	1·1250	·8587	27·8	1·7174	1·2173	33·1	2·4347
·5675	22·6	1·1350	·8649	27·9	1·7298	1·2247	33·2	2·4494
·5725	22·7	1·1451	·8711	28·0	1·7422	1·2321	33·3	2·4642
·5776	22·8	1·1552	·8773	28·1	1·7547	1·2395	33·4	2·4790
·5827	22·9	1·1654	·8836	28·2	1·7672	1·2469	33·5	2·4939
·5878	23·0	1·1756	·8899	28·3	1·7798	1·2544	33·6	2·5088
·5929	23·1	1·1858	·8962	28·4	1·7924	1·2619	33·7	2·5238
·5980	23·2	1·1961	·9025	28·5	1·8050	1·2694	33·8	2·5388
·6032	23·3	1·2064	·9088	28·6	1·8177	1·2769	33·9	2·5538
·6084	23·4	1·2168	·9152	28·7	1·8304	1·2844	34·0	2·5689
·6136	23·5	1·2272	·9216	28·8	1·8432	1·2920	34·1	2·5840
·6188	23·6	1·2377	·9280	28·9	1·8560	1·2996	34·2	2·5992
·6241	23·7	1·2482	·9344	29·0	1·8689	1·3072	34·3	2·6144
·6294	23·8	1·2588	·9409	29·1	1·8818	1·3148	34·4	2·6297
·6347	23·9	1·2694	·9474	29·2	1·8948	1·3225	34·5	2·6450
·6400	24·0	1·2800	·9539	29·3	1·9078	1·3302	34·6	2·6604
·6453	24·1	1·2907	·9604	29·4	1·9208	1·3379	34·7	2·6758
·6507	24·2	1·3014	·9669	29·5	1·9339	1·3456	34·8	2·6912
·6561	24·3	1·3122	·9735	29·6	1·9470	1·3533	34·0	2·7067

EXAMPLE.

Required the content of a cask whose length is 40 inches, and bung and head diameters 32 and 24 inches.

To the right of 32, the bung diameter, is	2·2756 the bung area;
and to the left of 24, the head diameter, is	·6400 the head area,
the sum	2·9156
multiplied by	40 the length
produces	116·624 wine gallons for the content required.

To work by the sliding rule, we may proceed as in prob. 1 chap. 3, using 30 as the gauge point instead of 29·7; thus, having set 40 on C to 30 on D, against

A still farther improvement is to be made of this method, by transferring the table to scales upon a rod, or the inside of a Brahan's rule, instead of the two useless diagonal scales, a scale of inches and tenths corresponding to the scale of areas, A part of such a scale is represented below.

And for the making of such a scale I shall here subjoin a table of diameters computed from assumed equi-different areas, which is much fitter for this purpose than the foregoing table of areas, which was computed from assumed equi-different diameters. This table of diameters was computed by *Mr.* AB. CROCKER, then of *Ilminster*, but now of *Frome*, *Somersetshire*, and who also communicated the substance of this chapter, together with the collection of casks at the end of the last.

A Table

against 32 and 24 on D, stand 45·6 and 25·5 on C; then $2 \times 45·6 + 25·5 = 91·2 + 25·5 = 116·7$ wine gallons the content nearly as before.

And for the content in ale gallons, we have $116·6 \times \frac{9}{11} = 95·4$

CHAP-

A Table of Diameters for Making Scales.

Head area	Diam-eter	Bung area	Head area	Diam-eter	Bung area	Head area	Diam-eter	Bung area	Head area	Diam-eter	Bung area
·30	16·43	·60	·58	22·85	1·16	·86	27·82	1·72	1·14	32·03	2·28
·31	16·70	·62	·59	23·04	1·18	·87	27·98	1·74	1·15	32·17	2·30
·32	16·97	·64	·60	23·24	1·20	·88	28·14	1·76	1·16	32·31	2·32
·33	17·24	·66	·61	23·43	1·22	·89	28·30	1·78	1·17	32·45	2·34
·34	17·50	·68	·62	23·62	1·24	·90	28·45	1·80	1·18	32·59	2·36
·35	17·76	·70	·63	23·81	1·26	·91	28·61	1·82	1·19	32·73	2·38
·36	18·00	·72	·64	24·00	1·28	·92	28·78	1·84	1·20	32·86	2·40
·37	18·26	·74	·65	24·18	1·30	·93	28·93	1·86	1·21	33·00	2·42
·38	18·50	·76	·66	24·37	1·32	·94	29·09	1·88	1·22	33·14	2·44
·39	18·73	·78	·67	24·56	1·34	·95	29·24	1·90	1·23	33·27	2·46
·40	18·97	·80	·68	24·74	1·36	·96	29·40	1·92	1·24	33·41	2·48
·41	19·21	·82	·69	24·92	1·38	·97	29·55	1·94	1·25	33·54	2·50
·42	19·44	·84	·70	25·10	1·40	·98	29·70	1·96	1·26	33·67	2·52
·43	19·67	·86	·71	25·28	1·42	·99	29·85	1·98	1·27	33·81	2·54
·44	19·90	·88	·72	25·45	1·44	1·00	30·00	2·00	1·28	33·94	2·56
·45	20·12	·90	·73	25·63	1·46	1·01	30·15	2·02	1·29	34·07	2·58
·46	20·35	·92	·74	25·81	1·48	1·02	30·30	2·04	1·30	34·20	2·60
·47	20·57	·94	·75	25·98	1·50	1·03	30·45	2·06	1·31	34·33	2·62
·48	20·79	·96	·76	26·15	1·52	1·04	30·60	2·08	1·32	34·47	2·64
·49	21·00	·98	·77	26·33	1·54	1·05	30·74	2·10	1·33	34·60	2·66
·50	21·21	1·00	·78	26·50	1·56	1·06	30·88	2·12	1·34	34·73	2·68
·51	21·42	1·02	·79	26·67	1·58	1·07	31·03	2·14	1·35	34·86	2·70
·52	21·63	1·04	·80	26·84	1·60	1·08	31·18	2·16	1.36	34·98	2·72
·53	21·84	1·06	·81	27·00	1·62	1·09	31·32	2·18	1·37	35·11	2·74
·54	22·05	1·08	·82	27·17	1·64	1·10	31·47	2·20	1·38	35·25	2·76
·55	22·25	1·10	·83	27·33	1·66	1·11	31·61	2·22	1·39	35·37	2·78
·56	22·45	1·12	·84	27·50	1·68	1·12	31·75	2·24	1·40	35·50	2·80
·57	22·65	1·14	·85	27·66	1·70	1·13	31·90	2·26	1·41	35·62	2·82

Now as to the method of forming this table; since any area is equal to the square of its diameter divided by 900, the diameter must be equal to the root of the product of 900 drawn into the

CHAPTER VII.

A new and very exact Method of Computing the Content of a Cask of any Form from Three Dimensions only.

*Add into one sum 39 times the square of the bung diameter, 25 times the square of the head diameter,

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the area; so that there will belong
to the first area ·30, the diam. $\sqrt{\cdot 30 \times 900}$ or $\sqrt{30 \times 9} = \sqrt{A}$;
to the 2d area ·31, the diam. $\sqrt{31 \times 9} = \sqrt{A + 9} = \sqrt{B}$;
to the 3d area ·32, the diam. $\sqrt{32 \times 9} = \sqrt{B + 9} = \sqrt{C}$, &c.

Hence it is evident that the successive products, out of which the square root is to be extracted, will be obtained by the continual addition of 9; and then extracting the roots, by means of the table of logarithms, will give the several diameters required.

But if you have a table of square roots by you, such as that at the end of my *Mathematical Miscellany*, since the root of 9 is 3, the most expeditious method of making the table, is to take the square roots of 30, 31, 32, &c, out of the table of roots, and then multiply them by 3 for the several corresponding diameters.

* This rule I have made out from a method of investigating the contents of casks, which was hinted by *Mr.* JAMES DAVIDSON, *Teacher of the Mathematics at Dundee, in North Britain*; the invention of which take as below.

It appearing reasonable that, in determining a proper rule for computing the contents of casks, we ought to have regard to the general methods which are used by coopers in constructing them; application was therefore made to several ingenious workmen in that way, from whose account it appeared that, ordinarily, the sides or edges of each stave of a cask, for about one-third of its length next each end, are made tapering in a straight line, and that for the middle third part they were curved, or made convex, to form the bulge or middle of the cask.

From this plain account then, we are naturally led to consider one-third of a cask, at each end, as the frustum of a cone; and the middle part is most properly considered as in the form of a parabolic curve, for these reasons; viz. 1st, that, on account of the shortness of this curved part, it may be considered but as a small part near the vertex of the curve; 2d, that all curves near the vertex differ insensibly from the parabola; and 3d, that its rules are

meter, and 26 times the product of the diameters; multiply the sum by the length, and the product by ·00034; then the last product divided by 9 will give the wine gallons, and divided by 11 will give the ale gallons.

E X-

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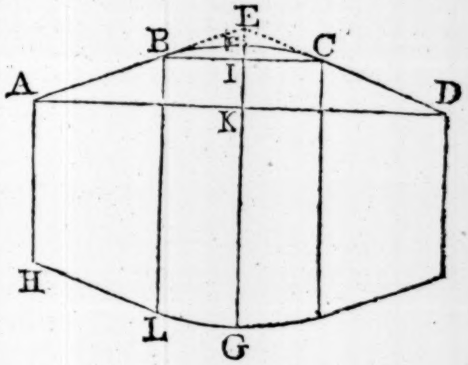
are most easily adapted to practical uses. We shall therefore suppose the middle third part of a cask to be the middle frustum of a parabolic spindle, and the ends as two frustums of a cone, and investigate its contents upon those principles in the following manner.



Let AB and CD be the two right-lined parts, and BC the parabolic part; then produce AB and DC to meet in E, and draw the lines as are evident by the figure. Put the length AD of the cask $= L$, the bung diameter $EG = B$, and the head diameter $AH = H$. Then since AB has the same direction as EB at E, ABE will be a tangent to the parabola EF, and therefore $FI = \frac{1}{2}EI$; but $BI = \frac{1}{3}AK$, and hence by similar triangles $EI = \frac{1}{3}EK$; consequently $FI = \frac{1}{2}EI = \frac{1}{6}EK = \frac{1}{5}FK = \frac{B - H}{10}$; so that the common diameter $BL = FG - 2FI = B - \frac{B - H}{5} = \frac{4B + H}{5}$; which call C.

Now, by the common rules for parabolic spindles and conic frustums, we obtain $\frac{8B^2 + 4BC + 3C^2}{15} \times \frac{Ln}{3} = \frac{328B^2 + 44BH + 3H^2}{25 \times 45} \times Ln$ for the parabolic, or middle part;
and $\frac{C^2 + CH + H^2}{3} \times \frac{2Ln}{3} = \frac{160B^2 + 280BH + 310H^2}{25 \times 45} \times Ln$ for the two ends; and the sum of these two give $(488B^2 + 324BH + 313H^2) \times \frac{Ln}{1125} = (39B^2 + 26BH + 25H^2) \times \frac{Ln}{90}$ nearly, for the content in inches; n being $= ·785398$. And the quantity

EXAMPLE

Required the content of a cask whose length is 40, bung diameter 32, and head diameter 24 inches.

Here $(39 \times 32^2 + 26 \times 32 \times 24 + 25 \times 24^2) \times 40 \times \cdot 00034 = 1010 \cdot 5$; which being divided by 9 and by 11, we obtain 112·3 wine gallons, or 91·9 ale gallons, for the content required. Agreeing nearly with the content found by the Diagonal Rod in page 573.

CHAPTER VIII.

Of the Ullage of Casks.

The ullage of a cask is now generally understood to be either the empty part, or the part filled, of a cask which is not quite full.

In

$\frac{\cdot 785398}{90}$ being divided by 231 gives $\frac{\cdot 00034}{9}$ the multiplier for wine gallons; and since 231 is to 282 as 9 to 11 very nearly, $\frac{\cdot 00034}{11}$ will be the multiplier for ale gallons, as in the rule.

And thus we have an easy and expeditious rule, which has not only the property of agreeing with the content, as computed from the method by four dimensions, when those dimensions are actually taken; but also agrees well with the real contents of casks; as hath been proved by several casks which have actually been filled with a true gallon measure, after their contents have been computed by this method.

To shew its agreement with the four-dimension method, take any one of the examples out of the collection at the end of chapter 5, as for instance No. 14, whose content by that method is 113·1 gallons; and by this method it will be found to come out nearly 112. Again, if we take any other, as suppose the last, whose content was 282·2, it will come out by this method 283·2. So that in these two instances, taken at random, although the contents be very large, they agree to within a gallon; the one method exceeding in the former case, and the other in the latter.

In this busineſs, caſks are conſidered as in two poſitions, viz. with their axes either parallel or perpendicular to the horizon, that is, either lying along, or ſtanding upright upon one end. The ullage of a caſk, either ſtanding or lying, may be determined for any particular variety, by the preceding parts of this work; but the rules thence obtained are too complex for ordinary practice, eſpecially for a lying caſk; I ſhall not, therefore, introduce thoſe rules into this place, but ſhall content myſelf with ſetting down ſuch approximations as have been found to agree beſt with expedition and accuracy; together with the univerſal method by three diameters, which is not only ſufficiently eaſy, but alſo the moſt accurate for this purpoſe.

PROBLEM I.

To find the Ullage of a Standing and Lying Caſk by the Lines SS *and* SL *on the Sliding Rule.*

By ſome of the preceding chapters, find the whole content of the caſk.

Then ſet the length on N to 100 on SS for a ſeg. ſtanding; or ſet the bung diameter on N to 100 on SL for a ſeg. lying; then againſt the wet or dry inches on N, is a number to be reſerved. Then ſet 100 on B to the reſerved number on A; and againſt the whole content on B, will be found the ullage on A. And this ullage will be either the empty part, or the part filled, according as the dry or wet inches were uſed, in finding the reſerved number.

EXAMPLE I.

Required the ullage of a ſtanding caſk, whoſe length is 40, bung diameter 32, head diameter 24, wet inches 10, and conſequently the dry inches 30.

By the laſt chapter, the content was nearly 92 ale gallons: Hence, having ſet 40 on N to 100 on SS, and againſt 10 on N found 23 on SS; then ſet 100 on B to 23 on A, and againſt 92 on B is found 21·2 ale gallons on A, for the quantity remaining in the caſk.

If the dry inches 30 be uſed, the reſerved ſegment will be found to be 76·7; and then the correſponding ullage is 70·2, for the gallons drawn off.

And the ſum of theſe two parts is 91·4 gallons, which is about half a gallon leſs than the whole content; which error would be inconſiderable if it were to be divided between the two parts; but inſtead of that, commonly the one part is too little, and the other too great, by which it happens that the error in each part moſtly exceeds that of their ſum.

EXAMPLE. II.

Let the dimenſions and content be the ſame in the caſe of the lying caſk, alſo the wet inches 8, and conſequently the dry inches 24.

Having ſet 32 on N to 100 on SL, againſt 8 and 24 on N, are the reſerved ſegments 17·8 and 82·5 on SL. Then having ſet 100 on B to 92 on A, againſt 17·8 and 82·5 on B, are 16·4 and 76 on A; which are the parts filled and empty reſpectively, and whoſe ſum is 92·4, near half a gallon too much.

PRO-

PROBLEM II.

To find the Ullage of a Standing Cask by the Pen.

RULE I.

*Add all together the ſquare of the diameter at the ſurface of the liquor, the ſquare of the diameter of the neareſt end, and the ſquare of double the diameter taken in the middle between the other two; then multiply the ſum by the length between the ſurface and neareſt end, and the product again by $\cdot0004\frac{2}{3}$ for ale gallons, or by $\cdot0005\frac{2}{3}$ for wine gallons, in the leſs part of the caſk, whether empty or filled.

EXAMPLE.

Taking the above diameters 24, 27, and 29 inches, and the wet inches 10; we ſhall have $(24^2 + 54^2 + 29^2) \times 10 \times \cdot0004\frac{2}{3} = 4333 \times \cdot004\frac{2}{3} = 20\cdot2$ gallons for the ullage required.

RULE II.

As the ſquare of the length of the caſk, is to the ſquare of the difference between the ſaid length and wet or dry inches, viz. the leſs of them; ſo is the difference between the bung and head diameters, to a number, which being taken from the bung diameter, will leave the diameter of a cylinder of the ſame length

* The diameters at the ſurface of the liquor, and in the middle between it and the neareſt head, will eaſily be obtained, by ſuſpending a plummet by a rule laid over the middle of the head, ſo as juſt to touch the bulge of the caſk, and then meaſuring the diſtance of the ſtring from the ſide of the caſk at the ſurface of the liquor and in the middle between it and the neareſt head; for then the doubles of theſe meaſures taken from the bung diameter, will leave the ſaid diameters required.

length with, and nearly equal to the part filled or empty, viz. the less of them*.

EXAMPLE.

Taking the same example as in problem 1, we have as $40^2 : (40 - 10)^2 :: 32 - 24 : 4\frac{1}{2}$; and $32 - 4\frac{1}{2} = 27\frac{1}{2}$ the mean diameter.

Then $27{\cdot}5^2 \times 10 \times {\cdot}002\frac{7}{9} = 21$ ale gallons is the ullage required.

Note. The number $\dot{\cdot}0027851$ or $002\frac{7}{9}$ nearly, is the constant multiplier $\frac{{\cdot}785398\ \&c}{282}$.

PROBLEM III.

To find the Ullage of a Lying Cask by the Pen.

Divide the wet inches by the bung diameter; find the quotient in the first column of the table of circular segments, at the end of the book, and take out the segment opposite to it; then multiply this segment by the whole content of the cask, and the product again by $1\frac{1}{4}$ for the ullage nearly†.

EX-

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* This rule is founded on the reasonable supposition, that for a small part of cask, the diameter in the middle may be considered as a mean diameter; which is strictly true when the cask is paraboloidal, and very near the truth when parabolic-spindular; but the said middle diameter, as given in the rule, is computed from the latter form, because the rule is thereby rendered not only easier, but generally more exact.

Thus, by the property of the parabola, $L^2 : (L - I)^2 :: B - H : \frac{B - H}{L^2} \times (L - I)^2$, and $B - \frac{B - H}{L^2} \times (L - I)^2 = M$; where B is the bung, H the head, and M the mean diameter; also L is the length of the cask, and I the wet or dry inches.

† This method is evidently nothing else than taking the whole content, in such proportion to the ullage, as the whole bung circle bears to the segment of it cut off by the surface of the liquor; and is nearer to the truth than any other practical rule that I can find.

EXAMPLE.

Taking the ſame caſk as before, whoſe length is 40, bung diameter 32, head diameter 24; and ſuppoſing the wet inches to be 8.

By chapter 7, the whole content is nearly 92 ale gallons. Then $\frac{8}{32} = \frac{1}{4} = \cdot 25$; oppoſite to which, in the table of areas, is the ſegment ·15354621; hence $92 \times \cdot 15354621 \times 1\frac{1}{4} = 18$ ale gallons, the ullage required.

SCHOLIUM.

Having delivered the neceſſary rules for meaſuring caſks, &c, I do not ſuppoſe that any thing more of the ſubject of gauging is neceſſary to be given in this book. For as to ciſterns, couches, &c, tuns, coolers, &c, coppers, ſtills, &c, which are firſt ſuppoſed to be in the form of ſome of the ſolids in the former parts of this work, and then meaſured accordingly, no perſon can be at a loſs concerning them, who knows any thing of ſuch ſolids in general; and to treat of them here, would induce me to a long and tedious repetition, only for the ſake of pointing out the proper multipliers or diviſors; which is, I think, a reaſon very inadequate to ſo cumberſome an increaſe of my book.

I ſhall only juſt obſerve, that when tuns, &c, of oval baſes are to be gauged; as thoſe baſes really meaſure to more than true ellipſes of the ſame length and breadth, they ought to be meaſured by the equi-diſtant ordinate method, delivered in ſection 2 of part 4.

And that when caſks are met with, which have different head diameters, they may be deemed incomplete caſks, and their contents conſidered and meaſured as the ullage of a caſk.

SECTION III.

OF THE WORKS OF ARTIFICERS.

THE artificers whose works are here to be treated of, are 1 Bricklayers, 2 Carpenters and Joiners, 3 Glaziers, 4 Masons, 5 Painters, 6 Pavers, 7 Plasterers, 8 Plumbers, and 9 Slaters and Tilers.

Artificers compute the contents of their works by several different measures.

As glazing and masonry by the foot:

Painting, plastering, paving, &c, by the yard of 9 square feet:

Flooring, partitioning, roofing, tiling, &c, by the square, of 100 square feet:

And brickwork either by the yard of 9 square feet, or by the perch, or square rod or pole, containing $272\frac{1}{4}$ square feet, or $30\frac{1}{4}$ square yards, being the square of the rod or pole of $16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards long.

As this number $272\frac{1}{4}$ is a troublesome number to divide by, the $\frac{1}{4}$ is often omitted in practice, and the content in feet divided only by the 272. But as this is not exact, it will be both better and easier to multiply the feet by 4, and then divide successively by 9, 11, and 11. Also to divide square yards by $30\frac{1}{4}$, first multiply them by 4, and then divide twice by 11.

All works, whether superficial or solid, are computed by the rules proper to the figure of them, whether it be a triangle, or rectangle, a parallelopiped, or any other figure.

For taking measures, the most common instrument is what is called the carpenters rule, of which it will be necessary here to give a description.

CHAP-

CHAPTER I.

Of the Common or Carpenters Rule.

This inſtrument is otherwiſe called the ſliding rule; and it is much uſed in timber meaſuring and artificers works, both for taking the dimenſions, and caſting up the contents.

The inſtrument conſiſts of two equal pieces, each a foot in length, which are connected together by a folding joint.

One ſide or face of the rule, is divided into inches, and half-quarters, or eighths. On the ſame face alſo are ſeveral plane ſcales, divided into twelfth parts by diagonal lines; which are uſed in planning dimenſions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely each foot into 10 equal parts, and each of theſe into 10 parts again: ſo that by means of this laſt ſcale, dimenſions are taken in feet and tenths and hundredths, and multiplied as common decimal numbers, which is the beſt way.

On the one part of the other face are four lines, marked A, B, C, D, the two middle ones B and C being on a ſlider, which runs in a groove made in the ſtock. The ſame numbers ſerve for both theſe two middle lines, the one being above the numbers, and the other below.

Theſe four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or loweſt line D, is a ſingle one, proceeding from 4 to 40 It is alſo called the girt line, from its uſe in caſting up the contents of trees and timber; and upon it are marked WG at 17·15, and AG at 18·95, the wine and ale gauge points, to make this inſtrument ſerve the purpoſe of a gauging rule.

Upon the other part of this face there is a table of the

the value of a load, or 50 cubit feet, of timber, at all prices, from 6 pence to 2 shillings a foot.

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; and when the 1 at the beginning is accounted 10, then 1 in the middle is 100, and the 10 at the end 1000; and so on. And all the smaller divisions are altered proportionally.

PROBLEM I.

To Multiply Numbers together.

Suppose the two numbers 24 and 13.—Set 1 on B to 13 on A; then against 24 on B stands 312 on A, which is the required product of the two given numbers 24 and 13.

Note. In any operations, when a number runs beyond the end of the line, seek it on the other radius, or other part of the line, that is, take the 10th part of it, or the 100th part of it, &c, and increase the result proportionally 10 fold, or 100 fold, &c.

In like manner the product of 35 and 19 is 665,
and the product of 270 and 54 is 14580.

PROBLEM II.

To Divide by the Sliding Rule.

As suppose to divide 312 by 24.—Set the divisor 24 on B to the dividend 312 on A; then against 1 on B stands 13 the quotient on A.

Also 396 divided by 27 gives 14·6,
and 741 divided by 42 gives 17·6.

PRO-

PROBLEM III.

To Square any Number.

Suppose to square 23.—Set 1 on B to 23 on A; then against 23 on B stands 529 on A, which is the square of 23.

Or, by the other two lines, set 1 or 100 on C to the 10 on D, then against every number on D stands its square in the line C. So against 23 stands 529,
against 20 stands 400,
against 30 stands 900,
and so on.

If the given number be hundreds, &c, reckon the 1 on D for 100, or 1000, &c; then the corresponding 1 on C is 10000, or 1000000, &c. So the square of 230 is found to be 52900.

PROBLEM IV.

To Extract the Square Root.

Set 1 or 100, &c, on, C to 1 or 10, &c, on D; then against every number found on C, stands its square root on D.

So, against 529 stands its root 23,
against 400 stands its root 20,
against 900 stands its root 30,
against 300 stands its root 17·3,
and so on.

PROBLEM V.

To find a Mean Proportional between two Numbers.

As suppose between 29 and 430.—Set the one number 29 on C to the same on D; then against the other

other number 430 on C, ſtands their mean proportional 111 on D.

Alſo, the mean between 29 and 320 is 96·3, and the mean between 71 and 274 is 139.

PROBLEM VI.

To find a Third Proportional to two Numbers.

Suppoſe to 21 and 32.—Set the firſt 21 on B to the ſecond 32 on A; then againſt the ſecond 32 on B, ſtand 48·8 on A, which is the third proportional ſought.

Alſo the 3d proportional to 17 and 29 is 49·4, and the 3d proportional to 73 and 14 is 2·5.

PROBLEM VII.

To find a Fourth Proportional to three Numbers.

Or, to perform the Rule-of-Three.

Suppoſe to find a fourth proportional to 12, 28, and 114.—Set the firſt term 12 on B to the 2d term 28 on A; then againſt the third term 114 on B, ſtands 266 on A, which is the 4th proportional ſought.

Alſo the 4th proportional to 6, 14, 29, is 67·6, and the 4th proportional to 27, 20, 73, is 54·0.

CHAPTER II.

Of the different Meaſures uſed by different Artificers.

144	ſquare inches	=	a ſquare foot,
9	ſquare feet	=	a ſquare yard,
63	ſquare feet	=	7 ſquare yards = a rood,
100	ſquare feet	=	a ſquare,
$272\frac{1}{4}$	ſquare feet	=	$30\frac{1}{4}$ ſquare yards = a rod,

perch, or ſquare pole.

The

The above denominations are those by which most kinds of work are valued ; but some particular articles are valued by the foot running, or lineal measure.

A Table for changing hundredth Parts of Feet into Inches and Parts, and the contrary; by Means of which, Dimensions taken in one of these, may be readily changed to the other.

Centesms, or hundredth parts of a foot	Inches	12th pts.	Inches	8th pts. or half quarters	Centesms
1	0	1	0	1	1
2	0	3	0	2	2
3	0	4	0	3	3
4	0	6	0	4	4
5	0	7	0	5	5
6	0	9	0	6	6
7	0	10	0	7	7
8	1	0	1	0	8
9	1	1	2	0	17
10	1	2	3	0	25
20	2	5	4	0	33
30	3	7	5	0	42
40	4	10	6	0	50
50	6	0	7	0	58
60	7	2	8	0	67
70	8	5	9	0	75
80	9	7	10	0	83
90	10	10	11	0	92
100	12	0	12	0	100

EXAMPLE I.

If it be required to change 44 centesms into inches and 12ths.——Against 4 and 40 in the first column, stand

ſtand reſpectively 6 parts, and 4 in. 10 pts, in the ſecond; then the ſum of theſe two is 5 inches 4 parts, the number required.

EXAMPLE II.

If it be required to change 5 inches and $\frac{3}{4}$ or 6 eighths to centeſms.——Againſt 6 eighths and 5 inches, in the third column, ſtand reſpectively 6 and 42 in the laſt column; the ſum of which is 48 centeſms.

The reaſon why I have ſet 8ths in the third column, and not 12ths, is that the meaſurers who calculate by 12ths take the dimenſions in 8ths, and change them into 12ths; becauſe the inches upon the rule are divided into 8ths, and not 12ths.

Note. You may convert centeſms into inches and parts, and the contrary, without the table, by the following rule, viz.

Multiply centeſms by 12 for inches, cutting off two figures; and multiply theſe two figures by 12 again for parts, cutting off two figures here alſo. And divide 8ths or 12ths, with two cyphers annexed, by 8 or by 12 for inches; and divide inches by 12 for centeſms.

EXAMPLE I.

Thus, taking the firſt example above.

	44 centeſms	
	12	
Inches	5·28	That is 44 centeſms =
	12	5 inches 3 parts.
Parts	3·36	

EXAMPLE II.

And in the ſecond example above,

$$\begin{array}{r|l} 8 & 6\cdot00 \\ 12 & 5\cdot75 \\ \hline & 48 \text{ centeſms.} \end{array}$$

I ſhall now proceed to the different kinds of works, in treating of which ſeparately, I ſhall note the methods of meaſuring the ſeveral articles, with the allowances, deductions, &c; and then, at the end of the whole, illuſtrate them all in the ſeveral parts of a real building, whoſe plan and elevation ſhall be laid down, with its ſcale annexed, for comparing the dimenſions, &c.

CHAPTER III.

Of Bricklayers Work.

Brickwork is eſtimated at the rate of a brick and a half thick; and if a wall be more or leſs than this ſtandard thickneſs, it muſt be reduced to it, as follows:

Multiply the ſuperficial content of the wall by the number of half bricks in the thickneſs, and divide the product by 3.

The dimenſions of a building are uſually taken by meaſuring half round on the outſide, and half round it on the inſide; the ſum of theſe two gives the compaſs of the wall, to be multiplied by the height for the content of the materials.

Chimneys are by ſome meaſured as if they were ſolid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them.

And by others they are girt or meaſured round for their

their breadth, and the height of the ſtory is their height, taking the depth of the jambs for their thickneſs. And in this caſe no reduction is made for the vacuity from the floor to the mantle-tree, becauſe of the gathering of the breaſt and wings, to make room for the hearth in the next ſtory.

To meaſure the chimney ſhafts, which appear above the building; girt them about with a line for the breadth, to multiply by their height. And account their thickneſs half a brick more than it really is, in conſideration of the plaſtering and ſcaffolding.

All windows, doors, &c, are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the whole meaſure is taken for workmanſhip, and that all outſide meaſure too, namely, meaſuring quite round the outſide of the building, being in conſideration of the trouble of the returns or angles. There are alſo ſome other allowances, ſuch as double meaſure for feathered gable ends, &c.

EXAMPLES.

1. How many yards and rods of ſtandard brick-work are in a wall, whoſe length or compaſs is 57 feet 3 inches, and height 24 feet 6 inches; the walls being $2\frac{1}{2}$ bricks, or 5 half bricks, thick?

	Decimals.			*Duodecimals.*
	57·25			57 3
	24·5			24 6
	28625			234 0
	22900			114
	11450			28 7 6
	1402·625			1402 7 6
	5	half bricks thick		5
3	7013·125		3	7013 1 6
9	2337·708$\frac{1}{3}$	sq. feet	9	2337 8 6
	259·745$\frac{10}{27}$	yds		259 6 8 6
	4			4
11	1038·981$\frac{13}{27}$		11	1036
11	94·4528		11	94 2
rods	8·5866 Ans.			8 r 17yds 6f 8′ 6″

By the Sliding Rule.

	B		A		B		A
As	1	:	24$\frac{1}{2}$	: :	57$\frac{1}{4}$	:	1403.

Ex. 2. Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and 2$\frac{1}{2}$ bricks thick. Ans. 169·753 yards.

Ex. 3. A triangular gable is raised 17$\frac{1}{2}$ feet high, on an end wall whose length is 24 feet 9 inches, the thickness being 2 bricks: required the reduced content. Ans. 32·08$\frac{1}{3}$ yds.

Ex. 4. The end wall of a house is 28 feet 10 inches long, and 55 feet 8 inches high to the eaves; 20 feet high is 2$\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is 1$\frac{1}{2}$ brick

 thick;

thick; above which is a triangular gable, which riſes 42 courſes of bricks, of which every 4 courſes make a foot. What is the whole content in ſtandard meaſure?

Anſ. 253·626 yards.

CHAPTER IV.

Of Maſons Work.

To maſonry belongs all ſort of ſtone work; and the meaſure made uſe of is a foot, either ſuperficial or ſolid.

Walls, columns, blocks of ſtone or marble, &c, are meaſured by the cubic foot; and pavements, ſlabs, chimney pieces, &c, by the ſuperficial or ſquare foot.

Cubic or ſolid meaſure is uſed for the materials, and ſquare meaſure for the workmanſhip.

In the ſolid meaſure, the true length, breadth, and thickneſs, are taken, and multiplied continually together. In the ſuperficial, there muſt be taken the length and breadth of every part of the projection, which is ſeen without the general upright face of the building.

EXAMPLES.

1. Required the ſolid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick.

Decimals.		*Duodecimals.*		
53·5		53	6	
$12\frac{1}{4}$		12	3	
642·0		642	0	
13·375		13	4	6
655·375		655	4	6
2				2
1310·750	anſ.	1310	9	0

By

By the Sliding Rule.

B		A		B		A
1	:	$53\frac{1}{2}$	::	$12\frac{1}{4}$	:	655
1	:	655	::	2	:	1310

Ex. 2. What is the solid content of a wall, the length being 12 feet 3 inches, height 10 feet 9 inches, and 2 feet thick? Ans. 521·375 feet.

Ex. 3. Required the value of a marble slab, at 8s per foot; the length being 5 feet 7 inches, and breadth 1 foot 10 inches. Ans. £4 1 $10\frac{1}{2}$.

Ex. 4. In a chimney-piece, suppose the

length of the mantle and slab, each	4f	6in
breadth of both together - -	3	2
length of each jamb - -	4	4
breadth of both together -	1	9

Required the superficial content. Ans. 21f 10in.

CHAPTER V.

Of Carpenters and Joiners Work.

To this branch belongs all the wood-work of a house, such as flooring, partitioning, roofing, &c.

Note. Large and plain articles are usually measured by the square foot or yard, &c; but inriched mouldings, and some other articles, are often estimated by running or lineal measure, and some things are rated by the piece.

In measuring of joists, it is to be observed, that only one of their dimensions is the same with that of the floor; and the other will exceed the length of the room by the thickness of the wall, and $\frac{1}{3}$ of the same, because each end is let into the wall about $\frac{2}{3}$ of its thickness.

 No

No deductions are made for hearths, on account of the additional trouble and waste of materials.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In measuring of joiners work, the string is made to ply close to every part of the work over which it passes.

The measure of centering for cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length; but in groin-centering, it is usual to allow double measure, on account of their extraordinary trouble.

In roofing, the length of the house in the inside, together with $\frac{2}{3}$ of the thickness of one gable, is to be considered as the length; and the breadth is equal to double the length of a string which is stretched from the ridge down the rafter, and along the eaves-board, till it meets with the top of the wall.

For stair-cases, take the breadth of all the steps, by making a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step for the whole area.—By the length of a step is meant the length of the front and the returns at the two ends; and by the breadth is to be understood the girt of its two upper surfaces, or the tread and riser.

For the balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel post, for the length; and twice the length of the baluster upon the landing, with the girt of the hand-rail, for the breadth.

For wainscotting, take the compass of the room for the length; and the height from the floor to the ceiling, making the string ply close into all the mouldings, for the breadth.—Out of this must be made deductions for windows, doors and chimneys, &c: but workmanship

is

is counted for the whole, on account of the extraordinary trouble.

For doors, it is usually to allow for their thickness, by adding it into both the dimensions of length and breadth, and then multiply them together for the area. —If the door be pannelled on both sides, take double its measure for the workmanship; but if one side only be pannelled, take the area and its half for the workmanship.—*For the surrounding architrave,* girt it about the outermost part for its length; and measure over it, as far as it can be seen when the door is open, for the breadth.

Window-shutters, bases, &c, are measured in the same manner.

In the measuring of roofing for workmanship alone, all holes for chimney shafts and sky-lights are generally deducted.

But in measuring for work and materials, they commonly measure in all sky-lights, luthern-lights, and holes for the chimney shafts, on account of their trouble and waste of materials.

EXAMPLES.

1. Required the content of a floor 48 feet 6 inches long, and 24 feet 3 inches broad.

Decimals.		*Duodecimals.*		
48·5		48	6	
$24\frac{1}{4}$		24	3	
1940		204	0	
970		96		
12·125		12	1	6
1176·125	feet	1176	1	6
11·76125	squares Ans.	11 76	1	6

Ex. 2. A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many squares are in it?

Ans. 5sq. $98\frac{1}{8}$ feet.

Ex. 3. How many ſquares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning? Anſ. 18·3972 ſquares.

Ex. 4. What coſt the roofing of a houſe at 10s 6d a ſquare; the length, within the walls, being 52 feet 8 inches, and the breadth 30 feet 6 inches; reckoning the roof $\frac{3}{2}$ of the flat? Anſ. £12 12 11$\frac{3}{4}$.

Ex. 5. To how much, at 6s per ſquare yard, amounts the wainſcotting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compaſs 83 feet 8 inches; alſo the three window ſhutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and ſhutters, being worked on both ſides, are reckoned work and half work? Anſ. £36 12 2$\frac{1}{2}$

CHAPTER VI.

Of Slaters and Tilers Work.

In theſe articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of ſlates at the bottom, or for how much one row of ſlates or tiles is laid over another.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip. And in tiling and ſlating, it is common to add the length of the valley or hip to the content in feet.

Deductions are ſeldom made for chimney ſhafts or ſmall window holes.

EXAMPLES.

1. Required the content of a ſlated roof, the length being 45 feet 9 inches, and whole girt 34 feet 3 inches.

Decimals.

Decimals.		*Duodecimals.*
45·75		45 9
$34\frac{1}{4}$		34 3
18300		205 6
13725		135
114375		11 5 3
9) 1566·9375 feet		9) 1566 11 3
yds 174·104	Anſwer	174^{y} 11^{i} $3''$

Ex. 2. To how much amounts the tiling of a houſe, at 25s 6d per ſquare; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches, alſo the eaves projecting 16 inches on each ſide?

Anſ. £24 9 $5\frac{1}{2}$.

CHAPTER VII.

Of Plaſterers Work.

Plaſterers work is of two kinds, namely, ceiling, which is plaſtering upon laths; and rendering, which is plaſtering upon walls: which are meaſured ſeparately.

The contents are eſtimated either by the foot, or yard, or ſquare of 100 feet. Inriched mouldings, &c, are rated by running or lineal meaſure.

Deductions are to be made for chimneys, doors, windows, &c. But the windows are ſeldom deducted, as the plaſtered returns at the top and ſides are allowed to compenſate for the window opening.

EXAMPLES.

1. How many yards contains the ceiling, which is 43 feet 3 inches long, and 25 feet 6 inches broad?

Decimals.		*Duodecimals.*			
43·25		43	3		
$25\frac{1}{2}$		25	6		
21625		221	3		
8650		86			
21625		21	7	6	
9) 1102·875		9) 1102	10	6	
yds 122·541	Anſwer	122^{y}	4^{f}	10^{i}	$6''$

Ex. 2. To how much amounts the ceiling of a room, at 10d per yard; the length being 21 feet 8 inches, and the breadth 14 feet 10 inches? Anſ. £1 9 $8\frac{1}{4}$

Ex. 3. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8d and the latter at 3d per yard; allowing for the door of 7 feet by 3 feet 8, and a fire place of 5 feet ſquare? Anſ. £8 5 $6\frac{1}{2}$.

Ex. 4. Required the quantity of plaſtering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under ſide of the cornice, which girts $8\frac{1}{2}$ inches, and projects 5 inches from the wall on the upper part next the ceiling: deducting only for a door 7 feet by 4.

Anſ.	53^{yd}	5^{f}	3^{i}	of rendering
	18	5	6	of ceiling
		39	$0\frac{11}{12}$	of cornice.

CHAPTER VIII.

Of Painters Work.

Painters work is computed in ſquare yards. Every part is meaſured where the colour lies; and the mea-

measuring line is forced into all the mouldings and corners.

Windows are done at so much a piece. And it is usual to allow double measure for carved mouldings, &c.

EXAMPLES.

1. How many yards of painting contains the room which is 65 feet 6 inches in compass, and 12 feet 4 inches high?

Decimals.		*Duodecimals.*		
65·5		65		6
$12\frac{1}{3}$		12		4
786·0		786		0
$21·8\dot{3}$		21		10
9) $807·8\dot{3}$		9) 807		10
89·788	Answer	89	6	10

Ex. 2. The length of room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Ans. $73\frac{2}{27}$ yards.

Ex. 3. What cost the painting of a room, at 6d per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; also the door is 7 feet by 3 feet 6, and the window shutters to two windows each 7 feet 9 by 3 feet 6, but the breaks of the windows themselves are 8 feet 6 inches high, and 1 foot 3 inches deep: deducting the fire-place of 5 feet by 5 feet 6? Ans. £3 3 $10\frac{1}{2}$.

CHAPTER IX.

Of Glaziers Work.

Glaziers take their dimenſions either in feet, inches, and parts, or feet, tenths, and hundredths. And compute their work in ſquare feet.

In taking the length and breadth of a window, the croſs bars between the ſquares are included. Alſo windows of round or oval forms are meaſured as ſquare, meaſuring them to their greateſt length and breadth, on account of the waſte in cutting the glaſs.

EXAMPLES.

1. How many ſquare feet contains the window which is 4·25 feet long, and 2·75 feet broad?

Decimals.		*Duodecimals.*		
2·75		2	9	
$4\frac{1}{4}$		4	3	
11·00		11	0	
6875			8	3
11·6875	Anſwer	11	8	3

2. What will the glazing a triangular ſky-light come to at 10d per foot; the baſe being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches?

Anſ. £4 7 $2\frac{1}{4}$.

3. There is a houſe with three tier of windows, three windows in each tier, their common breadth 3 feet 11 inches;

	f	in
now the height of the firſt tier is	7	10
of the ſecond	6	8
of the third	5	4

Required the expence of glazing at 14d per foot.

Anſ. £13 11 $10\frac{1}{2}$.

Ex. 4.

Ex. 4. Required the expence of glazing the windows of a house at 13d a foot; there being three stories, and three windows in each story;

the height of the lower tier is	7^f	9^{in}
of the middle - -	6	6
of the upper - -	5	$3\frac{1}{4}$
and of an oval window over the door	1	$10\frac{1}{2}$

The common breadth of all the windows being 3 feet 9 inches. Ans. £12 5 6.

CHAPTER X.

Of Pavers Work.

Pavers work is done by the square yard. And the content is found by multiplying the length by the breadth.

EXAMPLES.

1. What cost the paving a foot-path at 3s 4d a yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches?

Decimals.			*Duodecimals.*
	$35\cdot\dot{3}$		35 4
	$8\frac{1}{4}$		8 3
	$282\cdot6\dot{6}$		282 8
	$8\cdot8\dot{3}$		8 10
	9)291·5		9)291 6
	$32\cdot3\dot{8}$	Content	32 3 6
2s is $\frac{1}{10}$	$3\cdot238\dot{8}$		£0 3 4
1 is $\frac{1}{2}$	$1\cdot619\dot{4}$		4
4d is $\frac{1}{3}$	5398		0 13 4
	1 5·3981		8
	20		5 6 8
	s 7·9620	3^{rd} is $\frac{1}{3}$	1 $1\frac{1}{4}$
	12	6^f is $\frac{1}{6}$	$2\frac{1}{4}$
	d 11·5440	Answer £	5 7 $11\frac{1}{2}$

Ex. 2. What coſt the paving a court at 3s 2d per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? Anſ. £7 4 4$\frac{1}{7}$.

Ex. 3. What will be the expence of paving a rectangular court yard, whoſe length is 63 feet, and breadth 45 feet; in which there is laid a foot-path of 5 feet 3 inches broad, running the whole length, with broad ſtones at 3s a yard; the reſt being paved with pebbles, at 2s 6d a yard? Anſ. £40 5 10$\frac{1}{2}$.

CHAPTER XI.

Of Plumbers Work.

Plumbers work is commonly done by the pound or hundred weight; but I ſhall below inſert a method of diſcovering the weight by the meaſure of it.

Sheet-lead uſed in roofing, guttering, &c, is commonly between 7 and 12 pound weight to the ſquare foot; but the following table ſhews by inſpection the particular weight of a ſquare foot for each of ſeveral thickneſſes.

Thickneſs.	lb. to a ſq. foot	Thickneſs.	lb. to a ſq. foot	Thickneſs.	lb. to a ſq. foot.	Thickneſs.	lb. to a ſq. foot.
·10	5·899	$\frac{1}{8}$	7·373	·15	8·848	·18	10·618
·11	6·489	·13	7·668	·16	9·438	·19	11·207
$\frac{1}{9}$	6·554	·14	8·258	$\frac{1}{6}$	9·831	·2 = $\frac{1}{5}$	11·797
·12	7·078	$\frac{1}{7}$	8·427	·17	10·028	·21	12·387

In this table the thickneſs is ſet down in tenths and hundredths, &c, of an inch; and the annexed correſponding numbers are the weights in avoirdupois pounds, and thouſandth parts of a pound. So the weight of a ſquare foot of $\frac{1}{10}$ or $\frac{10}{100}$ of an inch thick,

thick, is 5 pounds and 899 thousandth parts of a pound; and the weight of a square foot to $\frac{1}{9}$ of an inch thickness, is 6 pounds and $\frac{554}{1000}$ of a pound.

Leaden pipe of an inch bore, is commonly 13 or 14 lb to the yard in length.

EXAMPLES.

1. How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at $8\frac{1}{2}$lb to the square foot?

Decimals.		*Duodecimals.*
39·5		39 6
$3\frac{1}{4}$		3 3
118·5		118 6
9·875		9 10 6
128·375		128 4 6
$8\frac{1}{2}$		$8\frac{1}{2}$
		1024
1027·000		64
64·1875		$2\frac{5}{6}$
		$0\frac{17}{48}$
1091·1875	Answer	$1091\frac{9}{48}$ lb.

Ex. 2. What cost the covering and guttering a roof with lead at 18s the cwt; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former 9·831lb, and the latter 7·373lb to the square foot? Ans. £115 9 $1\frac{1}{2}$.

CHAP-

CHAPTER XII.

OF VAULTED AND ARCHED ROOFS.

ARCHED roofs are either vaults, domes, ſaloons, or groins.

Vaulted roofs are formed by arches ſpringing from the oppoſite walls, and meeting in a line at the top.

Domes are made by arches ſpringing from a circular or polygonal baſe, and meeting in a point at the top.

Saloons are formed by arches connecting the ſide walls to a flat roof or ceiling in the middle.

Groins are formed by the interſection of vaults with each other.

Vaulted roofs are commonly of the three following ſorts:

1. Circular roofs, or thoſe whoſe arch is ſome part of the circumference of a circle.

2. Elliptical or oval roofs, or thoſe whoſe arch is an oval, or ſome part of the circumference of an ellipſis.

3. Gothic roofs, or thoſe which are formed by two circular arcs, ſtruck from different centers, and meeting in a point directly over the middle of the breadth, or ſpan of the arch.

PROBLEM I.

To find the Surface of a Vaulted Roof.

Multiply the length of the arch by the length of the vault, and the product will be the ſuperficies.

Note. To find the length of the arch, make a line or ſtring ply cloſe to it, quite acroſs from ſide to ſide.

EXAM-

EXAMPLES.

1. Required the surface of a vaulted roof, the length of the arch being 31·2 feet, and the length of the vault 120 feet.

$$\begin{array}{r} 31{\cdot}2 \\ 120 \\ \hline \text{Ans. } 3744{\cdot}0 \text{ square feet.} \end{array}$$

Ex. 2. How many square yards are in the vaulted roof, whose arch is 42·4 feet, and the length of the vault 106 feet? Ans. $499{\cdot}3\dot{7}$ yds.

PROBLEM II.

To find the Content of the Concavity of a Vaulted Roof.

Multiply the length of the vault by the area of one end, that is, by the area of a vertical transverse section, for the content.

NOTE. When the arch is an oval, multiply the span by the height, and the product by ·7854 for the area.

EXAMPLES.

1. Required the content of the concavity of a semicircular vaulted roof, the span or diameter being 30 feet, and the length of the vault 150 feet.

$$\begin{array}{rl} {\cdot}7854 & \\ 900 & \text{the square of 30} \\ \hline 2)706{\cdot}86 & \\ 353{\cdot}43 & \text{area of the end} \\ 150 & \text{the length} \\ \hline 1767150 & \\ 35343 & \\ \hline 5301{\cdot}450 & \text{the answer.} \end{array}$$

Ex. 2.

Ex. 2. What is the content of the vacuity of an oval vault, whoſe ſpan is 30 feet, and height 12 feet; the length of the vault being 60 feet? Anſ. ~~1694·64~~. 1696

Ex. 3. Required the content of the vacuity of a gothic vault, whoſe ſpan is 50 feet, the chord of each arc 50 feet, and the diſtance of each arc from the middle of theſe chords 15 feet; alſo the length of the vault 20. Anſ. 42988·8.

PROBLEM III.

To find the Superficies of a Dome.

Find the area of the baſe, and double it; then ſay, as the radius of the baſe, is to the height of the dome, ſo is the double area of the baſe, to the ſuperficies.

NOTE. For the ſuperficies of a hemiſpherical dome, take the double area of the baſe only.

EXAMPLES.

1. To how much comes the painting of an octagonal ſpherical dome, at 8d per yard; each ſide of the baſe being 20 feet?

	4·828427	tabular area
	400	ſquare of 20
	1931·3708	area of the baſe
	2	
9)	3862·7416	ſuperficies in feet
	429·1934	yards
	8	
13	3433·5472	
2,0	28·6 1½d	
	£14 6 1½	anſwer

Ex. 2.

Ex. 2. Required the superficies of a hexagonal spherical dome, each side of the base being 10 feet.

Ans. 519·6152.

Ex. 3. What is the superficies of a dome with a circular base, whose circumference is 100 feet, and height 20 feet? Ans. 2000 feet.

PROBLEM IV.

To find the Solid Content of a Dome.

Multiply the area of the base by $\frac{2}{3}$ of the height.

EXAMPLES.

1. Required the solid content of an octagonal dome, each side of the base being 20 feet, and the height 21 feet.

$$\begin{array}{rl} 4{\cdot}828427 & \\ 400 & \\ \hline 1931{\cdot}3708 & \text{area of the base} \\ 14 & \tfrac{2}{3} \text{ of the height} \\ \hline 77254832 & \\ 19313708 & \\ \hline 27039{\cdot}1912 & \text{answer} \\ \hline \end{array}$$

Ex. 2. What is the solid content of a spherical dome, the diameter of whose circular base is 30 feet?

Ans. 7068·6 feet.

PROBLEM V.

To find the Superficies of a Saloon.

Find its breadth by applying a string close to it

 across

across the surface. Find also its length by measuring along the middle of it, quite round the room.

Then multiply these two together for the surface.

EXAMPLE.

The girt across the face of a saloon being 5 feet, and its mean compass about 100 feet, required the area or superficies.

$$\begin{array}{r} 100 \\ 5 \\ \hline 500 \end{array}$$ answer.

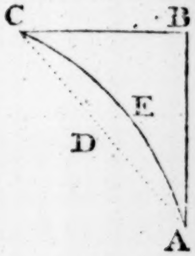
PROBLEM VI.

To find the Solid Content of a Saloon.

Multiply the area of a transverse section by the compass taken round the middle part. Subtract this product from the whole vacuity of the room, supposing the walls to go upright all the height to the flat ceiling. And the difference will be the answer.

EXAMPLE.

If the height AB of the saloon be 3·2 feet, the chord ADC of its front 4·5, and the distance DE of its middle part from the arch be 9 inches; required the solidity, supposing the mean compass to be 50 feet.



2) 4·5

2·25 AD
2·25

1125
450
450

$5·0625$ AD^2
$·5625$ DE^2

5·6250 (2·37 AE
4
43 | 162
3 | 129
46 | 33

0·75 DE
0·75

375
525

·5625

4
3) 9·48
$3·16 = \frac{4}{3}$ AE
4·50 AC

7·66
$·3 = \frac{4}{10}$ DE
2·298 area seg. ADCEA.

Again 3·2
3·2
64
96
10·24

4·5
4·5
225
180
$20·25$ AC^2
$10·24$ AB^2

10·01 ($3·16 = BC$
9
61 | 1·01
1 | 61
40

$1·6 = \frac{1}{2}AB$
1896
316
5·056 area of triangle ABC
2·298 area seg.
2·758 area of section AECBA
50 compass
137·900 content of the solid part

Then

Then this taken from the whole upright ſpace, will leave the content of the vacuity contained within the room.

PROBLEM VII.

To find the Concave Superficies of a Groin.

To the area of the baſe, add $\frac{1}{7}$ part of itſelf, for the ſuperficial content.

EXAMPLES.

1. What is the ſuperficial content of the groin arch, raiſed on a ſquare baſe of 15 feet on each ſide?

$$\begin{array}{rl} 15 & \\ 15 & \\ \hline 75 & \\ 15 & \\ \hline 7\,)\,225 & \text{area of the baſe} \\ 32\frac{1}{7} & \text{its 7th part} \\ \hline 257\frac{1}{7} & \text{anſwer} \\ \hline \end{array}$$

Ex. 2. Required the ſuperficies of a groin arch, raiſed on a rectangular baſe, whoſe dimenſions are 20 feet by 16. Anſ. $365\frac{5}{7}$.

PROBLEM VIII.

To find the Solid Content of a Groin Arch.

Multiply the area of the baſe by the height; from the product ſubtract $\frac{1}{10}$ of itſelf, and the remainder will be the content of the vacuity.

EXAM-

EXAMPLES.

1. Required the content of the vacuity within a groin arch, ſpringing from the ſides of a ſquare baſe, each ſide of which is 16 feet.

$$\begin{array}{rl} 16 & \\ 16 & \\ \hline 96 & \\ 16 & \\ \hline 256 & \text{area of baſe} \\ 8 & \text{height or radius} \\ \hline 2048 & \\ 204\frac{4}{5} & \frac{1}{10}\ \text{ſubtract} \\ \hline 1843\frac{1}{5} & \text{anſwer} \\ \hline \end{array}$$

2. What is the content of the vacuity below an oval groin, the ſide of its ſquare baſe being 24 feet, and its height 8 feet? Anſ. $4147\frac{1}{5}$.

NOTES.

1. To find the ſolid content of the brick or ſtone work, which forms any arch or vault: Multiply the area of the baſe by the height, including the work over the top of the arch; and from the product ſubtract the content of the vacuity found by the foregoing problems; and the remainder will be the content of the ſolid materials.

2. In groin arches however, it is uſual to take the whole as ſolid, without deducting the vacuity, on account of the trouble and waſte of materials, attending the cutting and fitting them to the arch.

CHAPTER XIII.

GENERAL ILLUSTRATION.

HAVING gone through the meaſurable parts of building, and noted the methods of meaſuring them, and of computing the contents; I proceed now to the general illuſtration of the whole, by aſſigning the dimenſions of a houſe, and from thence computing the contents of the ſeveral different kinds of works ſuppoſed to be uſed in it; in the performing of which are ſhewn the methods of ruling the book, and entering the dimenſions, with the contents; then the method of abſtracting the contents, and laſtly of forming the bills of expences of the work and materials.

At the end of this chapter is given a plate, containing the elevation and plans of the ſeveral ſtories of the houſe, accurately delineated from a particular ſcale, inſerted in the plate; by which the reader may, with a pair of compaſſes, meaſure and compare the dimenſions of moſt of the articles, as he proceeds through them. In this example is uſed a houſe of only two ſtories, and two rooms on each floor, as thoſe are very ſufficient for explaining the ſeveral methods and works here treated of.

The columns of numbers in the following forms, are ſufficiently explained by the titles at the tops of them; excepting the figures 2, 3, &c, in the firſt column, and prefixed to the dimenſions; the meaning of which figures is, that the contents ariſing from the dimenſions to which they are prefixed, are to be multiplied by 2, 3, &c, and then ſet down in the column of contents.

Of

Of the Digging *of the* Cellars.

Feet	*In.*	
54	0	length
28	0	breadth
8	0	depth

Which three dimensions, being multiplied continually together, produce 12096 cubic feet; which being divided by 27, the cubic feet in a yard, there result 448 cubic yards of digging.

Thus:

```
          54
          28
        ----
         432
        108
        ----
        1512
           8
       -----
27 ) 12096 ( 448
     108
     ----
      129
      108
      ----
       216
       216
       ---
```

The Brick Work.

	Dimensions. Feet	In.	Half bricks thick	Contents Feet	In.	Titles
	155 10	0 0	6	1550	0	The cellar walls
	23 10	6 0	8	470	0	Middle walls of ditto
2	6 3	8 8	8	48	10	Deduct doors in ditto
2	23 17	6 0	1	799	0	Paving the cellars, brick on edge
	23 9	6 0	1	211	6	Ditto between the cellars
2	24 17	0 6	8	840	0	Groin arches over the two cellars
	24 9	0 6	8	228	0	Common arch under the passage
	23 9	6 0	4	211	6	Deduct vacuity of ditto
	155 24	0 6	5	3797	6	Out walls of the principal and attic story
2	24 24	0 6	7	1176	0	Middle walls of ditto
2	7 8	0 0	6	112	0	Chimney shafts within the roof

Di-

	Dimensions. Feet	In.	Half bricks thick	Products Feet	In.	Titles
						Deduct as follows, viz.
2	8 4	8 8	5	80	10	Windows in front
3	4 4	8 8	5	65	4	Ditto
	11 6	0 0	5	66	0	Front door
4	8 4	0 0	5	128	0	End windows
2	4 4	0 0	5	32	0	Ditto
2	4 4	0 0	1	32	0	Ditto blind or falſe
	4 8	0 0	5	32	0	Stair-caſe window
6	6 3	8 8	7	146	8	Doors in middle walls
2	6 3	8 8	1	48	10	Ditto falſe
2	4 4	6 6	5	40	6	Receſs of chimneys
2	3 3	10 10	5	29	4	Ditto
2	3 3	6 6	5	24	6	Ditto

To

To abſtract the foregoing particulars, that is, to collect together the contents of the ſame kind, next to reduce their ſums to the ſtandard thickneſs of a brick and a half, and then, having made the deductions, to collect the reduced ſums, to obtain the whole in one ſum, I proceed thus: I make only two columns for the whole contents, and two for the deductions of the ſame thickneſs, viz. one column for the one-brick contents, and the other for the brick-and-half contents; and diſpoſe the ſuperior denominations in one or both of theſe columns, by placing them down more than once; thus, the contents of 2 bricks thick, I ſet down twice in the one-brick column; thoſe of $2\frac{1}{2}$ bricks, I ſet down once in the one-brick column, and once in the brick-and-half column; thoſe of 3 bricks, twice in the brick-and-half column; thoſe of $3\frac{1}{2}$ bricks, once in the brick-and-half, and twice in the one-brick column; and thoſe of four bricks, twice in the brick-and-half, and once in the one-brick column; and ſo on, if there were higher denominations; and, laſtly, for the contents of half a brick thick, I take the halves of them, and ſet in the one-brick column. Then, having added up every column, I reduce the ſums of the one-brick thick, to the brick-and-half thickneſs; and add this reduced content to the brick-and-half ſums, both in the ſolids and deductions; then, laſtly, taking the one ſum from the other, for the whole content of the brick work, as follows.

Abſtract

Abstract of the Brick Work.

1½ br. thick		1 brick thick		1½ deduct		1 br. deduct	
Feet	*In.*	*Feet*	*In.*	*Feet*	*In.*	*Feet*	*In.*
1550	0	470	0	48	10	48	10
1550	0	399	6	48	10	211	6
470	0	105	9	80	10	211	6
470	0	840	0	65	4	80	10
840	0	228	0	66	0	65	4
840	0	3797	6	128	0	66	0
228	0	1176	0	32	0	128	0
228	0	1176	0	32	0	32	0
3797	6	8192	9	146	8	16	0
1176	0		2	40	6	32	0
112	0	3)16385	6	29	4	146	8
112	0	5461	10	24	6	146	8
11373	6			742	10	24	5
5461	10			869	5	40	6
16835	4			1612	3	29	4
1612	3					24	6
9)15223	1					1304	1
1691$\frac{4}{9}$ yds							2
						3)2608	2
						869	5

Feet

Feet	In.	
2	0	thick. of walls
	4	multiply
8	0	
24	6	height of 2 ſtories
196	0	
	5	half bricks thick
3)980	0	
326	8	reduced cont. of the 4 quoins of the houſe
1612	3	the deductions
1933	11	ſum
282	0	vacuity of arch under the paſſage, deduct
9)1656	11	
184		yards to be added for workmanſhip only.

Feet	In.	
211	6	vacu. arch.
	4	half br. thick
3)846	0	
282	0	reduced content of the vacuity of the com. arch.

So that in the Bill we ſhall have

	£	s	d
1691 yards 4 feet of brick-and-half wall, work and materials at — per yard	—	—	—
184 yards for workmanſhip, in doors and windows, &c, at — per yard	—	—	—
	£ —	—	—

The

The MASON'S WORK.

	Dimentions Feet	Dimentions In.	Contents Feet	Contents In.	Titles.
	157 2	0 2	340	2	Stone base
	157 0	0 8	104	8	Facia
6	4 0	8 9	21	0	Window soles
	27 2	0 4	63	0	Jambs and head of front door, plain.
2	9 3	0 0	54	0	Columns to front door
4	3 1	8 6	22	0	Base and capitals to ditto (commonly taken double measure)
	11 0	6 11	10	6	Architrave
	9 0	6 10	7	11	Soffit of ditto
	11 0	6 8	7	8	Frize
	13 1	10 6	20	9	Level cornice
	12 2	0 6	30	0	Cornice of pediment
2	1 0	6 4	1	0	Returns of cima-recta

Di-

	Dimensions Feet	Dimensions In.	Contents Feet	Contents In.	Titles
2	22 2	8 2	98	2	Architraves of low windows
2	5 0	8 6½	6	1	Frize of ditto
2	7 1	9 5	21	11	Cornice of ditto
3	21 2	4 2	138	8	Architrave of attic windows
	120 2	2 1	250	4	Cornice to front and ends of the houſe
	109 1	6 10	200	9	Blocking courſe to front and ends
4	5 1	6 7	34	10	Steps to front
2	8 2	6 0	34	0	Ends of ditto
2	3 1	0 4	8	0	Ditto
	9 3	0 0	27	0	Top of the landing
2	14 3	0 6	98	0	Chimney tops of ſtone
2	15 1	4 8	51	1	Baſe and facias of ditto

	Dimenſions		Contents		Titles
	Feet	*In.*	*Feet*	*In.*	
2	5 2	0 2	21	8	Poliſhed ſlabs to the chimneys in the low rooms
2	3 1	9 10	13	9	Back ſlabs to ditto
2	12 0	1 11	22	1	Jambs and mantles to ditto
4	4 1	4 6	26	0	Coves to ditto
2	4 2	8 0	18	8	Slabs to chimneys in the attic ſtory
2	3 1	6 10	12	10	Back ſlabs
2	11 0	2 11	20	5	Jambs and mantles
4	4 1	0 6	24	0	Coves
2	4 2	2 0	16	8	Slabs
2	3 1	0 10	11	0	Back ſlabs
2	10 0	4 10	17	2	Jambs and mantles
4	3 1	6 6	21	0	Coves
12	3 1	6 4	56	0	Steps down to the cellar

Ab-

Abstract of the Masonry.

In making this abstract, the articles of the same price must be collected together. So in the first column, are collected together the base, facias, window soles, jambs and head of door, blocking course and chimney tops, as being of the same value; in the 2d are collected architraves, frizes, cornices, mouldings about the door, columns, bases, and capitals; and so for other things of another value.

Base, facias, &c		Architraves, &c		Chimney slabs, &c		Front steps		Chimney jambs, &c		Cornice of the house		Cellar steps	
Feet	*I*	*Feet*	*I*	*Feet*	*I*	*Feet*	*I*	*Feet*	*I*	*Feet*	*I*	*Feet*	*I*
340	2	54	0	21	8	34	10	22	1	250	4	56	0
104	8	22	0	13	9	34	0	20	5				
21	0	10	6	26	0	8	0	17	2				
63	0	7	11	18	8	27	0	59	8				
200	9	7	8	12	10	103	10						
98	0	20	9	24	0								
51	1	30	0	16	8								
878	8	1	0	11	0								
		98	2	21	0								
		6	1	165	7								
		21	11										
		138	8										
		418	8										

The Mason's Bill.

Feet	*In.*		
878	8	Base, facias, &c. at — a foot	- - -
418	8	Columns, mouldings, &c. at — a foot	- - -
165	7	Chimney slabs and coves, at — a foot	- - -
103	10	Steps to front door, at — a foot	- - -
59	8	Chim. jambs and mantles, at — a foot	- - -
250	4	Cornice of the house, at — a foot	- - -
56	0	Cellar steps, at — a foot	- - -
		£	- - -

The

The Carpenters *and* Joiners Work.

	Dimensions		Contents		Titles
	Feet	*In.*	*Feet*	*In.*	
2	24 17	0 6	840	0	Common joist. in the low rooms
2	24 17	6 6	857	6	Framing the upper floors, with girders, binding & bridging joist
2	11 12	9 0	282	0	Common joisting of the stair-case, allowing the joist one foot hold of the wall at each end
			180	0	Plates over doors and windows, running measure
	59 36	3 0	2133	0	Roofing
	109 2	0 0	218	0	Gutter boarding at front and ends
4	24 17	0 6	1680	0	Ceiling joists to both stories
2	17 11	6 0	385	0	Stoothed partitions
2	24 17	0 6	840	0	Deal flooring
2	23 17	6 6	822	6	Ditto upper story
2	12 9	0 9	234	0	Ditto in stair-case

	Dimensions.		Contents		Titles
	Feet	In.	Feet	In.	
11	4 1	8 0	51	4	Ditto in window ways
7	3 2	8 6	64	2	Ditto in door ways
11	4 1	0 6	66	0	Deal ſteps to ſtair-caſe, 1ſt and 2d flights
11	1 0	0 7	6	5	Ends of ditto
2	4 4	0 6	36	0	Half paces to ditto
9	4 3	0 0	108	0	Steps of the 3d flight
9	1 0	0 7	5	3	Ends of ditto
	9 0	9 7	5	8	Face of the landing
7	8 4	4 8	272	2	Saſhes
6	4 4	8 8	130	8	Ditto
6	16 2	8 6	250	0	Inſide door-caſings, pannelled.
3	16 0	8 8	33	4	Ditto, plain

Di-

	Dimensions Feet	Dimensions In.	Contents Feet	Contents In.	Titles
	25 0	0 9	18	9	Frame to front door
	25 1	0 2	29	2	Inside casing to ditto, pannelled
9	6 3	8 4	200	0	Doors, 6 pannelled
	10 5	0 0	50	0	Front door
7	8 4	0 0	224	0	Window shutters on the first floor and stair-case window
5	4 4	0 0	80	0	Ditto to attic windows
12	4 1	6 2	63	0	Soffits to windows, pannelled
7	22 0	8 11	145	5	Architraves to low windows and stair-case window
	28 1	4 0	28	4	Ditto to front door
5	20 0	8 11	94	8	Ditto to upper windows
15	18 0	9 9	210	11	Ditto to all the doors
2	72 2	0 0	288	0	Plain dados to low rooms

	Dimensions Feet	Dimensions In	Contents Feet	Contents In.	Titles
2	72 0	0 9	108	0	Base and sur-base to ditto
2	72 0	0 6	72	0	Plain plinth to ditto
	22 8	6 2	183	9	Balustrade in the stair-case
	68 0	0 3	17	0	Base moulding to stair-case
	68 0	0 6	34	0	Plain plinth to ditto
5	6 3	4 0	95	0	Plain backs and elbows to attic windows

Abstract of the Carpenters and Joiners Work.

Joisting	Ceiling joists, &c.	Flooring	Stairs	Sashes	Window shutters, &c.	Dados, &c.	6 pannelled doors	Architraves, &c.
Feet I	Feet I	Feet I	Feet I	Feet I	Feet I	Feet I	Feet I	Feet I
840 0	1680 0	840 0	66 0	272 2	250 0	33 4	200 0	145 5
282 0	385 0	822 6	6 5	130 8	29 2	18 9	50 0	28 4
		234 0	36 0		224 0	288 0		94 8
1122 0	2065 0	51 4	108 0	402 10	80 0	72 0	250 0	210 11
		64 2	5 3		63 0	34 0		108 0
			5 8			95 0		17 0
		2012 0			646 2			
			227 4			541 1		604 4

224 0
8 0

2)304 0

152 0 Half of the wind. shut, being work & half
250 0 Doors, being double work

402 0 Sum, to be charged for workmanship, besides being charged once for work and materials.

The

The Carpenters and Joiners Bill.

Sq.	*Feet*	*I*		*£*	*s*	*d*
11	22	0	Common joisting, at — a square	-	-	-
8	57	6	Framing the upper floors, at — a sq.	-	-	-
	180	0	Plates over doors and windows, running measure, at — a foot	-	-	-
21	33	0	Roofing, at — a square	-	-	-
	218	0	Gutter boarding, at — a foot	-	-	-
20	65	0	Ceiling joists and stoothed partitions, at — a square	-	-	-
20	12	0	Flooring, at — a square	-	-	-
	227	4	Steps in the stair-case, at — a foot	-	-	-
	402	10	Sashes, at — a foot	-	-	-
	646	2	Window-shutters, soffits, & pannelled door-cases, at — a foot	-	-	-
	541	1	Plain door-cases, frame to front door, plain plinths, and plain backs and elbows, at — a foot	-	-	-
	250	0	Doors, 6 pannelled, at — a foot	-	-	-
	604	4	Architraves, &c at — a foot	-	-	-
	402	0	Workmanship only in doors and window-shutters	-	-	-
				£		

Note. All the articles except the last are for both work and materials.

The SLATERS WORK.

The Slater's work will be the same with the article of roofing in the carpenter's work, which is 2133 square feet, or

Sqrs.	*Feet*				
21	33	at — a square	£ -	-	-

 The

The PLASTERER'S WORK.

	Dimenſions		Contents		Titles
	Feet	*In.*	*Feet*	*In.*	
2	23 17	0 0	782	0	Ceil. to the low rooms, 3 coats
2	83 1	0 0	166	0	Plain cornice to ditto
8	83	0	664	0	Enriched mouldings in ditto, running meaſure
	9 12	0 0	108	0	Ceiling in the lower part of the ſtair-caſe, 3 coats
	43 0	6 9	32	7	Plain cornice in ditto
	83 9	0 0	747	0	Walls in one of the low rooms, hard-finiſhing
2	4 4	4 4	37	6	Deduct doors from ditto
3	8 5	8 4	138	8	Deduct windows
	5 2	0 0	10	0	Deduct part of chimney-piece
			747	0	The walls plaſtering of the other low rooms, 2 coats, the dimenſions and deductions the ſame as in the former
			186	2	The deductions together
	22 11	2 6	254	11	Walls of the ſtair-caſe, hard-finiſhing.

Di-

	Dimensions Feet	Dimensions In.	Contents Feet	Contents In.	Titles
	7	6	22	6	Walls of the stair-case, hard-finishing
	3	0			
	7	6	71	3	Ditto
	9	6			
	6	0	34	6	Ditto
	5	9			
	17	9	124	3	Ditto
	7	0			
	67	0	569	6	Walls of upper part, ditto
	8	6			
	10	0	66	8	Deduct front door from ditto
	6	8			
7	6	0	168	0	Ditto other doors
	4	0			
	8	8	46	2	Ditto window
	5	4			
	7	4	39	1	Ditto attic window
	5	4			
	23	0	201	3	Stair-case ceiling, 3 coats
	8	9			
	67	8	67	8	Plain cornice to ditto
	1	0			

	Dimensions		Contents		Titles
	Feet	In.	Feet	In.	
2	17 14	0 6	493	0	Ceilings of upper rooms, 3 coats
2	17 7	0 3	246	6	Ditto
2	65 0	0 10	108	4	Plain cornices
2	50 0	6 10	84	2	Ditto
4	17 8	6 6	595	0	Partitions, 2 coats on laths
2	47 8	6 6	807	6	Ditto walls, 2 coats
2	33 8	6 6	569	6	Ditto
4	7 5	4 4	156	5	Deduct windows from ditto
4	6 4	0 4	104	0	Deduct doors
2	4 4	6 0	36	0	Deduct chimney-pieces
	4 3	0 4	13	4	Ditto

The

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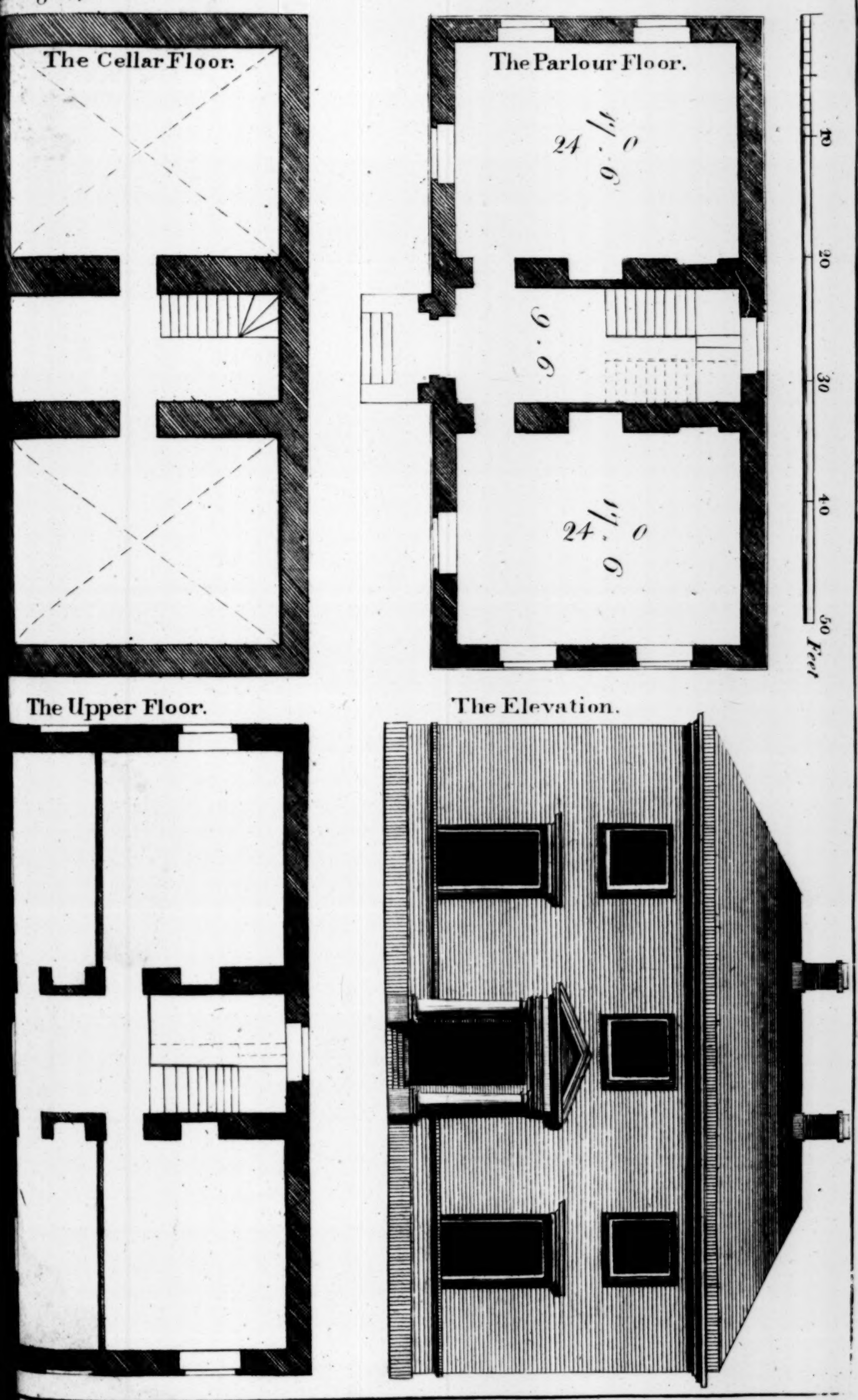

Sculp

The Abstract of the Plastering.

Ceiling		Rendering		Deductions		Plain Cornices	Enrich'd Mouldings
3 Coats	2 Coats	Hard-finishing	2 Coats	Hard-finish.	2 Coats Walls		
Feet I	*Feet*	*Feet I*	*Feet I*	*Feet I*	*Feet I*	*Feet I*	*Feet I*
782 0	9) 595 *I*	747 0	747 0	37 6	186 2	166 0	664 0
108 0	0	254 11	807 6	138 8	156 5	32 7	
201 3	*Y F*	22 6	569 0	10 0	104 0	67 8	
493 0	66 1	71 3		66 8	36 0	108 4	
246 6		34 6	2123 6	168 0	13 4	84 2	
9) 1830 9		124 3	495 11	46 2	495 11	458 9	
Y F I		569 6	9) 1627 7	39 1			
203 3 9		1823 11	*Y F I*	506 1			
		506 1	180 7 7				
		9) 1317 10					
		Y F I					
		146 3 10					

The Plasterer's Bill.

Yds	*F*	*I*		£	s	d
203	3	9	Ceiling, 3 coats, at — a yard	-	-	-
66	1	0	Ditto, 2 coats, at — a yard	-	-	-
146	3	10	Plastering, hard-finishing, at - a yd.	-	-	-
180	7	7	Ditto, 2 coats, at — a yard	-	-	-
	458	9	Plain cornice, at — a foot	-	-	-
	664	0	Enriched mouldings, at — a foot	-	-	-
			£	-	-	-

The annexed plate contains the elevation and plans of the house to which all the foregoing examples belong.

I shall now deliver a short but complete tract on Timber Measuring, with which, I shall conclude these applications, as proposed.

SEC-

SECTION IV.

TIMBER MEASURING.

PROBLEM I.

To find the Area or Superficial Feet in a Board or Plank.

MULTIPLY the length by the mean breadth.

Note. When the board is tapering, add the breadths at the two ends together, and take half the ſum for the mean breadth.

By the Sliding Rule.

Set 12 on B to the breadth in inches on A; then againſt the length in feet on B, is the content on A, in feet and fractional parts.

EXAMPLES.

1. What is the value of a plank, whoſe length is 12 feet 6 inches, and mean breadth 11 inches?

By Decimals.

	12·5
	11
12	137·5
$1\frac{1}{2}$d is $\frac{1}{8}$	11·46
	1s 5d anſ.

By Duodecimals.

	12	6	
		11	
$1\frac{1}{2}$d is $\frac{1}{8}$	11	5	6
	1s	$4\frac{1}{2}$d	
5^{i} is	0	$\frac{1}{2}$	
	1s	5d anſ.	

BY

BY THE SLIDING RULE.

As 12 B : 11 A :: $12\frac{1}{2}$ B : $11\frac{1}{2}$ A.

That is, as 12 on B is to 11 on A, so is $12\frac{1}{2}$ on B to $11\frac{1}{2}$ on A.

Ex. 2. Required the content of a board, whose length is 11 feet 2 inches, and breadth 1 foot 10 inches. Ans. $20^f\ 5^i\ 8''$.

Ex. 3. What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at $2\frac{1}{2}$d a foot? Ans. 3s $3\frac{3}{4}$d.

Ex. 4. Required the value of 5 oaken planks, at 3d per foot, each of them being $17\frac{1}{2}$ feet long; and their several breadths are as follows, namely, two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower. Ans. £1 5 $8\frac{1}{4}$.

PROBLEM II.

To find the Solid Content of Squared or Four-sided Timber.

Multiply the mean breadth by the mean thickness, and the product again by the length, and the last product will give the content.

BY THE SLIDING RULE.

C D D C
As length : 12 or 10 : : quarter girt : ſolidity.

That is, as the length in feet on C, is to 12 on D when the quarter girt is in inches, or to 10 on D when it is in tenths of feet; ſo is the quarter girt on D, to the content on C.

Note 1. If the tree taper regularly from the one end to the other, either take the mean breadth and thickneſs in the middle, or take the dimenſions at the two ends, and half their ſum will be the mean dimenſions.

2. If the piece do not taper regularly, but is unequally thick in ſome parts and ſmall in others, take ſeveral different dimenſions, add them all together, and divide their ſum by the number of them, for the mean dimenſions.

3. The quarter girt is a geometrical mean proportional between the mean breadth and thickneſs, that is the ſquare root of their product. Sometimes unſkilful meaſurers uſe the arithmetical mean inſtead of it, that is, half their ſum; but this is always attended with error, and the more ſo as the breadth and depth differ the more from each other.

EXAMPLES.

1. The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and leſs end 1 foot 6 inches and 1 foot 3 inches, and the thickneſs at the greater and leſs end 1 foot 3 inches and 1 foot: required the ſolid content.

Decimals.

Decimals.		*Duodecimals.*
1·5		1 6
1·25		1 3
2) 2·75		2) 2 9
1·375	mean breadth	1 4 6
1·25		1 3
1·0		1 0
2) 2·25		2) 2 3
1·125	mean depth	1 1 6
1·375	mean breadth	1 4 6
5625		1 1 6
7875		4 6
3375		6 6
1125		
1·546875		1 6 6 6
18·5	length	18 6
7734375		27 9 9 0
12375000		9 3 3
1546875		
28·6171875	content	28 7 0 3

BY THE SLIDING RULE.

B A B A

As 1 : $13\frac{1}{2}$ :: $16\frac{1}{2}$: 223, the mean square.

C D C D

As 1 : 1 :: 223 : 14·9, quarter girt.

C D D C

As $18\frac{1}{2}$: 12 :: 14·9 : 28·6, the content.

Ex. 2. What is the content of the piece of timber, whose length is $24\frac{1}{2}$ feet, and the mean breadth and thickness each 1·04 feet? Ans. $26\frac{1}{2}$ feet.

Ex. 3 Required the content of a piece of timber, whose length is 20·38 feet, and its ends unequal squares, the side of the greater being $19\frac{1}{8}$, and the side of the less $9\frac{7}{8}$. Ans. 29·838 feet.

Ex. 4. Required the content of the piece of timber, whose length is 27·36 feet; at the greater end the breadth is 1·78, and thickness 1·23; and at the less end the breadth is 1·04, and thickness 0·91.
Ans. 41·278 feet.

PROBLEM III.

To find the Solidity of Round or Unsquared Timber.

RULE I, OF COMMON RULE.

Multiply the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference, by the length, for the content.

BY THE SLIDING RULE.

As the length upon C : 12 or 10 upon D ::
quarter girt, in 12ths or 10ths, on D : content on C.

Note 1. When the tree is tapering, take the mean dimensions as in the former problems, either by girting it in the middle for the mean girt, or at the two ends, and take half the sum of the two. But when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

2. This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or

or nearly what the quantity would be after the tree is hewed square in the usual way: so that it seems intended to make an allowance for the squaring of the tree. When the true quantity is desired, use the 2d rule, given below.

EXAMPLES.

1. A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content?

Decimals.		*Duodecimals.*		
3·5	quarter girt	3	6	
3·5		3	6	
175		10	6	
105		1	9	
12·25		12	3	
9·5	length	9	6	
6125		110	3	
11025		6	1	6
116·375	content	116	4	6

BY THE SLIDING RULE.

	C		D		D		C
As	9·5	:	10	::	35	:	$116\frac{1}{3}$
Or	9·5	:	12	::	42	:	$116\frac{1}{3}$

Ex. 2. The length of a tree is 24 feet, its girt at the thicker end 6 feet, and at the smaller end 2 feet; required the content. Ans. 96 feet.

Ex. 3. What is the content of a tree, whose mean girt is 3·15 feet, and length 14 feet 6 inches?

Ans. 8·9929 feet.

Ex. 4.

Ex. 4. Required the content of a tree, whose length is $17\frac{1}{4}$ feet, and which girts in five different places as follows, namely, in the first place 9·43 feet, in the second 7·92, in the third 6·15, in the fourth 4·74, and in the fifth 3·16. Ans. 42·6075.

RULE II.

Multiply the square of $\frac{1}{5}$ of the mean girt by double the length, and the product will be the content, very near the truth.

BY THE SLIDING RULE.

As the double length on C : 12 or 10 on D :: $\frac{1}{5}$ of the girt, in 12ths or 10ths, on D : content on C.

EXAMPLES.

1. What is the content of a tree, its length being 9 feet 6 inches, and its mean girt 14 feet?

Decimals.		*Duodecimals.*		
2·8	$\frac{1}{5}$ of girt	2	9	7
2·8		2	9	7
224		5	7	2
56		2	1	3
			1	8
7·84				
19		7	10	1
7056		19		
784				
148·96	content	148	11	7

BY THE SLIDING RULE.

	C		D		D		C
As	19	:	10	: :	28	:	149.
Or	19	:	12	: :	$33\frac{6}{10}$	:	149.

Ex. 2. Required the content of a tree, which is 24 feet long, and mean girt 8 feet. Anf. 122·88 feet.

Ex. 3. The length of a tree is $14\frac{1}{2}$ feet, and mean girt 3·15 feet, what is the content? Anfw. 11·51 feet.

Ex. 4. The length of a tree is $17\frac{1}{4}$ feet, and its mean girt 6·28; what is the content?

Anf. 54·4065 feet.

Ex. 5. Sold an oak tree whofe girt at the low end was $9\frac{1}{4}$ feet, and its length, to the part where it becomes of 2 feet girt, is $27\frac{1}{2}$ feet; it hath alfo two boughs, the girt at the thicker end of the one is 4·3, and at the thicker end of the other 3·94, the length of the timber of the former, that is to the part where it becomes of only 2 feet girt, is 9 feet; and the length of the latter $7\frac{3}{4}$ feet; required the price at 2*l.* 3*s.* 9*d.* a load of 50 feet, allowing $\frac{1}{10}$ of the mean girt for the bark of the trunk, and $\frac{1}{12}$ of the fame in the boughs, by both the rules. Anf. true value £2 18 6.
falfe value £2 5 $9\frac{1}{2}$.

Note 1. That part of a tree or of the branches, which is lefs than 2 feet in circumference, or 6 inches quarter girt, is cut off; not being accounted timber.

Note 2. A cuftom has of late been creeping into ufe, where the buyers of timber can introduce it, of allowing an inch on every foot of quarter girt, for bark. This practice, however, is unreafonable, and ought to be difcouraged. Elm timber is the chief kind in which an allowance ought to be made, and it will be found on examination, that the common allowance of one inch on the tree, is abundantly fufficient for an average allowance.

Note 3. Fifty cubic feet of timber make a load; and therefore to reduce feet to loads, divide them by 50.

A TABLE

For readily finding the Content of Trees, according to the common Method of measuring Timber.

RULE.

Seek the quarter girt in the first column towards the left-hand, and take out the number opposite. Multiply that number by the length of the tree in feet &c, and the product will be the content in solid feet &c.

Ins.	Feet	In.	''	'''	''''	Ins.	Feet	In.	''	'''	''''	Ins.	Feet	In.	''	'''	''''
6	0	3	0	0	0	15	1	6	9	0	0	24	4	0	0	0	0
¼	0	3	3	0	9	¼	1	7	4	6	9	¼	4	1	0	0	9
½	0	3	6	3	0	½	1	8	0	3	0	½	4	2	0	3	0
¾	0	3	9	6	9	¾	1	8	8	0	9	¾	4	3	0	6	9
7	0	4	1	0	0	16	1	9	4	0	0	25	4	4	1	0	0
¼	0	4	4	6	9	¼	1	10	0	0	9	¼	4	5	1	6	9
½	0	4	8	3	0	½	1	10	8	3	0	½	4	6	2	3	0
¾	0	5	0	0	9	¾	1	11	4	6	9	¾	4	7	3	0	9
8	0	5	4	0	0	17	2	0	1	0	0	26	4	8	4	0	0
¼	0	5	8	0	9	¼	2	0	9	6	9	¼	4	9	5	0	9
½	0	6	0	3	0	½	2	1	6	3	0	½	4	10	6	3	0
¾	0	6	4	6	9	¾	2	2	3	0	9	¾	4	11	7	6	9
9	0	6	9	0	0	18	2	3	0	0	0	27	5	0	9	0	0
¼	0	7	1	6	9	¼	2	3	9	0	9	¼	5	1	10	6	9
½	0	7	6	3	0	½	2	4	6	3	0	½	5	3	0	3	0
¾	0	7	11	0	9	¾	2	5	3	6	9	¾	5	4	2	0	9
10	0	8	4	0	0	19	2	6	1	0	0	28	5	5	4	0	0
¼	0	8	9	0	9	¼	2	6	10	6	9	¼	5	6	6	0	9
½	0	9	2	3	0	½	2	7	8	3	0	½	5	7	8	3	0
¾	0	9	7	6	9	¾	2	8	6	0	9	¾	5	8	10	6	9
11	0	10	1	0	0	20	2	9	4	0	0	29	5	10	1	0	0
¼	0	10	6	6	9	¼	2	10	2	0	9	¼	5	11	3	6	9
½	0	11	0	3	0	½	2	11	0	3	0	½	6	0	6	3	0
¾	0	11	6	0	9	¾	2	11	10	6	9	¾	6	1	9	0	9
12	1	0	0	0	0	21	3	0	9	0	0	30	6	3	0	0	0
¼	1	0	6	0	9	¼	3	1	7	6	9	¼	6	4	3	0	9
½	1	1	0	3	0	½	3	2	6	3	0	½	6	5	6	3	0
¾	1	1	6	6	9	¾	3	3	5	0	9	¾	6	6	9	6	9
13	1	2	1	0	0	22	3	4	4	0	0	31	6	8	1	0	0
¼	1	2	7	6	9	¼	3	5	3	0	9	¼	6	9	4	6	9
½	1	3	2	3	0	½	3	6	2	3	0	½	6	10	8	3	0
¾	1	3	9	0	9	¾	3	7	1	6	9	¾	7	0	0	0	9
14	1	4	4	0	0	23	3	8	1	0	0	32	7	1	4	0	0
¼	1	4	11	0	9	¼	3	9	0	6	9	¼	7	2	8	0	9
½	1	5	6	3	0	½	3	10	0	3	0	½	7	4	0	3	0
¾	1	6	1	6	9	¾	3	11	0	0	9	¾	7	5	4	6	9

SCHOLIUM.

In measuring squared timber, unskilful measurers usually take $\frac{1}{4}$ of the circumference, or girt, for the side of a mean square; which quarter girt therefore multiplied by itself, and the product multiplied by the length, they account the solidity, or content: but it is really always above the truth. Indeed when the breadth and thickness are nearly equal, this method will give the solidity pretty near the truth; but if the breadth and thickness differ considerably from each other, the error will be so great as that it ought not by any means to be neglected.

Thus, suppose we take, for an example, a balk of 24 feet long, and a foot square throughout; and consequently its solidity 24 cubic feet. Now if this balk be slit exactly in two from end to end, making each piece 6 inches broad, and 12 inches thick; it is evident that the true solidity of each will be 12 feet; but by the quarter girt method, they would amount to much more; for the false quarter girt, being equal to half the sum of the breadth and thickness, in this case will be 9 inches, the square of which is 81, which being divided by 144, and the quotient multiplied by 24 the length, we obtain $13\frac{1}{2}$ feet for the solidity of each part, and consequently the two solidities together make 27 feet, instead of 24.

Again, suppose the balk to be so cut, as that the breadth of the one piece may be only 4 inches, and consequently that of the other 8 inches. Here the true content of the less piece will be 8 feet, and that of the greater 16 feet. But, proceeding by the other method, the quarter girt of the less piece will be 8, whose square 64, multiplied by 24, and the product divided by 144, gives $10\frac{2}{3}$ feet instead of 8. And, by the same method, the content of the greater piece will be $16\frac{2}{3}$ feet, instead of 16. And the sum of both is $27\frac{1}{3}$ feet, instead of 24.

Farther, if the less piece be cut only 2 inches broad, and consequently the greater 10 inches; the

 true

true content of the leſs piece would be 4 feet, and that of the greater 20. But, by the other method, the quarter girt of the leſs piece would be 7 inches, whoſe ſquare 49 being divided by 6, gives $8\frac{1}{6}$ feet, inſtead of 4, for the content. And by the ſame method the content of the greater piece would be $20\frac{1}{6}$, inſtead of 20 feet. So that their ſum would be $28\frac{1}{3}$, inſtead of 24 feet.

Hence it is evident, that the greater the proportion is between the breadth and depth, the greater will the error be, by uſing the falſe method; that the ſum of the two parts, by the ſame method, is greater, as the difference of the ſame two parts is greater, and conſequently the ſum is leaſt when the two parts are equal to each other, or when the balk is cut equally in two; and laſtly, that when the ſides of a balk differ not above an inch or two from each other, the quarter girt method may then be uſed, without inducing an error that will be of any material conſequence.

From the preceding examples it appears that this new method, which is very near the truth, is full as eaſy in practice as the common falſe one. But there are many other reaſons for changing the method; and one in particular, is the preventing of the ſellers from playing any tricks with their timber, by cutting trees into different lengths, ſo as to make them meaſure to more than the whole did; for, by the falſe method, this may be done in many reſpects, as will appear in the three following problems, which contain the chief caſes of this artifice, but which however I do not explain to teach them to uſe theſe means.

PROBLEM

PROBLEM IV.

To find where a Piece of Round Tapering Timber must be cut, so that the Two Parts, measured separately, according to the Common Method of measuring, shall produce a Greater Solidity than when cut in any other Part, and Greater than the Whole.

* Cut it through exactly in the middle, or at $\frac{1}{2}$ of the length, and the two parts will measure to the most possible by the common method.

EXAMPLE.

Supposing a tree to girt 14 feet at the greater end, 2 feet at the less, and consequently 8 feet in the middle; and that the length is 32 feet.

Then, by the common method, the whole tree measures to only 128 feet; but when cut through at the middle, the greater part measures to 121, and the less part to 25 feet; whose sum is 146 feet; which exceeds the whole by 18 feet, and is the most that it can be made to measure to by cutting it into two parts.

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* DEMONSTRATION.

Put G = the greatest girt, g = the least, and x = the girt at the section; also L = the whole length, and z = the length to be cut off the less end.

Then, by similar figures, $L : z :: G-g : x-g$, hence $x = \frac{Gz-gz}{L} + g$. But $(g+x)^2 . z + (G+x)^2 . (L-z) =$ a maximum; whose fluxion being put equal to nothing, and the value of x substituted instead of it, there results $z = \frac{1}{2}L$. *Q. E. D.*

COROLLARY.

By thus bisecting the length of a tree, and then each of the parts, and so on, continually bisecting the lengths of the several parts, the measure of the whole will be continually increased.

PROBLEM V.

To find where a Tree should be Cut, so that the Part next the Greater End may measure to the Most Possible.

* From the greater girt take 3 times the less; then, as the difference of the girts is to the remainder, so is $\frac{1}{3}$ of the whole length, to the length from the less end to be cut off.

Or, cut it where the girt is $\frac{1}{3}$ of the greatest girt.

Note. If the greatest girt do not exceed 3 times the least, the tree cannot be cut as is required by this problem. For, when the least girt is exactly equal to $\frac{1}{3}$ of the greater, the tree already measures to the greatest possible; that is, none can be cut off, nor indeed added to it, continuing the same taper, that the remainder or sum may measure to so much as the whole: And when the least girt exceeds $\frac{1}{3}$ of the greater, the result by the rule shews how much in length must be *added*, that the result may measure to the most possible.

EXAMPLE.

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* DEMONSTRATION.

Using the same notation as in the last demonstration; we shall have here also $x = \frac{Gz - gz}{L} + g$, and $(G+x)^2 . (L-x) =$ a maximum; which, treated as before, gives $z = \frac{G-3g}{G-g} \times \frac{1}{2} L$. And $x = \frac{G-g}{L} z + g = \frac{1}{3} G$, by substituting the above value of z.

Q.E.D

EXAMPLE.

Taking here the ſame example as before, we ſhall have as $12 : 8 :: \frac{32}{3} : 7\frac{1}{9} =$ the length to be cut off; and conſequently the length of the remaining part is $24\frac{8}{9}$; alſo $\frac{14}{3} = 4\frac{2}{3}$ is the girt at the ſection. Hence the content of the remaining part is $135\frac{41}{81}$ feet; whereas the whole tree, by the ſame method, meaſures only to 128 feet.

PROBLEM VI.

To Cut a Tree ſo as that the Part next the Greater End may meaſure, by the Common Method, to exactly the ſame Quantity as the Whole meaſures to.

* Call the ſum of the girts of the two ends s, and their difference d. Then multiply d by the ſum of d and $4s$, and from the root of the product take the difference between d and $2s$; then, as $2d$ is to the remainder, ſo is the whole length, to the length to be cut off the ſmall end.

And if s be taken from the ſaid root, half the remainder will be the girt at the ſection.

* DEMONSTRATION.

Uſing ſtill the ſame notation, we ſhall have $s^2L = (L-z).(G+x)^2$; hence, inſtead of x, ſubſtituting its value $\frac{dz}{L} + g$, we obtain $z = \frac{L}{2d} \times (\sqrt{(4s+d).d} - 2s + d)$. And hence $x = \frac{1}{2}\sqrt{(4s+d).d} - \frac{1}{2}s$. Q.E.D.

EXAMPLE.

Using still the same example, we have $s = 16$, $d = 12$, and the length $L = 32$; hence $\frac{L}{2d} \times (\sqrt{(4s+d).d} - 2s + d) = \frac{32}{24} \times (\sqrt{76 \times 12} - 20) = \frac{16}{3}\sqrt{57} - 26\frac{2}{3} = 13.599118$ = the length to be cut off; and consequently 18.400882 is the length of the remaining part. Also $\frac{1}{2}\sqrt{(4s+d).d} - \frac{1}{2}s = 2\sqrt{57} - 8 = 7.099669$ is the girt at the section. Hence the girt in the middle of the greater part is $\frac{14+7.099669}{2} = 10.549834$, whose $\frac{1}{4}$th part is 2.637458; and consequently the content of the same part is $2.637458^2 \times 18.400882 = 128$, the very same as the whole tree measures to, notwithstanding above $\frac{1}{3}$ part is cut off the length.

A NEW

A

NEW AND ACCURATE

T A B L E

O F T H E

AREAS OF CIRCULAR SEGMENTS,

F O R

EVERY TEN-THOUSANDTH PART

O F T H E

D I A M E T E R.

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
0001	00000133	0046	00041540	0091	00115427
0002	00000377	0047	00042901	0092	00117332
0003	00000693	0048	00044277	0093	00119247
0004	00001067	0049	00045666	0094	00121172
0005	00001490	0050	00047070	0095	00123107
0006	00001959	0051	00048487	0096	00125052
0007	00002469	0052	00049919	0097	00127007
0008	00003016	0053	00051364	0098	00128972
0009	00003599	0054	00052823	0099	00130947
0010	00004215	0055	00054296	0100	00132932
0011	00004863	0056	00055781	0101	00134927
0012	00005541	0057	00057281	0102	00136932
0013	00006247	0058	00058793	0103	00138947
0014	00006981	0059	00060318	0104	00140971
0015	00007742	0060	00061856	0105	00143005
0016	00008529	0061	00063407	0106	00145048
0017	00009341	0062	00064971	0107	00147101
0018	00010177	0063	00066547	0108	00149163
0019	00011036	0064	00068135	0109	00151235
0020	00011919	0065	00069736	0110	00153317
0021	00012823	0066	00071350	0111	00155407
0022	00013749	0067	00072975	0112	00157507
0023	00014697	0068	00074613	0113	00159617
0024	00015665	0069	00076263	0114	00161735
0025	00016654	0070	00077924	0115	00163863
0026	00017663	0071	00079597	0116	00166000
0027	00018691	0072	00081282	0117	00168146
0028	00019738	0073	00082979	0118	00170301
0029	00020805	0074	00084688	0119	00172466
0030	00021889	0075	00086407	0120	00174639
0031	00022992	0076	00088139	0121	00176821
0032	00024113	0077	00089881	0122	00179012
0033	00025251	0078	00091635	0123	00181212
0034	00026407	0079	00093400	0124	00183421
0035	00027579	0080	00095176	0125	00185639
0036	00028769	0081	00096963	0126	00187865
0037	00029975	0082	00098762	0127	00190100
0038	00031197	0083	00100571	0128	00192344
0039	00032436	0084	00102391	0129	00194597
0040	00033690	0085	00104221	0130	00196858
0041	00034961	0086	00106063	0131	00199128
0042	00036247	0087	00107915	0132	00201406
0043	00037547	0088	00109777	0133	00203693
0044	00038864	0089	00111651	0134	00205988
0045	00040195	0090	00113534	0135	00208292

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
0136	00210604	0181	00322912	0226	00449919
0137	00212925	0182	00325582	0227	00452895
0138	00215254	0183	00328259	0228	00455877
0139	00217591	0184	00330943	0229	00458866
0140	00219937	0185	00333635	0230	00461861
0141	00222291	0186	00336333	0231	00464862
0142	00224653	0187	00339039	0232	00467869
0143	00227024	0188	00341752	0233	00470883
0144	00229402	0189	00344472	0234	00473904
0145	00231789	0190	00347199	0235	00476930
0146	00234184	0191	00349933	0236	00479963
0147	00236587	0192	00352674	0237	00483002
0148	00238998	0193	00355422	0238	00486047
0149	00241417	0194	00358177	0239	00489099
0150	00243844	0195	00360939	0240	00492157
0151	00246279	0196	00363708	0241	00495221
0152	00248722	0197	00366484	0242	00498291
0153	00251173	0198	00369266	0243	00501368
0154	00253631	0199	00372056	0244	00504451
0155	00256098	0200	00374853	0245	00507539
0156	00258573	0201	00377656	0246	00510634
0157	00261055	0202	00380466	0247	00513736
0158	00263545	0203	00383283	0248	00516843
0159	00266043	0204	00386107	0249	00519956
0160	00268549	0205	00388938	0250	00523076
0161	00271062	0206	00391775	0251	00526201
0162	00273583	0207	00394620	0252	00529333
0163	00276112	0208	00397471	0253	00532470
0164	00278648	0209	00400328	0254	00535614
0165	00281192	0210	00403193	0255	00538764
0166	00283744	0211	00406064	0256	00541920
0167	00286303	0212	00408941	0257	00545081
0168	00288870	0213	00411826	0258	00548249
0169	00291444	0214	00414717	0259	00551423
0170	00284025	0215	00417614	0260	00554603
0171	00296614	0216	00420518	0261	00557788
0172	00299211	0217	00423429	0262	00560980
0173	00301815	0218	00426346	0263	00564178
0174	00304427	0219	00429270	0264	00567381
0175	00307045	0220	00432201	0265	00570590
0176	00309672	0221	00435137	0266	00573806
0177	00312305	0222	00438081	0267	00577027
0178	00314946	0223	00441031	0268	00580254
0179	00317594	0224	00443987	0269	00583487
0180	00320249	0225	00446950	0270	00586726

Verf. fine	Seg. area	Verf. fine	Seg. area.	Verf. fine	Seg. area.
0271	00589970	0316	00741838	0361	00904564
0272	00593221	0317	00745339	0362	00908297
0273	00596477	0318	00748846	0363	00912036
0274	00599739	0319	00752358	0364	00915779
0275	00603007	0320	00755875	0365	00919527
0276	00606280	0321	00759398	0366	00923280
0277	00609560	0322	00762926	0367	00927038
0278	00612845	0323	00766459	0368	00930801
0279	00616135	0324	00769997	0369	00934569
0280	00619432	0325	00773541	0370	00938342
0281	00622734	0326	00777090	0371	00942119
0282	00626042	0327	00780645	0372	00945902
0283	00629356	0328	00784204	0373	00949686
0284	00632676	0329	00787769	0374	00953482
0285	00636001	0330	00791339	0375	00957279
0286	00639331	0331	00794915	0376	00961081
0287	00642668	0332	00798495	0377	00964888
0288	00646010	0333	00802081	0378	00968700
0289	00649358	0334	00805672	0379	00972517
0290	00652711	0335	00809268	0380	00976338
0291	00656070	0336	00812869	0381	00980164
0292	00659434	0337	00816476	0382	00983996
0293	00662804	0338	00820088	0383	00987831
0294	00666180	0339	00823705	0384	00991672
0295	00669561	0340	00827327	0385	00995517
0296	00672948	0341	00830954	0386	00999368
0297	00676341	0342	00834586	0387	01003224
0298	00679739	0343	00838224	0388	01007083
0299	00683142	0344	00841867	0389	01010948
0300	00686551	0345	00845514	0390	01014818
0301	00689966	0346	00849166	0391	01018692
0302	00693386	0347	00852824	0392	01022571
0303	00696811	0348	00856487	0393	01026455
0304	00700242	0349	00860155	0394	01030343
0305	00703679	0350	00863828	0395	01034237
0306	00707120	0351	00867506	0396	01038135
0307	00710568	0352	00871190	0397	01042038
0308	00714021	0353	00874878	0398	01045945
0309	00717479	0354	00878571	0399	01049857
0310	00720942	0355	00882269	0400	01053774
0311	00724411	0356	00885973	0401	01057695
0312	00727886	0357	00889681	0402	01061622
0313	00731366	0358	00893394	0403	01065553
0314	00734851	0359	00897113	0404	01069488
0315	00738342	0360	00900836	0405	01073428

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
0406	01077373	0451	01259617	0496	01450745
0407	01081323	0452	01263770	0497	01455090
0408	01085277	0453	01267927	0498	01459438
0409	01089236	0454	01272088	0499	01463791
0410	01093199	0455	01276254	0500	01468148
0411	01097167	0456	01280424	0501	01472509
0412	01101140	0457	01284599	0502	01476874
0413	01105117	0458	01288778	0503	01481243
0414	01109099	0459	01292961	0504	01485616
0415	01113086	0460	01297148	0505	01489994
0416	01117077	0461	01301340	0506	01494375
0417	01121073	0462	01305536	0507	01498761
0418	01125073	0463	01309737	0508	01503151
0419	01129078	0464	01313942	0509	01507544
0420	01133088	0465	01318151	0510	01511942
0421	01137102	0466	01322364	0511	01516344
0422	01141120	0467	01326582	0512	01520750
0423	01145144	0468	01330804	0513	01525161
0424	01149171	0469	01335031	0514	01529575
0425	01153203	0470	01339261	0515	01533993
0426	01157240	0471	01343496	0516	01538415
0427	01161282	0472	01347735	0517	01542842
0428	01165328	0473	01351979	0518	01547272
0429	01169378	0474	01356226	0519	01551707
0430	01173433	0475	01360478	0520	01556145
0431	01177492	0476	01364735	0521	01560588
0432	01181556	0477	01368995	0522	01565034
0433	01185625	0478	01373260	0523	01569485
0434	01189697	0479	01377529	0524	01573940
0435	01193775	0480	01381802	0525	01578398
0436	01197857	0481	01386079	0526	01582861
0437	01201943	0482	01390361	0527	01587328
0438	01206034	0483	01394647	0528	01591798
0439	01210129	0484	01398937	0529	01596273
0440	01214229	0485	01403231	0530	01600752
0441	01218333	0486	01407530	0531	01605234
0442	01222441	0487	01411833	0532	01609720
0443	01226554	0488	01416140	0533	01614211
0444	01230672	0489	01420451	0534	01618706
0445	01234794	0490	01424766	0535	01623205
0446	01238920	0491	01429085	0536	01627707
0447	01243050	0492	01433409	0537	01632214
0448	01247186	0493	01437737	0538	01636724
0449	01251325	0494	01442069	0539	01641239
0450	01255469	0495	01446405	0540	01645757

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
0541	01650279	0586	01857800	0631	02073936
0542	01654806	0587	01862500	0632	02078801
0543	01659336	0588	01867203	0633	02083669
0544	01663870	0589	01871910	0634	02088541
0545	01668408	0590	01876621	0635	02093416
0546	01672950	0591	01881335	0636	02098295
0547	01677496	0592	01886053	0637	02103178
0548	01682046	0593	01890775	0638	02108064
0549	01686600	0594	01895500	0639	02112953
0550	01691157	0595	01900230	0640	02117847
0551	01695719	0596	01904963	0641	02122744
0552	01700284	0597	01909700	0642	02127644
0553	01704854	0598	01914440	0643	02132548
0554	01709427	0599	01919184	0644	02137455
0555	01714004	0600	01923932	0645	02142366
0556	01718585	0601	01928684	0646	02147281
0557	01723170	0602	01933439	0647	02152199
0558	01727759	0603	01938198	0648	02157121
0559	01732351	0604	01942961	0649	02162046
0560	01736948	0605	01947727	0650	02166975
0561	01741548	0606	01952497	0651	02171906
0562	01746152	0607	01957271	0652	02176842
0563	01750760	0608	01962048	0653	02181781
0564	01755372	0609	01966829	0654	02186724
0565	01759988	0610	01971614	0655	02191671
0566	01764608	0611	01976403	0656	02196620
0567	01769231	0612	01981195	0657	02201574
0568	01773859	0613	01986990	0658	02206531
0569	01778490	0614	01991790	0659	02211491
0570	01783125	6615	01996593	0660	02216455
0571	01787763	0616	02001400	0661	02220422
0572	01792406	0617	02006210	0662	02225393
0573	01797052	0618	02011024	0663	02230368
0574	01801702	0619	02015842	0664	02235346
0575	01806356	0620	02020663	0665	02240327
0576	01811014	0621	02025488	0666	02245312
0577	01815676	0622	02030317	0667	02250300
0578	01820341	0623	02035149	0668	02255292
0579	01825010	0624	02039985	0669	02260287
0580	01829683	0625	02044824	0670	02265286
0581	01834360	0626	02049667	0671	02270288
0582	01839041	0627	02054514	0672	02275294
0583	01843725	0628	02059364	0673	02280303
0584	01848413	0629	02064218	0674	02285316
0585	01853105	0630	02069075	0675	02290332

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
0676	02295351	0721	02524745	0766	02760840
0677	02300374	0722	02529920	0767	02766161
0678	02305401	0723	02535098	0768	02771485
0679	02310431	0724	02540279	0769	02776812
0680	02315484	0725	02545464	0770	02782142
0681	02320500	0726	02550652	0771	02787475
0682	02325540	0727	02555843	0772	02792812
0683	02330584	0728	02561037	0773	02798152
0684	02335631	0729	02566235	0774	02803495
0685	02340681	0730	02571437	0775	02808841
0686	02345735	0731	02576641	0776	02814190
0687	02350792	0732	02581848	0777	02819543
0688	02355852	0733	02587059	0778	02824898
0689	02360916	0734	02592273	0779	02830257
0690	02365984	0735	02597491	0780	02835619
0691	02371055	0736	02602712	0781	02840984
0692	02376129	0737	02607936	0782	02846352
0693	02381206	0738	02613163	0783	02851723
0694	02386287	0739	02618393	0784	02857097
0695	02391372	0740	02623627	0785	02862475
0696	02396459	0741	02628864	0786	02867856
0697	02401550	0742	02634105	0787	02873240
0698	02406646	0743	02639348	0788	02878627
0699	02411743	0744	02644598	0789	02884017
0700	02416845	0745	02649845	0790	02889410
0701	02421949	0746	02655098	0791	02894806
0702	02427057	0747	02660355	0792	02900206
0703	02432169	0748	02665614	0793	02905608
0704	02437283	0749	02670877	0794	02911014
0705	02442401	0750	02676144	0795	02916423
0706	02447523	0751	02681413	0796	02921835
0707	02452648	0752	02686686	0797	02927250
0708	02457776	0753	02691962	0798	02932668
0709	02462908	0754	02697241	0799	02938089
0710	02468042	0755	02702526	0800	02943513
0711	02473180	0756	02707809	0801	02948941
0712	02478322	0757	02713097	0802	02954371
0713	02483467	0758	02718389	0803	02959805
0714	02488615	0759	02723684	0804	02965241
0715	02493766	0760	02728982	0805	02970681
0716	02498921	0761	02734284	0806	02976124
0717	02504079	0762	02739589	0807	02981570
0718	02509241	0763	02744897	0808	02987019
0719	02514405	0764	02750208	0809	02992471
0720	02519574	0765	02755523	0810	02997926

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
0811	03003385	0856	03252145	0901	03506905
0812	03008846	0857	03257742	0902	03512633
0813	03014310	0858	03263342	0903	03518364
0814	03019778	0859	03268945	0904	03524098
0815	03025248	0860	03274550	0905	03529834
0816	03030722	0861	03280159	0906	03535574
0817	03036198	0862	03285771	0907	03541316
0818	03041678	0863	03291386	0908	03547061
0819	03047161	0864	03297003	0909	03552809
0820	03052646	0865	03302624	0910	03558560
0821	03058135	0866	03308247	0911	03564313
0822	03063627	0867	03313874	0912	03570070
0823	03069122	0868	03319503	0913	03575829
0824	03074620	0869	03325135	0914	03581591
0825	03080121	0870	03330771	0915	03587356
0826	03085625	0871	03336409	0916	03593124
0827	03091132	0872	03342050	0917	03598895
0828	03096642	0873	03347694	0918	03604668
0829	03102155	0874	03353341	0919	03610444
0830	03107671	0875	03358991	0920	03616223
0831	03113190	0876	03364643	0921	03622005
0832	03118713	0877	03370299	0922	03627790
0833	03124238	0878	03375958	0923	03633578
0834	03129766	0879	03381619	0924	03639368
0835	03135297	0880	03387284	0925	03645161
0836	03140832	0881	03392951	0926	03650957
0837	03146369	0882	03398621	0927	03656756
0838	03151909	0883	03404294	0928	03662558
0839	03157452	0884	03409970	0929	03668362
0840	03162999	0885	03415649	0930	03674169
0841	03168548	0886	03421331	0931	03679980
0842	03174100	0887	03427016	0932	03685792
0843	03179655	0888	03432704	0933	03691608
0844	03185214	0889	03438394	0934	03697426
0845	03190775	0850	03444088	0935	03703248
0846	03196339	0891	03449784	0936	03709072
0847	03201906	0892	03455483	0937	03714899
0848	03207476	0893	03461185	0938	03720728
0849	03213050	0894	03466890	0939	03726561
0850	03218626	0895	03472598	0940	03732396
0851	03224205	0896	03478309	0941	03738234
0852	03229787	0897	03484022	0942	03744074
0853	03235372	0898	03489739	0943	03749918
0854	03240960	0899	03495458	0944	03755764
0855	03246551	0900	03501180	0945	03761613

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
0946	03767465	0991	04033636	1036	04305241
0947	03773320	0992	04039613	1037	04311338
0948	03779177	0993	04045593	1038	04317436
0949	03785037	0994	04051576	1039	04323538
0950	03790900	0995	04057561	1040	04329642
0951	03796766	0996	04063549	1041	04335748
0952	03802634	0997	04069540	1042	04341857
0953	03808506	0998	04075533	1043	04347969
0954	03814380	0999	04081529	1044	04354083
0955	03820256	1000	04087528	1045	04360200
0956	03826136	1001	04093529	1046	04366319
0957	03832018	1002	04099533	1047	04372442
0958	03837903	1003	04105540	1048	04378566
0959	03843791	1004	04111549	1049	04384693
0960	03849681	1005	04117561	1050	04390823
0961	03855574	1006	04123576	1051	04396955
0962	03861470	1007	04129593	1052	04403090
0963	03867369	1008	04135613	1053	04409228
0964	03873270	1009	04141635	1054	04415368
0965	03879174	1010	04147661	1055	04421511
0966	03885081	1011	04153689	1056	04427656
0967	03890991	1012	04159719	1057	04433804
0968	03896903	1013	04165752	1058	04439954
0969	03902818	1014	04171788	1059	04446107
0970	03908736	1015	04177827	1060	04452262
0971	03914657	1016	04183868	1061	04458420
0972	03920580	1017	04189912	1062	04464581
0973	03926506	1018	04195958	1063	04470744
0974	03932435	1019	04202007	1064	04476910
0975	03938366	1020	04208059	1065	04483078
0976	03944300	1021	04214113	1066	04489249
0977	03950237	1022	04220170	1067	04495422
0978	03956176	1023	04226229	1068	04501598
0979	03962119	1024	04232291	1069	04507777
0980	03968064	1025	04238356	1070	04513958
0981	03974011	1026	04244424	1071	04520141
0982	03979962	1027	04250494	1072	04526327
0983	03985915	1028	04256566	1073	04532516
0984	03991870	1029	04262642	1074	04538707
0985	03997829	1030	04268719	1075	04544901
0986	04003790	1031	04274800	1076	04551097
0987	04009754	1032	04280883	1077	04557296
0988	04015720	1033	04286969	1078	04563497
0989	04021689	1034	04293057	1079	04569701
0990	04027661	1035	04299148	1080	04575907

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
1081	04582116	1126	04864103	1171	05151054
1082	04588327	1127	04870426	1172	05157486
1083	04594541	1128	04876752	1173	05163920
1084	04600758	1129	04883080	1174	05170357
1085	04606977	1130	04889411	1175	05176796
1086	04613198	1131	04895744	1176	05183237
1087	04619422	1132	04902079	1177	05189681
1088	04625649	1133	04908417	1178	05196127
1089	04631878	1134	04914758	1179	05202576
1090	04638109	1135	04921100	1180	05209027
1091	04644343	1136	04927446	1181	05215480
1092	04650580	1137	04933793	1182	05221936
1093	04656819	1138	04940144	1183	05228394
1094	04663060	1139	04946496	1184	05234855
1095	04669304	1140	04952851	1185	05241317
1096	04675551	1141	04959209	1186	05247783
1097	04681800	1142	04965568	1187	05254250
1098	04688052	1143	04971931	1188	05260720
1099	04694306	1144	04978295	1189	05267192
1100	04700562	1145	04984663	1190	05273667
1101	04706821	1146	04991032	1191	05280144
1102	04713083	1147	04997404	1192	05286623
1103	04719347	1148	05003779	1193	05293105
1104	04725613	1149	05010155	1194	05299589
1105	04731882	1150	05016535	1195	05306075
1106	04738154	1151	05022916	1196	05312564
1107	04744428	1152	05029300	1197	05319055
1108	04750704	1153	05035687	1198	05325548
1109	04756983	1154	05042076	1199	05332044
1110	04763265	1155	05048467	1200	05338542
1111	04769548	1156	05054861	1201	05345042
1112	04775835	1157	05061257	1202	05351545
1113	04782124	1158	05067655	1203	05358050
1114	04788415	1159	05074056	1204	05364558
1115	04794709	1160	05080459	1205	05371067
1116	04801005	1161	05086865	1206	05377579
1117	04807304	1162	05093273	1207	05384094
1118	04813605	1163	05099684	1208	05390610
1119	04819908	1164	05106097	1209	05397130
1120	04826215	1165	05112512	1210	05403651
1121	04832523	1166	05118930	1211	05410175
1122	04838834	1167	05125350	1212	05416701
1123	04845148	1168	05131772	1213	05423229
1124	04851464	1169	05138197	1214	05429760
1125	04857782	1170	05144624	1215	05436293

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
1216	05442828	1261	05739292	1306	06040318
1217	05449366	1262	05745932	1307	06047058
1218	05455906	1263	05752575	1308	06053801
1219	05462448	1264	05759220	1309	06060546
1220	05468992	1265	05765867	1310	06067293
1221	05475539	1266	05772516	1311	06074042
1222	05482088	1267	05779168	1312	06080793
1223	05488640	1268	05785822	1313	06087546
1224	05495194	1269	05792478	1314	06094302
1225	05501750	1270	05799136	1315	06101060
1226	05508308	1271	05805797	1316	06107820
1227	05514869	1272	05812460	1317	06114582
1228	05521432	1273	05819125	1318	06121347
1229	05527997	1274	05825792	1319	06128113
1230	05534565	1275	05832461	1320	06134882
1231	05541135	1276	05839133	1321	06141653
1232	05547707	1277	05845807	1322	06148426
1233	05554281	1278	05852483	1323	06155201
1234	05560858	1279	05859162	1324	06161978
1235	05567437	1280	05865843	1325	06168758
1236	05574019	1281	05872525	1326	06175540
1237	05580602	1282	05879211	1327	06182324
1238	05587188	1283	05885898	1328	06189110
1239	05593776	1284	05892588	1329	06195898
1240	05600367	1285	05899279	1330	06202688
1241	05606960	1286	05905973	1331	06209481
1242	05613555	1287	05912670	1332	06216276
1243	05620152	1288	05919368	1333	06223073
1244	05626752	1289	05926069	1334	06229872
1245	05633353	1290	05932772	1335	06236673
1246	05639958	1291	05939477	1336	06243476
1247	05646564	1292	05946184	1337	06250282
1248	05653173	1293	05952894	1338	06257090
1249	05659784	1294	05959605	1339	06263899
1250	05666397	1295	05966319	1340	06270711
1251	05673012	1296	05973035	1341	06277525
1252	05679730	1297	05979754	1342	06284342
1253	05686250	1298	05986474	1343	06291160
1254	05692873	1299	05993197	1344	06297981
1255	05699497	1300	05999922	1345	06304803
1256	05706124	1301	06006649	1346	06311628
1257	05712753	1302	06013379	1347	06318455
1258	05719384	1303	06020110	1348	06325284
1259	05726018	1304	06026844	1349	06332116
1260	05732654	1305	06033580	1350	06338949

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
1351	06345785	1396	06655575	1441	06969577
1352	06352622	1397	06662507	1442	06976602
1353	06359462	1398	06669442	1443	06983629
1354	06366304	1399	06676378	1444	06990657
1355	06373148	1400	06683317	1445	06997688
1356	06379994	1401	06690258	1446	07004721
1357	06386843	1402	06697201	1447	07011756
1358	06393693	1403	06704146	1448	07018793
1359	06400546	1404	06711093	1449	07025832
1360	06407400	1405	06718042	1450	07032873
1361	06414257	1406	06724993	1451	07039916
1362	06421116	1407	06731946	1452	07046961
1363	06427977	1408	06738901	1453	07054008
1364	06434840	1409	06745859	1454	07061057
1365	06441706	1410	06752818	1455	07068108
1366	06448573	1411	06759780	1456	07075162
1367	06455443	1412	06766743	1457	07082217
1368	06462314	1413	06773709	1458	07089274
1369	06469188	1414	06780676	1459	07096333
1370	06476064	1415	06787646	1460	07103394
1371	06482942	1416	06794618	1461	07110457
1372	06489822	1417	06801592	1462	07117522
1373	06496704	1418	06808567	1463	07124589
1374	06503589	1419	06815545	1464	07131659
1375	06510475	1420	06822525	1465	07138730
1376	06517364	1421	06829507	1466	07145803
1377	06524254	1422	06836491	1467	07152878
1378	06531147	1423	06843478	1468	07159955
1379	06538042	1424	06850466	1469	07167034
1380	06544939	1425	06857456	1470	07174115
1381	06551838	1426	06864448	1471	07181198
1382	06558739	1427	06871443	1472	07188284
1383	06565642	1428	06878439	1473	07195371
1384	06572548	1429	06885437	1474	07202460
1385	06579455	1430	06892438	1475	07209551
1386	06586365	1431	06899440	1476	07216644
1387	06593276	1432	06906445	1477	07223739
1388	06600190	1433	06913451	1478	07230836
1389	06607106	1434	06920460	1479	07237935
1390	06614024	1435	06927470	1480	07245036
1391	06620944	1436	06934483	1481	07252139
1392	06627866	1437	06941498	1482	07259244
1393	06634790	1438	06948515	1483	07266351
1394	06641716	1439	06955533	1484	07273460
1395	06648644	1440	06962554	1485	07280571

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
1486	07287684	1531	07609792	1576	07935802
1487	07294798	1532	07616994	1577	07943090
1488	07301915	1533	07624199	1578	07950380
1489	07309034	1534	07631406	1579	07957672
1490	07316155	1535	07638614	1580	07964966
1491	07323278	1536	07645824	1581	07972262
1492	07330402	1537	07653037	1582	07979560
1493	07337529	1538	07660251	1583	07986859
1494	07344658	1539	07667467	1584	07994161
1495	07351788	1540	07674685	1585	08001464
1496	07358921	1541	07681905	1586	08008769
1497	07366056	1542	07689127	1587	08016076
1498	07373192	1543	07696350	1588	08023385
1499	07380331	1544	07703576	1589	08030696
1500	07387471	1545	07710804	1590	08038008
1501	07394613	1546	07718033	1591	08045323
1502	07401758	1547	07725265	1592	08052639
1503	07408904	1548	07732498	1593	08059957
1504	07416052	1549	07739733	1594	08067277
1505	07423203	1550	07746970	1595	08074599
1506	07430355	1551	07754209	1596	08081923
1507	07437509	1552	07761450	1597	08089248
1508	07444665	1553	07768693	1598	08096576
1509	07451823	1554	07775938	1599	08103905
1510	07458983	1555	07783185	1600	08111236
1511	07466145	1556	07790433	1601	08118569
1512	07473309	1557	07797684	1602	08125904
1513	07480475	1558	07804936	1603	08133241
1514	07487643	1559	07812190	1604	08140580
1515	07494812	1560	07819446	1605	08147920
1516	07501984	1561	07826704	1606	08155262
1517	07509158	1562	07833964	1607	08162607
1518	07516333	1563	07841226	1608	08169953
1519	07523511	1564	07848490	1609	08177300
1520	07530690	1565	07855756	1610	08184650
1521	07537872	1566	07863023	1611	08192002
1522	07545055	1567	07870292	1612	08199355
1523	07552240	1568	07877564	1613	08206710
1524	07559427	1569	07884837	1614	08214067
1525	07566616	1570	07892112	1615	08221426
1526	07573808	1571	07899389	1616	08228789
1527	07581001	1572	07906668	1617	08236150
1528	07588195	1573	07913949	1618	08243514
1529	07595392	1574	07921231	1619	08250880
1530	07602591	1575	07928516	1620	08258248

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
1621	08265618	1666	08599148	1711	08936302
1622	08272990	1667	08606601	1712	08943835
1623	08280364	1668	08614056	1713	08951369
1624	08287739	1669	08621513	1714	08958906
1625	08295116	1670	08628972	1715	08966444
1626	08302495	1671	08636432	1716	08973983
1627	08309876	1672	08643894	1717	08981525
1628	08317259	1673	08651358	1718	08989068
1629	08324644	1674	08658824	1719	08996613
1630	08332030	1675	08666292	1720	09004160
1631	08339418	1676	08673761	1721	09011709
1632	08346808	1677	08681232	1722	09019259
1633	08354200	1678	08688705	1723	09026811
1634	08361594	1679	08696180	1724	09034364
1635	08368989	1680	08703656	1725	09041920
1636	08376387	1681	08711134	1726	09049477
1637	08383786	1682	08718614	1727	09057036
1638	08391187	1683	08726096	1728	09064596
1639	08398590	1684	08733580	1729	09072159
1640	08405994	1685	08741065	1730	09079723
1641	08413401	1686	08748552	1731	09087289
1642	08420809	1687	08756041	1732	09094856
1643	08428219	1688	08763532	1733	09102425
1644	08435631	1689	08771024	1734	09109996
1645	08443044	1690	08778518	1735	09117569
1646	08450460	1691	08786014	1736	09125143
1647	08457877	1692	08793512	1737	09132720
1648	08465296	1693	08801011	1738	09140298
1649	08472717	1694	08808512	1739	09147877
1650	08480140	1695	08816015	1740	09155458
1651	08487564	1696	08823520	1741	09163042
1652	08494991	1697	08831027	1742	09170626
1653	08502419	1698	08838535	1743	09178213
1654	08509849	1699	08846045	1744	09185801
1655	08517281	1700	08853557	1745	09193391
1656	08524714	1701	08861070	1746	09200983
1657	08532149	1702	08868585	1747	09208576
1658	08539587	1703	08876103	1748	09216171
1659	08547025	1704	08883621	1749	09223768
1660	08554466	1705	08891142	1750	09231366
1661	08561909	1706	08898664	1751	09238966
1662	08569353	1707	08906188	1752	09246568
1663	08576799	1708	08913714	1753	09254172
1664	08584247	1709	08921242	1754	09261777
1665	08591697	1710	08928771	1755	09269384

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
1756	09276993	1801	09621138	1846	09968654
1757	09284603	1802	09628824	1847	09976414
1758	09292216	1803	09636512	1848	09984176
1759	09299829	1804	09644201	1849	09991939
1760	09307445	1805	09651893	1850	09999704
1761	09315062	1806	09659585	1851	10007471
1762	09322681	1807	09667280	1852	10015240
1763	09330302	1808	09674976	1853	10023010
1764	09337924	1809	09682674	1854	10030781
1765	09345548	1810	09690374	1855	10038555
1766	09353174	1811	09698075	1856	10046330
1767	09360802	1812	09705778	1857	10054106
1768	09368431	1813	09713482	1858	10061884
1769	09376062	1814	09721188	1859	10069664
1770	09383694	1815	09728896	1860	10077445
1771	09391328	1816	09736606	1861	10085228
1772	09398964	1817	09744317	1862	10093012
1773	09406602	1818	09752029	1863	10100799
1774	09414241	1819	09759744	1864	10108586
1775	09421882	1820	09767460	1865	10116376
1776	09429525	1821	09775178	1866	10124167
1777	09437169	1822	09782897	1867	10131959
1778	09444815	1823	09790618	1868	10139754
1779	09452463	1824	09798341	1869	10147549
1780	09460112	1825	09806065	1870	10155347
1781	09467764	1826	09813791	1871	10163146
1782	09475416	1827	09821519	1872	10170947
1783	09483071	1828	09829248	1873	10178749
1784	09490727	1829	09836979	1874	10186553
1785	09498385	1830	09844711	1875	10194358
1786	09506044	1831	09852445	1876	10202165
1787	09513705	1832	09860181	1877	10209974
1788	09521368	1833	09867919	1878	10217784
1789	09529033	1834	09875658	1879	10225596
1790	09536699	1835	09883398	1880	10233409
1791	09544367	1836	09891141	1881	10241224
1792	09552036	1837	09898885	1882	10249041
1793	09559708	1838	09906630	1883	10256859
1794	09567381	1839	09914377	1884	10264679
1795	09575055	1840	09922126	1885	10272500
1796	09582731	1841	09929877	1886	10280323
1797	09590409	1842	09937629	1887	10288148
1798	09598089	1843	09945383	1888	10295974
1799	09605770	1844	09953138	1889	10303802
1800	09613453	1845	09960895	1890	10311631

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
1891	10319462	1936	10673487	1981	11030652
1892	10327295	1937	10681390	1982	11038624
1893	10335129	1938	10689295	1983	11046598
1894	10342965	1939	10697201	1984	11054573
1895	10350802	1940	10705109	1985	11062550
1896	10358641	1941	10713018	1986	11070528
1897	10366481	1942	10720929	1987	11078507
1898	10374324	1943	10728842	1988	11086489
1899	10382167	1944	10736756	1989	11094471
1900	10390013	1945	10744671	1990	11102456
1901	10397859	1946	10752588	1991	11110441
1902	10405708	1947	10760507	1992	11118429
1903	10413558	1948	10768427	1993	11126417
1904	10421409	1949	10776349	1994	11134407
1905	10429262	1950	10784272	1995	11142399
1906	10437117	1951	10792197	1996	11150392
1907	10444973	1952	10800123	1997	11158387
1908	10452831	1953	10808051	1998	11166383
1909	10460691	1954	10815980	1999	11174381
1910	10468552	1955	10823911	2000	11182380
1911	10476414	1956	10831844	2001	11190381
1912	10484278	1957	10839778	2002	11198383
1913	10492144	1958	10847713	2003	11206387
1914	10500011	1959	10855650	2004	11214392
1915	10507880	1960	10863589	2005	11222399
1916	10515751	1961	10871529	2006	11230407
1917	10523623	1962	10879471	2007	11238417
1918	10531496	1963	10887414	2008	11246428
1919	10539371	1964	10895359	2009	11254441
1920	10547248	1965	10903305	2010	11262455
1921	10555126	1966	10911253	2011	11270471
1922	10563006	1967	10919202	2012	11278488
1923	10570887	1968	10927153	2013	11286507
1924	10578770	1969	10935105	2014	11294527
1925	10586655	1970	10943059	2015	11302549
1926	10594541	1971	10951014	2016	11310572
1927	10602428	1972	10958971	2017	11318597
1928	10610318	1973	10966930	2018	11326623
1929	10618208	1974	10974890	2019	11334650
1930	10626101	1975	10982851	2020	11342679
1931	10633994	1976	10990814	2021	11350710
1932	10641890	1977	10998779	2022	11358742
1933	10649787	1978	11006745	2023	11366776
1934	10657685	1979	11014713	2024	11374811
1935	10665585	1980	11022682	2025	11382847

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
2026	11390885	2071	11754115	2116	12.20272
2027	11398925	2072	11762220	2117	12128442
2028	11406966	2073	11770327	2118	12136613
2029	11415008	2074	11778435	2119	12144785
2030	11423052	2075	11786545	2120	12152959
2031	11431097	2076	11794656	2121	12161134
2032	11439144	2077	11802768	2122	12169311
2033	11447193	2078	11810882	2123	12177489
2034	11455242	2079	11818998	2124	12185668
2035	11463294	2080	11827114	2125	12193849
2036	11471346	2081	11835233	2126	12202031
2037	11479401	2082	11843352	2127	12210215
2238	11487456	2083	11851474	2128	12218400
2039	11495514	2084	11859596	2129	12226587
2040	11503572	2085	11867720	2130	12234774
2041	11511632	2086	11875846	2131	12242964
2042	11519694	2087	11883972	2132	12251154
2043	11527757	2088	11892101	2133	12259346
2044	11535821	2089	11900230	2134	12267540
2045	11543887	2090	11908362	2135	12275735
2046	11551955	2091	11916494	2136	12283931
2047	11560024	2092	11924628	2137	12292129
2048	11568094	2093	11932764	2138	12300328
2049	11576166	2094	11940901	2139	12308528
2050	11584239	2095	11949039	2140	12316730
2051	11592314	2096	11957179	2141	12324933
2052	11600390	2097	11965320	2142	12333138
2053	11608468	2098	11973462	2143	12341344
2054	11616547	2099	11981606	2144	12349551
2055	11624628	2100	11989752	2145	12357760
2056	11632710	2101	11997899	2146	12365970
2057	11640793	2102	12006047	2147	12374182
2058	11648878	2103	12014197	2148	12382395
2059	11656965	2104	12022348	2149	12390609
2060	11665053	2105	12030501	2150	12398825
2061	11673142	2106	12038655	2151	12407042
2062	11681233	2107	12046810	2152	12415261
2063	11689325	2108	12054967	2153	12423481
2064	11697419	2109	12063125	2154	12431702
2065	11705514	2110	12071285	2155	12439924
2066	11713610	2111	12079446	2156	12448149
2067	11721708	2112	12087608	2157	12456374
2068	11729808	2113	12095772	2158	12464601
2069	11737909	2114	12103938	2159	12472829
2070	11746011	2115	12112104	2160	12481059

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
2161	12489290	2206	12861101	2251	13235641
2162	12497522	2207	12869394	2252	13243994
2163	12505756	2208	12877689	2253	13252349
2164	12513991	2209	12885986	2254	13260705
2165	12522227	2210	12894283	2255	13269063
2166	12530465	2211	12902583	2256	13277422
2167	12538704	2212	12910883	2257	13285782
2168	12546945	2213	12919185	2258	13294143
2169	12555187	2214	12927488	2259	13302506
2170	12563430	2215	12935792	2260	13310870
2171	12571675	2216	12944098	2261	13319236
2172	12579921	2217	12952405	2262	13327603
2173	12588169	2218	12960714	2263	13335971
2174	12596418	2219	12969024	2264	13344340
2175	12604668	2220	12977335	2265	13352711
2176	12612919	2221	12985647	2266	13361083
2177	12621172	2222	12993961	2267	13369456
2178	12629427	2223	13002276	2268	13377831
2179	12637682	2224	13010593	2269	13386206
2180	12645939	2225	13018911	2270	13394584
2181	12654198	2226	13027230	2271	13402962
2182	12662458	2227	13035550	2272	13411342
2183	12670719	2228	13043872	2273	13419723
2184	12678981	2229	13052195	2274	13428105
2185	12687245	2230	13060520	2275	13436489
2186	12695511	2231	13068846	2276	13444874
2187	12703777	2232	13077173	2277	13453260
2188	12712045	2233	13085501	2278	13461648
2189	12720314	2234	13093831	2279	13470037
2190	12728585	2235	13102162	2280	13478427
2191	12736857	2236	13110495	2281	13486819
2192	12745131	2237	13118828	2282	13495212
2193	12753405	2238	13127164	2283	13503606
2194	12761682	2239	13135500	2284	13512001
2195	12769959	2240	13143838	2285	13520398
2196	12778238	2241	13152177	2286	13528796
2197	12786518	2242	13160517	2287	13537195
2198	12794800	2243	13168859	2288	13545595
2199	12803082	2244	13177202	2289	13553997
2200	12811367	2245	13185546	2290	13562400
2201	12819652	2246	13193892	2291	13570805
2202	12827939	2247	13202239	2292	13579211
2203	12836228	2248	13210588	2293	13587618
2204	12844517	2249	13218937	2294	13596026
2205	12852808	2250	13227288	2295	13604435

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
2296	13612846	2341	13992656	2386	14375009
2297	13621259	2342	14001126	2387	14383535
2298	13629672	2343	14009596	2388	14392061
2299	13638087	2344	14018068	2389	14400589
2300	13646503	2345	14026541	2390	14409117
2301	13654920	2346	14035015	2391	14417647
2302	13663339	2347	14043491	2392	14426179
2303	13671758	2348	14051968	2393	14434711
2304	13680180	2349	14060446	2394	14443245
2305	13688602	2350	14068925	2395	14451780
2306	13697026	2351	14077406	2396	14460316
2307	13705451	2352	14085888	2397	14468854
2308	13713877	2353	14094371	2398	14477392
2309	13722304	2354	14102855	2399	14485932
2310	13730733	2355	14111341	2400	14494473
2311	13739163	2356	14119828	2401	14503015
2312	13747595	2357	14128316	2402	14511559
2313	13756027	2358	14136805	2403	14520103
2314	13764461	2359	14145296	2404	14528649
2315	13772896	2360	14153787	2405	14537197
2316	13781333	2361	14162281	2406	14545745
2317	13789771	2362	14170775	2407	14554294
2318	13798210	2363	14179270	2408	14562845
2319	13806650	2364	14187767	2409	14571397
2320	13815091	2365	14196265	2410	14579950
2321	13823534	2366	14204764	2411	14588505
2322	13831978	2367	14213265	2412	14597060
2323	13840424	2368	14221767	2413	14605617
2324	13848870	2369	14230270	2414	14614175
2325	13857318	2370	14238774	2415	14622735
2326	13865767	2371	14247279	2416	14631295
2327	13874218	2372	14255786	2417	14639857
2328	13882669	2373	14264294	2418	14648419
2329	13891123	2374	14272803	2419	14656984
2330	13899577	2375	14281314	2420	14665549
2331	13908032	2376	14289825	2421	14674115
2332	13916489	2377	14298338	2422	14682683
2333	13924947	2378	14306852	2423	14691252
2334	13933406	2379	14315368	2424	14699822
2335	13941867	2380	14323884	2425	14708393
2336	13950328	2381	14332402	2426	14716966
2337	13958791	2382	14340921	2427	14725540
2338	13967256	2383	14349441	2428	14734114
2339	13975721	2384	14357963	2429	14742691
2340	13984188	2385	14366485	2430	14751268

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
2431	14759846	2476	15147109	2521	15536740
2432	14768426	2477	15155742	2522	15545425
2433	14777007	2478	15164376	2523	15554111
2434	14785589	2479	15173011	2524	15562798
2435	14794172	2480	15181648	2525	15571487
2436	14802757	2481	15190285	2526	15580176
2437	14811342	2482	15198924	2527	15588867
2438	14819929	2483	15207564	2528	15597559
2439	14828517	2484	15216205	2529	15606252
2440	14837107	2485	15224848	2530	15614946
2441	14845697	2486	15233491	2531	15623641
2442	14854289	2487	15242136	2532	15632337
2443	14862882	2488	15250781	2533	15641035
2444	14871476	2489	15259428	2534	15649733
2445	14880071	2490	15268077	2535	15658433
2446	14888667	2491	15276726	2536	15667134
2447	14897265	2492	15285376	2537	15675836
2448	14905864	2493	15294028	2538	15684539
2449	14914464	2494	15302681	2539	15693243
2450	14923065	2495	15311334	2540	15701949
2451	14931667	2496	15319989	2541	15710655
2452	14940271	2497	15328646	2542	15719363
2453	14948875	2498	15337303	2543	15728071
2454	14957481	2499	15345961	2544	15736781
2455	14966088	2500	15354621	2545	15745492
2456	14974696	2501	15363282	2546	15754205
2457	14983306	2502	15371944	2547	15762918
2458	14991917	2503	15380607	2548	15771632
2459	15000528	2504	15389271	2549	15780348
2460	15009141	2505	15397937	2550	15789064
2461	15017755	2506	15406603	2551	15797782
2462	15026371	2507	15415271	2552	15806501
2463	15034987	2508	15423940	2553	15815221
2464	15043605	2509	15432610	2554	15823942
2465	15052224	2510	15441281	2555	15832665
2466	15060844	2511	15449954	2556	15841388
2467	15069465	2512	15458627	2557	15850113
2468	15078088	2513	15467302	2558	15858838
2469	15086711	2514	15475978	2559	15867565
2470	15095336	2515	15484655	2560	15876293
2471	15103962	2516	15493333	2561	15885022
2472	15112589	2517	15502012	2562	15893752
2473	15121217	2518	15510692	2563	15902483
2474	15129847	2519	15519374	2564	15911215
2475	15138477	2520	15528056	2565	15919949

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
2566	15928684	2611	16322884	2656	16719287
2567	15937419	2612	16331669	2657	16728121
2568	15946157	2613	16340456	2658	16736956
2569	15954894	2614	16349243	2659	16745791
2570	15963633	2615	16358032	2660	16754628
2571	15972373	2616	16366821	2661	16763466
2572	15981114	2617	16375612	2662	16772305
2573	15989856	2618	16384404	2663	16781145
2574	15998600	2619	16393196	2664	16789986
2575	16007345	2620	16401990	2665	16798828
2576	16016091	2621	16410785	2666	16807671
2577	16024837	2622	16419581	2667	16816515
2578	16033585	2623	16428379	2668	16825360
2579	16042334	2624	16437177	2669	16834207
2580	16051084	2625	16445976	2670	16843054
2581	16059836	2626	16454777	2671	16851902
2582	16068588	2627	16463578	2672	16860752
2583	16077341	2628	16472381	2673	16869602
2584	16086096	2629	16481184	2674	16878454
2585	16094852	2630	16489989	2675	16881306
2586	16103608	2631	16498795	2676	16896160
2587	16112366	2632	16507602	2677	16905015
2588	16121125	2633	16516410	2678	16913870
2589	16129885	2634	16525219	2679	16922727
2590	16138646	2635	16534029	2680	16931585
2591	16147409	2636	16542840	2681	16940444
2592	16156172	2637	16551652	2682	16949304
2593	16164937	2638	16560465	2683	16958165
2594	16173702	2639	16569280	2684	16967027
2595	16182469	2640	16578095	2685	16975890
2596	16191237	2641	16586912	2686	16984754
2597	16200005	2642	16595729	2687	16993619
2598	16208775	2643	16604548	2688	17002485
2599	16217546	2644	16613368	2689	17011352
2600	16226319	2645	16622188	2690	17020221
2601	16235092	2646	16631010	2691	17029090
2602	16243866	2647	16639833	2692	17037960
2603	16252641	2648	16648657	2693	17046832
2604	16261418	2649	16657482	2694	17055704
2605	16270196	2650	16666308	2695	17064578
2606	16278974	2651	16675136	2696	17073452
2607	16287754	2652	16683964	2697	17082328
2608	16296535	2653	16692793	2698	17091204
2609	16305317	2654	16701624	2699	17100082
2610	16314100	2655	16710455	2700	17108961

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
2701	17117840	2746	17518490	2791	17921186
2702	17126721	2747	17527417	2792	17930157
2703	17135603	2748	17536345	2793	17939130
2704	17144486	2749	17545274	2794	17948104
2705	17153370	2750	17554203	2795	17957079
2706	17162254	2751	17563134	2796	17966054
2707	17171140	2752	17572066	2797	17975030
2708	17180027	2753	17580999	2798	17984008
2709	17188915	2754	17589933	2799	17992986
2710	17197804	2755	17598867	2800	18001966
2711	17206694	2756	17607803	2801	18010946
2712	17215585	2757	17616740	2802	18019928
2713	17224477	2758	17625678	2803	18028910
2714	17233371	2759	17634617	2804	18037894
2715	17242265	2760	17643556	2805	18046878
2716	17251160	2761	17652497	2806	18055863
2717	17260056	2762	17661439	2807	18064850
2718	17268953	2763	17670382	2808	18073837
2719	17277852	2764	17679326	2809	18082825
2720	17286751	2765	17688271	2810	18091815
2721	17295651	2766	17697217	2811	18100805
2722	17304553	2767	17706163	2812	18109796
2723	17313455	2768	17715111	2813	18118788
2724	17322358	2769	17724060	2814	18127781
2725	17331263	2770	17733010	2815	18136775
2726	17340168	2771	17741961	2816	18145771
2727	17349075	2772	17750913	2817	18154767
2728	17357982	2773	17759865	2818	18163764
2729	17366891	2774	17768819	2819	18172762
2730	17375800	2775	17777774	2820	18181761
2731	17384711	2776	17786730	2821	18190761
2732	17393622	2777	17795687	2822	18199762
2733	17402535	2778	17804644	2823	18208763
2734	17411448	2779	17813603	2824	18217766
2735	17420363	2780	17822563	2825	18226770
2736	17429278	2781	17831524	2826	18235775
2737	17438195	2782	17840486	2827	18244781
2738	17447113	2783	17849448	2828	18253787
2739	17456031	2784	17858412	2829	18262795
2740	17464951	2785	17867377	2830	18271804
2741	17473872	2786	17876342	2831	18280813
2742	17482793	2787	17885309	2832	18289824
2743	17491716	2788	17894277	2833	18298835
2744	17500640	2789	17903245	2834	18307848
2745	17509565	2790	17912215	2835	18316861

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area.	Verſ. ſine	Seg. area.
2836	18325876	2881	18732510	2926	19141040
2837	18334891	2882	18741568	2927	19150139
2838	18343907	2883	18750627	2928	19159240
2839	18352925	2884	18759687	2929	19168341
2840	18361943	2885	18768748	2930	19177443
2841	18370962	2886	18777810	2931	19186547
2842	18379982	2887	18786872	2932	19195651
2843	18389004	2888	18795936	2933	19204756
2844	18398026	2889	18805001	2934	19213862
2845	18407049	2890	18814066	2935	19222969
2846	18416073	2891	18823132	2936	19232076
2847	18425098	2892	18832200	2937	19241185
2848	18434124	2893	18841268	2938	19250295
2849	18443150	2894	18850337	2939	19259405
2850	18452178	2895	18859407	2940	19268517
2851	18461207	2896	18868479	2941	19277629
2852	18470237	2897	18877551	2942	19286742
2853	18479267	2898	18886623	2943	19295856
2854	18488299	2899	18895697	2944	19304971
2855	18497332	2900	18904772	2945	19314087
2856	18506365	2901	18913848	2946	19323204
2857	18515400	2902	18922924	2947	19332321
2858	18524435	2903	18932002	2948	19341440
2859	18533471	2904	18941080	2949	19350560
2860	18542509	2905	18950160	2950	19359680
2861	18551547	2906	18959240	2951	19368801
2862	18560586	2907	18968321	2952	19377924
2863	18569626	2908	18977404	2953	19387047
2864	18578667	2909	18986487	2954	19396171
2865	18587709	2910	18995571	2955	19405296
2866	18596752	2911	19004656	2956	19414421
2867	18605796	2912	19013741	2957	19423548
2868	18614841	2913	19022828	2958	19432676
2869	18623887	2914	19031916	2959	19441804
2870	18632934	2915	19041005	2960	19450933
2871	18641981	2916	19050094	2961	19460064
2872	18651030	2917	19059184	2962	19469195
2873	18660080	2918	19068276	2963	19478327
2874	18669130	2919	19077368	2964	19487460
2875	18678182	2920	19086461	2965	19496594
2876	18687234	2921	19095555	2966	19505728
2877	18696287	2922	19104650	2967	19514864
2878	18705342	2923	19113746	2968	19524001
2879	18714397	2924	19122843	2969	19533138
2880	18723453	2925	19131941	2970	19542276

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
2971	19551415	3016	19963589	3061	20377514
2972	19560556	3017	19972769	3062	20386732
2973	19569696	3018	19981949	3063	20395951
2974	19578838	3019	19991130	3064	20405170
2975	19587981	3020	20000313	3065	20414391
2976	19597125	3021	20009495	3066	20423612
2977	19606269	3022	20018679	3067	20432834
2978	19615415	3023	20027864	3068	20442057
2979	19624561	3024	20037049	3069	20451281
2980	19633708	3025	20046236	3070	20460505
2981	19642856	3026	20055423	3071	20469731
2982	19652005	3027	20064611	3072	20478957
2983	19661155	3028	20073800	3073	20488184
2984	19670305	3029	20082990	3074	20497412
2985	19679457	3030	20092181	3075	20506641
2986	19688609	3031	20101372	3076	20515870
2987	19697763	3032	20110565	3077	20525101
2988	19706917	3033	20119758	3078	20534332
2989	19716072	3034	20128952	3079	20543564
2990	19725228	3035	20138147	3080	20552797
2991	19734385	3036	20147343	3081	20562031
2992	19743542	3037	20156539	3082	20571265
2993	19752701	3038	20165737	3083	20580501
2994	19761861	3039	20174935	3084	20589737
2995	19771021	3040	20184134	3085	20598974
2996	19780182	3041	20193335	3086	20608212
2997	19789344	3042	20202535	3087	20617451
2998	19798507	3043	20211737	3088	20626690
2999	19807671	3044	20220940	3089	20635930
3000	19816836	3045	20230143	3090	20645172
3001	19826001	3046	20239348	3091	20654414
3002	19835168	3047	20248553	3092	20663657
3003	19844335	3048	20257759	3093	20672900
3004	19853503	3049	20266966	3094	20682145
3005	19862672	3050	20276173	3095	20691390
3006	19871842	3051	20285382	3096	20700636
3007	19881013	3052	20294591	3097	20709883
3008	19890185	3053	20303802	3098	20719131
3009	19899357	3054	20313013	3099	20728380
3010	19908531	3055	20322225	3100	20737629
3011	19917705	3056	20331438	3101	20746879
3012	19926880	3057	20340651	3102	20756131
3013	19936056	3058	20349866	3103	20765382
3014	19945233	3059	20359081	3104	20774635
3015	19954411	3060	20368297	3105	20783889

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
3106	20793143	3151	21210430	3196	21629328
3107	20802398	3152	21219721	3197	21638655
3108	20811654	3153	21229013	3198	21647983
3109	20820911	3154	21238307	3199	21657311
3110	20830169	3155	21247600	3200	21666640
3111	20839427	3156	21256895	3201	21675970
3112	20848687	3157	21266191	3202	21685301
3113	20857947	3158	21275487	3203	21694632
3114	20867208	3159	21284784	3204	21703964
3115	20876469	3160	21294082	3205	21713297
3116	20885732	3161	21303980	3206	21722631
3117	20894995	3162	21312680	3207	21731966
3118	20904259	3163	21321380	3208	21741301
3119	20913524	3164	21331281	3209	21750637
3120	20922790	3165	21340583	3210	21759974
3121	20932057	3166	21349886	3211	21769311
3122	20941324	3167	21359189	3212	21778650
3123	20950592	3168	21368493	3213	21787989
3124	20959861	3169	21377798	3214	21797329
3125	20969131	3170	21387104	3215	21806669
3126	20978402	3171	21396410	3216	21816011
3127	20987673	3172	21405718	3217	21825353
3128	20996946	3173	21415026	3218	21834696
3129	21006219	3174	21424335	3219	21844040
3130	21015493	3175	21433644	3220	21853384
3131	21024767	3176	21442955	3221	21862729
3132	21034043	3177	21452266	3222	21872076
3133	21043319	3178	21461578	3223	21881422
3134	21052596	3179	21470891	3224	21890770
3135	21061874	3180	21480205	3225	21900118
3136	21071153	3181	21489519	3226	21909467
3137	21080432	3182	21498834	3227	21918817
3138	21089713	3183	21508150	3228	21928168
3139	21098994	3184	21517467	3229	21937519
3140	21108276	3185	21526784	3230	21946871
3141	21117558	3186	21536103	3231	21956224
3142	21126842	3187	21545422	3232	21965577
3143	21136126	3188	21554742	3233	21974932
3144	21145411	3189	21564062	3234	21984287
3145	21154697	3190	21573384	3235	21993643
3146	21163984	3191	21582706	3236	22002999
3147	21173272	3192	21592029	3237	22012357
3148	21182560	3193	21601352	3238	22021715
3149	21191849	3194	21610677	3239	22031074
3150	21201139	3195	21620002	3240	22040433

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
3241	22049794	3286	22471781	3331	22895246
3242	22059155	3287	22481175	3332	22904673
3243	22068517	3288	22490571	3333	22914101
3244	22077879	3289	22499966	3334	22923529
3245	22087243	3290	22509363	3335	22932958
3246	22096607	3291	22518761	3336	22942388
3247	22105972	3292	22528159	3337	22951818
3248	22115337	3293	22537557	3338	22961249
3249	22124704	3294	22546957	3339	22970681
3250	22134071	3295	22556357	3340	22980113
3251	22143439	3296	22565758	3341	22989546
3252	22152807	3297	22575160	3342	22998980
3253	22162177	3298	22584562	3343	23008415
3254	22171547	3299	22593966	3344	23017850
3255	22180918	3300	22603370	3345	23027286
3256	22190289	3301	22612774	3346	23036722
3257	22199661	3302	22622179	3347	23046160
3258	22209035	3303	22631586	3348	23055598
3259	22218408	3304	22640992	3349	23065037
3260	22227783	3305	22650400	3350	23074476
3261	22237158	3306	22659808	3351	23083916
3262	22246534	3307	22669217	3352	23093357
3263	22255911	3308	22678627	3353	23102799
3264	22265289	3309	22688037	3354	23112241
3265	22274667	3310	22697448	3355	23121684
3266	22284046	3311	22706860	3356	23131128
3267	22293426	3312	22716273	3357	23140572
3268	22302806	3313	22725686	3358	23150017
3269	22312187	3314	22735100	3359	23159463
3270	22321569	3315	22744514	3360	23168909
3271	22330952	3316	22753930	3361	23178356
3272	22340336	3317	22763346	3362	23187804
3273	22349720	3318	22772763	3363	23197253
3274	22359105	3319	22782180	3364	23206702
3275	22368490	3320	22791599	3365	23216152
3276	22377877	3321	22801018	3366	23225602
3277	22387264	3322	22810437	3367	23235054
3278	22396652	3323	22819858	3368	23244505
3279	22406040	3324	22829279	3369	23253958
3280	22415430	3325	22838701	3370	23263411
3281	22424820	3326	22848123	3371	23272866
3282	22434211	3327	22857546	3372	23282320
3283	22443602	3328	22866970	3373	23291776
3284	22452994	3329	22876395	3374	23301232
3285	22462387	3330	22885820	3375	23310689

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
3376	23320146	3421	23746437	3466	24174075
3377	23329604	3422	23755925	3467	24183593
3378	23339063	3423	23765414	3468	24193112
3379	23348523	3424	23774904	3469	24202631
3380	23357983	3425	23784395	3470	24212151
3381	23367444	3426	23793886	3471	24221672
3382	23376905	3427	23803378	3472	24231193
3383	23386368	3428	23812871	3473	24240715
3384	23395831	3429	23822364	3474	24250238
3385	23405294	3430	23831858	3475	24259761
3386	23414759	3431	23841352	3476	24269285
3387	23424224	3432	23850848	3477	24278809
3388	23433689	3433	23860343	3478	24288334
3389	23443156	3434	23869840	3479	24297860
3390	23452623	3435	23879337	3480	24307386
3391	23462090	3436	23888835	3481	24316913
3392	23471559	3437	23898334	3482	24326441
3393	23481028	3438	23907833	3483	24335969
3394	23490498	3439	23917333	3484	24345498
3395	23499968	3440	23926833	3485	24355028
3396	23509439	3441	23936334	3486	24364558
3397	23518911	3442	23945836	3487	24374089
3398	23528384	3443	23955339	3488	24383621
3399	23537857	3444	23964842	3489	24393153
3400	23547331	3445	23974345	3490	24402685
3401	23556805	3446	23983850	3491	24412219
3402	23566280	3447	23993355	3492	24421753
3403	23575756	3448	24002861	3493	24431288
3404	23585233	3449	24012367	3494	24440823
3405	23594710	3450	24021874	3495	24450359
3406	23604188	3451	24031382	3496	24459895
3407	23613666	3452	24040890	3497	24469432
3408	23623146	3453	24050399	3498	24478970
3409	23632626	3454	24059909	3499	24488509
3410	23642106	3455	24069419	3500	24498048
3411	23651587	3456	24078930	3501	24507588
3412	23661069	3457	24088442	3502	24517128
3413	23670552	3458	24097954	3503	24526669
3414	23680035	3459	24107467	3504	24536210
3415	23689519	3460	24116980	3505	24545753
3416	23699004	3461	24126494	3506	24555295
3417	23708489	3462	24136009	3507	24564839
3418	23717975	3463	24145525	3508	24574383
3419	23727461	3464	24155041	3509	24583928
3420	23736949	3465	24164558	3510	24593473

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
3511	24603019	3556	25033227	3601	25464656
3512	24612566	3557	25042801	3602	25474257
3513	24622113	3558	25052376	3603	25483858
3514	24631661	3559	25061951	3604	25493460
3515	24641209	3560	25071527	3605	25503063
3516	24650758	3561	25081104	3606	25512666
3517	24660308	3562	25090681	3607	25522270
3518	24669858	3563	25100259	3608	25531874
3519	24679409	3564	25109837	3609	25541479
3520	24688961	3565	25119416	3610	25551085
3521	24698513	3566	25128996	3611	25560691
3522	24708066	3567	25138576	3612	25570297
3523	24717619	3568	25148157	3613	25579905
3524	24727173	3569	25157738	3614	25589512
3525	24736728	3570	25167320	3615	25599121
3526	24746283	3571	25176903	3616	25608730
3527	24755839	3572	25186486	3617	25618339
3528	24765396	3573	25196070	3618	25627950
3529	24774953	3574	25205654	3619	25637560
3530	24784511	3575	25215239	3620	25647172
3531	24794069	3576	25224824	3621	25656783
3532	24803628	3577	25234411	3622	25666396
3533	24813187	3578	25243997	3623	25676009
3534	24822748	3579	25253585	3624	25685622
3535	24832308	3580	25263173	3625	25695237
3536	24841870	3581	25272761	3626	25704851
3537	24851432	3582	25282350	3627	25714467
3538	24860995	3583	25291940	3628	25724082
3539	24870558	3584	25301530	3629	25733699
3540	24880122	3585	25311121	3630	25743316
3541	24889686	3586	25320713	3631	25752933
3542	24899251	3587	25330305	3632	25762552
3543	24908817	3588	25339898	3633	25772170
3544	24918383	3589	25349491	3634	25781790
3545	24927950	3590	25359085	3635	25791410
3546	24937518	3591	25368679	3636	25801030
3547	24947086	3592	25378274	3637	25810651
3548	24956655	3593	25387870	3638	25820272
3549	24966224	3594	25397466	3639	25829895
3550	24975794	3595	25407063	3640	25839517
3551	24985365	3596	25416660	3641	25849141
3552	24994936	3597	25426258	3642	25858764
3553	25004508	3598	25435857	3643	25868389
3554	25014080	3599	25445456	3644	25878014
3555	25023653	3600	25455055	3645	25887639

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
3646	25897265	3691	26331014	3736	26765861
3647	25906892	3692	26340666	3737	26775536
3648	25916519	3693	26350318	3738	26785212
3649	25926147	3694	26359970	3739	26794889
3650	25935775	3695	26369623	3740	26804566
3651	25945404	3696	26379277	3741	26814243
3652	25955034	3697	26388931	3742	26823921
3653	25964664	3698	26398586	3743	26833601
3654	25974294	3699	26408241	3744	26843280
3655	25983925	3700	26417897	3745	26852960
3656	25993557	3701	26427553	3746	26862640
3657	26003189	3702	26437210	3747	26872320
3658	26012822	3703	26446868	3748	26882001
3659	26022455	3704	26456526	3749	26891683
3660	26032089	3705	26466184	3750	26901365
3661	26041724	3706	26475843	3751	26911048
3662	26051359	3707	26485503	3752	26920731
3663	26060994	3708	26495163	3753	26930415
3664	26070630	3709	26504824	3754	26940099
3665	26080267	3710	26514485	3755	26949784
3666	26089904	3711	26524147	3756	26959469
3667	26099542	3712	26533809	3757	26969155
3668	26109180	3713	26543472	3758	26978841
3669	26118819	3714	26553135	3759	26988528
3670	26128459	3715	26562799	3760	26998216
3671	26138099	3716	26572463	3761	27007903
3672	26147739	3717	26582128	3762	27017592
3673	26157381	3718	26591793	3763	27027281
3674	26167022	3719	26601459	3764	27036970
3675	26176664	3720	26611125	3765	27046660
3676	26186307	3721	26620792	3766	27056350
3677	26195950	3722	26630460	3767	27066041
3678	26205594	3723	26640128	3768	27075733
3679	26215239	3724	26649796	3769	27085425
3680	26224884	3725	26659465	3770	27095117
3681	26234529	3726	26669135	3771	27104810
3682	26244175	3727	26678805	3772	27114503
3683	26253822	3728	26688476	3773	27124198
3684	26263469	3729	26698147	3774	27133892
3685	26273117	3730	26707818	3775	27143587
3686	26282765	3731	26717491	3776	27153282
3687	26292414	3732	26727164	3777	27162978
3688	26302063	3733	26736838	3778	27172675
3689	26311713	3734	26746511	3779	27182372
3690	26321363	3735	26756186	3780	27192069

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
3781	27201767	3826	27638691	3871	28076594
3782	27211466	3827	27648412	3872	28086336
3783	27221165	3828	27658133	3873	28096078
3784	27230864	3829	27667855	3874	28105821
3785	27240565	3830	27677577	3875	28115564
3786	27250265	3831	27687300	3876	28125308
3787	27259966	3832	27697023	3877	28135052
3788	27269668	3833	27706746	3878	28144797
3789	27279370	3834	27716470	3879	28154542
3790	27289072	3835	27726195	3880	28164288
3791	27298775	3836	27735920	3881	28174034
3792	27308479	3837	27745645	3882	28183781
3793	27318183	3838	27755371	3883	28193528
3794	27327887	3839	27765098	3884	28203275
3795	27337592	3840	27774825	3885	28213023
3796	27347298	3841	27784352	3886	28222772
3797	27357004	3842	27794280	3887	28232520
3798	27366710	3843	27804008	3888	28242270
3799	27376417	3844	27813737	3889	28252019
3800	27386125	3845	27823466	3890	28261770
3801	27395833	3846	27833196	3891	28271520
3802	27405541	3847	27842926	3892	28281272
3803	27415259	3848	27852657	3893	28291023
3804	27424959	3849	27862388	3894	28300775
3805	27434669	3850	27872120	3895	28310528
3806	27444380	3851	27881852	3896	28320281
3807	27454091	3852	27891585	3897	28330034
3808	27463802	3853	27901318	3898	28339788
3809	27473514	3854	27911051	3899	28349342
3810	27483226	3855	27920785	3900	28359297
3811	27492939	3856	27930520	3901	28369052
3812	27502653	3857	27940255	3902	28378808
3813	27512367	3858	27949990	3903	28388564
3814	27522081	3859	27959726	3904	28398321
3815	27531796	3860	27969463	3905	28408078
3816	27541511	3861	27979199	3906	28417835
3817	27551227	3862	27988937	3907	28427593
3818	27560943	3863	27998675	3908	28437351
3819	27570660	3864	28008413	3909	28447110
3820	27580377	3865	28018152	3910	28456870
3821	27590095	3866	28027891	3911	28466629
3822	27599813	3867	28037630	3912	28476389
3823	27609532	3868	28047370	3913	28486150
3824	27619251	3869	28057111	3914	28495911
3825	27628971	3870	28066852	3915	28505673

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
3916	28515435	3961	28955175	4006	29395776
3917	28525197	3962	28964957	4007	29405576
3918	28534960	3963	28974739	4008	29415378
3919	28544723	3964	28984522	4009	29425179
3920	28554487	3965	28994305	4010	29434981
3921	28564251	3966	29004089	4011	29444783
3922	28574015	3967	29013873	4012	29454585
3923	28583781	3968	29023657	4013	29464388
3924	28593546	3969	29033442	4014	29474192
3925	28603312	3970	29043228	4015	29483996
3926	28613078	3971	29053013	4016	29493800
3927	28622845	3972	29062799	4017	29503605
3928	28632612	3973	29072586	4018	29513410
3929	28642380	3974	29082373	4019	29523215
3930	28652148	3975	29092160	4020	29533021
3931	28661917	3976	29101948	4021	29542827
3932	28671686	3977	29111736	4022	29552634
3933	28681455	3978	29121525	4023	29562441
3934	28691225	3979	29131314	4024	29572248
3935	28700995	3980	29141104	4025	29582056
3936	28710766	3981	29150894	4026	29591864
3937	28720537	3982	29160684	4027	29601673
3938	28730309	3983	29170475	4028	29611482
3939	28740081	3984	29180266	4029	29621291
3940	28749853	3985	29190057	4030	29631101
3941	28759626	3986	29199849	4031	29640911
3942	28769400	3987	29209642	4032	29650722
3943	28779173	3988	29219435	4033	29660533
3944	28788948	3989	29229228	4034	29670344
3945	28798722	3990	29239022	4035	29680156
3946	28808497	3991	29248816	4036	29689968
3947	28818273	3992	29258610	4037	29699781
3948	28828049	3993	29268405	4038	29709594
3949	28837825	3994	29278200	4039	29719407
3950	28847602	3995	29287996	4040	29729221
3951	28857379	3996	29297792	4041	29739035
3952	28867157	3997	29307589	4042	29748850
3953	28876935	3998	29317386	4043	29758665
3954	28886713	3999	29327183	4044	29768480
3955	28896492	4000	29336981	4045	29778296
3956	28906272	4001	29346779	4046	29788112
3957	28916051	4002	29356577	4047	29797928
3958	28925832	4003	29366376	4048	29807745
3959	28935612	4004	29376176	4049	29817562
3960	28945393	4005	29385976	4050	29827380

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
4051	29837198	4096	30279403	4141	30722353
4052	29847017	4097	30289239	4142	30732205
4053	29856835	4098	30299075	4143	30742057
4054	29866655	4099	30308911	4144	30751909
4055	29876474	4100	30318747	4145	30761761
4056	29886294	4101	30328584	4146	30771614
4057	29896114	4102	30338421	4147	30781467
4058	29905935	4103	30348259	4148	30791321
4059	29915756	4104	30358097	4149	30801175
4060	29925578	4105	30367935	4150	30811029
4061	29935400	4106	30377774	4151	30820884
4062	29945222	4107	30387613	4152	30830739
4063	29955045	4108	30397452	4153	30840594
4064	29964868	4109	30407292	4154	30850450
4065	29974691	4110	30417132	4155	30860306
4066	29984515	4111	30426973	4156	30870162
4067	29994339	4112	30436813	4157	30880019
4068	30004164	4113	30446655	4158	30889876
4069	30013988	4114	30456496	4159	30899733
4070	30023814	4115	30466338	4160	30909591
4071	30033639	4116	30476180	4161	30919449
4072	30043466	4117	30486023	4162	30929307
4073	30053292	4118	30495866	4163	30939166
4074	30063119	4119	30505709	4164	30949025
4075	30072946	4120	30515553	4165	30958884
4076	30082774	4121	30525397	4166	30968744
4077	30092602	4122	30535242	4167	30978604
4078	30102430	4123	30545086	4168	30988465
4079	30112259	4124	30554932	4169	30998325
4080	30122088	4125	30564777	4170	31008186
4081	30131917	4126	30574623	4171	31018048
4082	30141747	4127	30584469	4172	31027910
4083	30151577	4128	30594316	4173	31037772
4084	30161408	4129	30604163	4174	31047634
4085	30171239	4130	30614010	4175	31057497
4086	30181070	4131	30623858	4176	31067360
4087	30190902	4132	30633706	4177	31077223
4088	30200734	4133	30643554	4178	31087087
4089	30210566	4134	30653403	4179	31096951
4090	30220399	4135	30663252	4180	31106816
4091	30230232	4136	30673101	4181	31116681
4092	30240066	4137	30682951	4182	31126546
4093	30249899	4138	30692801	4183	31136411
4094	30259734	4139	30702651	4184	31146277
4095	30269[illegible]68	4140	30712502	4185	31156143

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
4186	31166009	4231	31610334	4276	32055289
4187	31175876	4232	31620215	4277	32065183
4188	31185743	4233	31630096	4278	32075078
4189	31195611	4234	31639978	4279	32084974
4190	31205478	4235	31649860	4280	32094869
4191	31215346	4236	31659743	4281	32104765
4192	31225215	4237	31669625	4282	32114661
4193	31235084	4238	31679509	4283	32124558
4194	31244953	4239	31689392	4284	32134455
4195	31254822	4240	31699276	4285	32144352
4196	31264692	4241	31709159	4286	32154249
4197	31274562	4242	31719044	4287	32164147
4198	31284482	4243	31728928	4288	32174045
4199	31294303	4244	31738813	4289	32183943
4200	31304174	4245	31748698	4290	32193842
4201	31314045	4246	31758584	4291	32203740
4202	31323917	4247	31768470	4292	32213640
4203	31333789	4248	31778356	4293	32223539
4204	31343661	4249	31788242	4294	32233439
4205	31353534	4250	31798129	4295	32243339
4206	31363407	4251	31808016	4296	32253239
4207	31373280	4252	31817903	4297	32263139
4208	31383153	4253	31827791	4298	32273040
4209	31393027	4254	31837679	4299	32282941
4210	31402902	4255	31847567	4300	32292843
4211	31412776	4256	31857455	4301	32302744
4212	31422651	4257	31867344	4302	32312646
4213	31432526	4258	31877233	4303	32322548
4214	31442402	4259	31887123	4304	32332451
4215	31452278	4260	31897013	4305	32342354
4216	31462154	4261	31906903	4306	32352257
4217	31472030	4262	31916793	4307	32362160
4218	31481907	4263	31926684	4308	32372064
4219	31491784	4264	31936574	4309	32381968
4220	31501661	4265	31946466	4310	32391872
4221	31511539	4266	31956357	4311	32401776
4222	31521417	4267	31966249	4312	32411681
4223	31531296	4268	31976141	4313	32421586
4224	31541174	4269	31986033	4314	32431491
4225	31551053	4270	31995926	4315	32441397
4226	31560933	4271	32005819	4316	32451303
4227	31570812	4272	32015712	4317	32461209
4228	31580692	4273	32025606	4318	32471115
4229	31590572	4274	32035500	4319	32481022
4230	31600453	4275	32045394	4320	32490929

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
4321	32500836	4366	32946939	4411	33393559
4322	32510744	4367	32956858	4412	33403490
4323	32520651	4368	32966778	4413	33413420
4324	32530559	4369	32976698	4414	33423351
4325	32540468	4370	32986618	4415	33433282
4326	32550376	4371	32996538	4416	33443214
4327	32560285	4372	33006459	4417	33453146
4328	32570194	4373	33016380	4418	33463078
4329	32580104	4374	33026301	4419	33473010
4330	32590013	4375	33036223	4420	33482942
4331	32599923	4376	33046144	4421	33492875
4332	32609834	4377	33056066	4422	33502807
4333	32619744	4378	33065988	4423	33512741
4334	32629655	4379	33075911	4424	33522674
4335	32639566	4380	33085834	4425	33532607
4336	32649477	4381	33095757	4426	33542541
4337	32659389	4382	33105680	4427	33552475
4338	32669301	4383	33115603	4428	33562409
4339	32679213	4384	33125527	4429	33572344
4340	32689125	4385	33135451	4430	33582279
4341	32699038	4386	33145375	4431	33592213
4342	32708951	4387	33155299	4432	33602149
4343	32718864	4388	33165224	4433	33612084
4344	32728777	4389	33175149	4434	33622020
4345	32738691	4390	33185074	4435	33631955
4346	32748605	4391	33195000	4436	33641892
4347	32758529	4392	33204925	4437	33651828
4348	32768434	4393	33214851	4438	33661764
4349	32778348	4394	33224777	4439	33671701
4350	32788263	4395	33234704	4440	33681638
4351	32798179	4396	33244630	4441	33691575
4352	32808094	4397	33254557	4442	33701513
4353	32818010	4398	33264484	4443	33711450
4354	32827926	4399	33274412	4444	33721388
4355	32837842	4400	33284339	4445	33731326
4356	32847759	4401	33294267	4446	33741265
4357	32857676	4402	33304195	4447	33751203
4358	32867593	4403	33314124	4448	33761142
4359	32877510	4404	33324052	4449	33771081
4360	32887428	4405	33333981	4450	33781020
4361	32897346	4406	33343910	4451	33790959
4362	32907264	4407	33353840	4452	33800899
4363	32917182	4408	33363769	4453	33810839
4364	32927101	4409	33373699	4454	33820779
4365	32937020	4410	33382629	4455	33830719

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
4456	33840660	4501	34288204	4546	34736154
4457	33850601	4502	34298154	4547	34746113
4458	33860542	4503	34308105	4548	34756072
4459	33870483	4504	34318055	4549	34766031
4460	33880424	4505	34328006	4550	34775991
4461	33890366	4506	34337957	4551	34785950
4462	33900308	4507	34347908	4552	34795910
4463	33910250	4508	34357859	4553	34805870
4464	33920192	4509	34367811	4554	34815830
4465	33930134	4510	34377763	4555	34825790
4466	33940077	4511	34387715	4556	34835750
4467	33950020	4512	34397667	4557	34845711
4468	33959963	4513	34407619	4558	34855672
4469	33969907	4514	34417572	4559	34865633
4470	33979850	4515	34427525	4560	34875594
4471	33989794	4516	34437478	4561	34885555
4472	33999738	4517	34447431	4562	34895516
4473	34009682	4518	34457384	4563	34905478
4474	34019626	4519	34467338	4564	34915440
4475	34029571	4520	34477291	4565	34925402
4476	34039516	4521	34487245	4566	34935364
4477	34049461	4522	34497199	4567	34945326
4478	34059406	4523	34507154	4568	34955289
4479	34069352	4524	34517108	4569	34965252
4480	34079297	4525	34527063	4570	34975215
4481	34089243	4526	34537018	4571	34985178
4482	34099189	4527	34546973	4572	34995141
4483	34109136	4528	34556928	4573	35005104
4484	34119082	4529	34566883	4574	35015068
4485	34129029	4530	34576839	4575	35025031
4486	34138976	4531	34586795	4576	35034995
4487	34148923	4532	34596751	4577	35044959
4488	34158870	4533	34606707	4578	35054924
4489	34168818	4534	34616663	4579	35064888
4490	34178765	4535	34626620	4580	35074853
4491	34188713	4536	34636577	4581	35084817
4492	34198662	4537	34646534	4582	35094782
4493	34208610	4538	34656491	4583	35104747
4494	34218558	4539	34666448	4584	35114713
4495	34228507	4540	34676406	4585	35124678
4496	34238456	4541	34686363	4586	35134644
4497	34248405	4542	34696321	4587	35144609
4498	34258355	4543	34706279	4588	35154575
4499	34268304	4544	34716237	4589	35164541
4500	34278254	4545	34726196	4590	35174508

Verf. fine	Seg. area	Verf. fine	Seg. area	Verf. fine	Seg. area
4591	35184474	4636	35633126	4681	36082074
4592	35194441	4637	35643099	4682	36092053
4593	35204407	4638	35653073	4683	36102033
4594	35214374	4639	35663047	4684	36112013
4595	35224341	4640	35673021	4685	36121993
4596	35234308	4641	35682995	4686	36131973
4597	35244276	4642	35692969	4687	36141954
4598	35254243	4643	35702944	4688	36151934
4599	35264211	4644	35712918	4689	36161915
4600	35274179	4645	35722893	4690	36171895
4601	35284147	4646	35732868	4691	36181876
4602	35294115	4647	35742843	4692	36191857
4603	35304084	4648	35752818	4693	36201838
4604	35314052	4649	35762793	4694	36211820
4605	35324021	4650	35772769	4695	36221801
4606	35333990	4651	35782744	4696	36231783
4607	35343958	4652	35792720	4697	36241764
4608	35353928	4653	35802696	4698	36251746
4609	35363897	4654	35812672	4699	36261728
4610	35373866	4655	35822648	4700	36271710
4611	35383836	4656	35832624	4701	36281691
4612	35393806	4657	35842600	4702	36291673
4613	35403776	4658	35852577	4703	36301656
4614	35413746	4659	35862553	4704	36311638
4615	35423716	4660	35872530	4705	36321621
4616	35433686	4661	35882507	4706	36331603
4617	35443657	4662	35892484	4707	36341586
4618	35453628	4663	35902461	4708	36351569
4619	35463598	4664	35912439	4709	36361552
4620	35473569	4665	35922416	4710	36371535
4621	35483541	4666	35932394	4711	36381518
4622	35493512	4667	35942372	4712	36391501
4623	35503483	4668	35952349	4713	36401485
4624	35513455	4669	35962327	4714	36411469
4625	35523427	4670	35972306	4715	36421452
4626	35533399	4671	35982284	4716	36431436
4627	35543371	4672	35992262	4717	36441420
4628	35553343	4673	36002241	4718	36451404
4629	35563315	4674	36012219	4719	36461388
4630	35573288	4675	36022198	4720	36471372
4631	35583260	4676	36032177	4721	36481357
4632	35593233	4677	36042156	4722	36491341
4633	35603206	4678	36052135	4723	36501326
4634	35613179	4679	36062115	4724	36511310
4635	35623153	4680	36072094	4725	36521295

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
4726	36531280	4772	36990699	4818	37450310
4727	36541265	4773	37000688	4819	37460304
4728	36551250	4774	37010678	4820	37470297
4729	36561236	4775	37020668	4821	37480291
4730	36571221	4776	37030658	4822	37490284
4731	36581206	4777	37040648	4823	37500278
4732	36591192	4778	37050638	4824	37510272
4733	36601178	4779	37060628	4825	37520266
4734	36611163	4780	37070618	4826	37530259
4735	36621149	4781	37080609	4827	37540253
4736	36631135	4782	37090599	4828	37550247
4737	36641121	4783	37100590	4829	37560242
4738	36651108	4784	37110580	4830	37570236
4739	36661094	4785	37120571	4831	37580230
4740	36671080	4786	37130562	4832	37590224
4741	36681067	4787	37140553	4833	37600219
4742	36691053	4788	37150544	4834	37610213
4743	36701040	4789	37160535	4835	37620208
4744	36711027	4790	37170526	4836	37630202
4745	36721014	4791	37180517	4837	37640197
4746	36731001	4792	37190508	4838	37650192
4747	36740988	4793	37200500	4839	37660186
4748	36750976	4794	37210491	4840	37670181
4749	36760963	4795	37220483	4841	37680176
4750	36770950	4796	37230474	4842	37690171
4751	36780938	4797	37240466	4843	37700166
4752	36790925	4798	37250458	4844	37710161
4753	36800913	4799	37260450	4845	37720156
4754	36810901	4800	37270442	4846	37730152
4755	36820889	4801	37280434	4847	37740147
4756	36830877	4802	37290426	4848	37750142
4757	36840865	4803	37300418	4849	37760138
4758	36850853	4804	37310410	4850	37770133
4759	36860842	4805	37320403	4851	37780129
4760	36870830	4806	37330395	4852	37790124
4761	36880819	4807	37340387	4853	37800120
4762	36890807	4808	37350380	4854	37810116
4763	36900796	4809	37360373	4855	37820111
4764	36910785	4810	37370365	4856	37830107
4765	36920774	4811	37380358	4857	37840103
4766	36930763	4812	37390351	4858	37850099
4767	36940752	4813	37400344	4859	37860095
4768	36950741	4814	37410337	4860	37870091
4769	36960730	4815	37420330	4861	37880087
4770	36970720	4816	37430324	4862	37890083
4771	36980709	4817	37440317	4863	37900080

Vers. sine	Seg. area	Vers. sine	Seg. area	Vers. sine	Seg. area
4864	37910076	4910	38369956	4956	38829914
4865	37920072	4911	38379955	4957	38839913
4866	37930069	4912	38389953	4958	38849913
4867	37940065	4913	38399952	4959	38859913
4868	37950062	4914	38409950	4960	38869912
4869	37960058	4915	38419949	4961	38879912
4870	37970055	4916	38429947	4962	38889912
4871	37980051	4917	38439946	4963	38899912
4672	37990048	4918	38449945	4964	38909911
4873	38000045	4919	38459943	4965	38919911
4874	38010042	4920	38469942	4966	38929911
4875	38020038	4921	38479941	4967	38939911
4876	38030035	4922	38489939	4968	38949910
4877	38040032	4923	38499938	4969	38959910
4878	38050029	4924	38509937	4970	38969910
4879	38060026	4925	38519936	4971	38979910
4880	38070023	4926	38529935	4972	38989910
4881	38080020	4927	38539934	4973	38999909
4882	38090018	4928	38549933	4974	39009909
4883	38100015	4929	38559932	4975	39019909
4884	38110012	4930	38569931	4976	39029909
4885	38120010	4931	38579930	4977	39039909
4886	38130007	4932	38589929	4978	39049909
4887	38140004	4933	38599928	4979	39059909
4888	38150002	4934	38609927	4980	39069909
4889	38159999	4935	38619926	4981	39079909
4890	38169997	4936	38629926	4982	39089908
4891	38179994	4937	38639925	4983	39099908
4892	38189992	4938	38649924	4984	39109908
4893	38199990	4939	38659923	4985	39119908
4894	38209988	4940	38669923	4986	39129908
4895	38219985	4941	38679922	4987	39139908
4896	38229983	4942	38689921	4988	39149908
4897	38239981	4943	38699920	4989	39159908
4898	38249979	4944	38709920	4990	39169908
4899	38259977	4945	38719919	4991	39179908
4900	38269975	4946	38729919	4992	39189908
4901	38279973	4947	38739918	4993	39199908
4902	38289971	4948	38749918	4994	39209908
4903	38299969	4949	38759917	4995	39219908
4904	38309967	4950	38769917	4996	39229908
4905	38319965	4951	38779916	4997	39239908
4906	38329963	4952	38789916	4998	39249908
4907	38339961	4953	38799915	4999	39259908
4908	38349960	4954	38809915	5000	39269908
4909	38359958	4955	38819914		

2

In the preceding table, each number in the column of ſegments is the area of the circular ſegment, whoſe height, or verſed ſine of its half arc, is the number immediately on the left of it, the diameter of the circle being 1, and the whole area 78539816; and therefore all the numbers in the table are to be accounted decimals.

This table is here extended to ten times the length which it had uſually been, the diameter being divided into ten thouſand equal parts, inſtead of one thouſand: And this ſize of it alſo allowed me to carry each number out to eight places of figures, inſtead of ſix; for when the diameter is divided only into one thouſand parts, the proportional part for the figures in the verſed ſine, beyond the third, will not give the area true beyond the 6th figure; but in this table it is always found true in the 8th place.

Rule the 9th to the circular ſegment in page 143 explains the uſe of it; to which it may be only neceſſary to add there, when a ſegment greater than a ſemicircle is to be found, ſubtract the quotient of its verſed ſine, divided by its diameter, from 1; then ſubtract the tabular ſegment which correſponds to the remainder, from 78539816, the whole tabular circle, and the remainder will be the tabular circle correſponding to the ſegment required; and which muſt then be multiplied by the ſquare of the diameter.

F I N I S.

Having the Plan of a Field, and its true Area, to find the scale by which it has been constructed.
Rule. By any scale whatever, measure such lines as will give you the area of the figure; then say As this area is to the square of the scale by which it was found, so is the true area, to the square of the scale required.

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9a. 1r. 32p. and that by a ~~Chain~~ scale of 2 Chains to an inch I find the area to be 4a. 0r. 32p. required the scale by which the plan was Constructed
First 9.1.32 = 945000 sqr Links and 4.0.32 = 420000 sqr links then
As 420000 : 4 :: 945000 : 9 the square root of which is 3 Hence it appears the plan was Constructed by a scale of 3 Chains to an inch

On the Strength of Columns; or Power of Resistance to Compressive Force

Table of Practical Formulae by which to determine the amount of Weight a Column of given dimensions will support in ~~lbs.~~

For a rectangular column of cast iron	$W = \dfrac{15300\, l\, b^3}{4b^2 + .18\, l^2}$.
For a rectangular column of Malleable iron	$W = \dfrac{17800\, l\, b^3}{4b^2 + .16\, l^2}$.
For a rectangular column of Oak	$W = \dfrac{3960\, l\, b^3}{4b^2 + .5\, l^2}$.
For a solid cylinder of Cast iron	$W = \dfrac{9562\, d^4}{4d^2 + .18\, l^2}$.
For a solid cylinder of Malleable iron	$W = \dfrac{11125\, d^4}{4d^2 + .16\, l^2}$.
For a solid cylinder of Oak	$W = \dfrac{2470\, d^4}{4d^2 + .5\, l^2}$.

Note:— W = the weight the column will support ~~in lbs.~~ lbs.
b = the breadth in inches.
l = the length in feet.
d = the diameter in inches.

Ex. 1 A rectangular column of oak 6 inches on the side and 12 feet in length, what weight will it support

$$\frac{3960 \times 12 \times 6^3}{4 \times 6^2 + .5 \times 12^2} = \frac{10264320}{216} = 47520 \text{ lbs}$$

Ex 2. What weight will a cast iron cylinder support, whose diameter is ~~4~~ 5 inches and length 10 feet

$$\frac{9562 \times 5^4}{4 \times 5^2 + .18 \times 10^2} = \frac{5976250}{118} = 50646 \text{ lbs. Answer.}$$

or thus

10^2 length: 10 × 10 = 100; 100 × .18 = 18.00

5^2 diameter: 5 × 5 = 25; 25 × 4 = 100; 100 + 18 = 118

5^4 diameter: 5 × 5 = 25; 25 × 5 = 125; 125 × 5 = 625

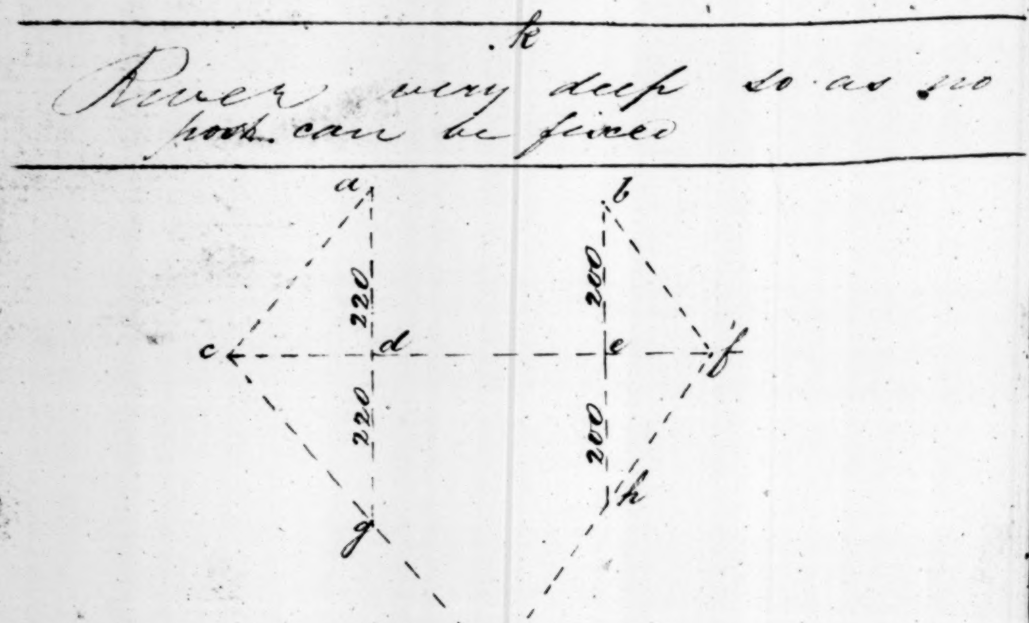
9562 × 625: 47810, 19124, 57372 = 5976250

118) 5976250 (50646 pounds Ans.

From Engineers Pocket Book 1847

Problem for finding the length of line when the Angle or Station falls in a River so that you cannot get to it.

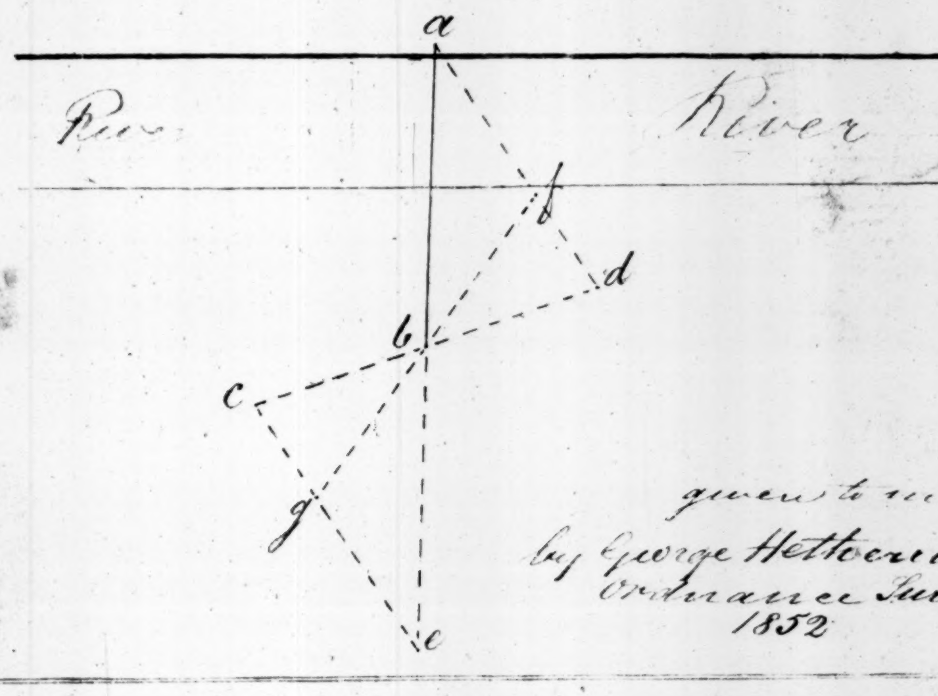
First run a line c f at any distance you please from the River. then erect perpendiculars at a and b and produce g and h. draw the two lines c i and f i then c i will be equal to c k and f i equal f k.



given to me by Mr George Hetherington
Ordnance Surveyor 185

Problem for Measuring across Rivers &c

I want to know how far it is from a to b. Run a line across at b no matter if its ever so obtuse as bd and bc, make bd and bc equal say 250 links each place a post in a direct line betwixt d and a at f next produce f g through b, then the angle bfd is equal to bcg and produce ce then the line be will be equal to ba



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1852